Calculus I Practice problems

Chap 1: Sec. 1.2-Sec. 1.5:

1. Let $f(x) = \begin{cases} |x+2| & \text{for } x \le 0; \\ 2+x^2 & \text{for } 0 < x < 2; \\ x^3 & \text{for } x \ge 2 \end{cases}$. Find (a) $\lim_{x\to 0^-} f(x)$, (b)$\lim_{x\to 0^+} f(x)$, (c) $\lim_{x\to 2^-} f(x)$, (d)$\lim_{x\to 2^+} f(x)$, (e) $\lim_{x\to 0} f(x)$, (f) $\lim_{x\to 2} f(x)$.
   Ans: (a) 2 (b) 2 (c) 6 (d) 8 (e) 2 (f) DNE

2. Compute $\lim_{x\to 0} \frac{1-\cos 4x}{9x^2}$ Ans: 8/9

3. Let $f(x) = \begin{cases} cx-2 & \text{for } x \le 2; \\ cx^2+2 & \text{for } x > 2 \end{cases}$ Find c such that $f(x)$ is continuous.
   Ans: $c = -2$

4. Determine the intervals on which $f(x) = \ln(1-x^2)$ is continuous. Ans: $1 - x^2 > 0$ or $(-1, 1)$

5. Compute
   (i) $\lim_{x\to 0} \frac{\sqrt{x+9}-3}{x}$
   (ii) $\lim_{x\to 1^-} \frac{2x}{x^2-1}$
   (iii) $\lim_{x\to\infty} \frac{x+\sin x}{4x+999}$
   Ans: (i) 6 (ii) $-\infty$ (iii) 1/4

Chap 2: Sec. 2.3-Sec. 2.9:

1. $\frac{d}{dx}[\frac{x^2-x}{3x+1}]$ =?

2. Find the tangent line to the curve $y = x^3 - 4x^2 + 2x + 1$ at the point $(1, 0)$.

3. (a) Let $y = \ln\sqrt{\frac{3x+1}{5x+2}}$. Find $\frac{dy}{dx}$. Ans: $\frac{(3x-1)(x+1)}{(3x+1)^2}$
   (b) Let $y = e^{x^2}\sin(x^2+x+1)\cdot\sqrt{3x+1}/(x^2-1)$. Find $\frac{dy}{dx}$.
   Hint: Take ln on bouth side.

4. The equation $7x^2y^3 - 5xy^2 - 4y = 7$ defines y implicitly as a function of x. Find $\frac{dy}{dx}$.
   Ans: $\frac{14xy^3-5y^2}{4+10xy-21x^2y^2}$

5. Find the detivative of $f(x) = x^{2x}$ Ans: $2(\ln x + 1)x^{2x}$

6. Compute $\frac{d}{dx}\cos^{-1}(2x^3)$ Ans: $\frac{-6x^2}{\sqrt{1-(2x^3)}^2}$

7. Determine if $f(x) = x^7 + 2x^3 - 2006$ is increasing, decreasing or neither. Prove $f(x) = 0$ has exactly one solution. Hint: Sec. 2.9 example 9.1

Chap 3: Sec. 3.1-Sec. 3.8:

1. Estimate $\tan((\pi/4) + 0.05)$ by the method of linear approximation (i.e., by differentials). Ans: $1+2*(0.05) = 1.1$

2. Compute $\lim_{x\to 1^+} \frac{\ln x}{(x-1)^2}$ Ans: ∞

3. Find the asymptotes of
   (i) $f(x) = \frac{(3x-1)^2}{9x^2-4}$. Ans: V:$x = 2/3$, $x = -2/3$; H:$y = 1$
   (ii) $f(x) = \frac{(3x-1)^2}{9x^2-1}$.Ans: V:$x = -1/3$; H:$y = 1$
   (iii)$f(x) = \frac{(3x-1)^2}{x-1}$ Ans: V:$x = 1$; H:none; S:$y = 9x + 3$.

4. Let $f(x) = 2x^3 - 3x^2 - 12x$. Find the relative extrema of $f(x)$. Ans: local max at $x = -1$; local min at $x = 2$; no abs. max/min

5. Find the absolute maximum and minimum values of the function $f(x) = 2x^3 - 9x^2 + 12x$ over the interval $[0, 2]$. Ans: Abs max: $f(1) = 5$; abs min: $f(0) = 0$

6. Determine the concavity of $f(x) = 4x^3 - x^4$. Ans: Concave up: $(-\infty, 0) \cup (2, \infty)$; Concave down: $(0, 2)$

7. If 300 cm^2 of material is available to make a box with square base and an open top, find the largest possible volume of the box. Explain why your answer is the absolute maximum.

8. Sketch the graph of the continuous function f that satisfies the conditions:

$$\begin{aligned} f''(x) &> 0 \quad \text{if } |x| > 2, \quad f''(x) < 0 \quad \text{if } |x| < 2; \\ f'(0) &= 0, \quad f'(x) > 0, \quad \text{if } x < 0, \quad f'(x) < 0, \quad \text{if } x > 0; \\ f(0) &= 1, \quad f(2) = \frac{1}{2}, \quad f(x) > 0 \quad \text{for all } x, \text{and } f \text{ is and even function.} \end{aligned}$$

Ans: The graph looks like $e^{-x^2/8}$

9. An automobile dealer is selling cars at a price of \$12,000. The demand function is $D(p) = 2(15 - 0.001p)^2$, where p is the price of a car. Should the dealer raise or lower the price to increase the revenue? Hint: Find the detivative of the revenue function, $R(p) = p \cdot D(p)$

10. Compute:
(i) $\lim_{x \to 0}(\frac{1}{\ln(x+1} - \frac{1}{x})$
(ii) $\lim_{x \to 0^+}(\cos x)^{1/x}$
(iii) $\lim_{x \to \infty}(1 + \frac{1}{x})^x$
Ans (i) Sec 3.2 example 2.7 (ii) Sec 3.2 exercise 38 (iii) e

Chap 4: Sec. 4.2-Sec. 4.7, Sec. 4.10.

1. Let $f(x) = x + 1$
(a) Divide the interval $[0, 5]$ into n equal parts, and using right endpoints find an expression for the Riemann sum R_n.
Hint: $1 + 2 + ... + n = \frac{n(n+1)}{2}$
(b) Using the answer you got from part(a), calculate $\lim\limits_{n \to \infty} R_n$ (without using antiderivatives).

2. Find the derivatives of the following functions. It is not necessary to simplify your answer:
(a) $f(x) = ((x^2 + 1)^3 + 1)^4$
(b) $G(x) = \int_0^{x^2} \sqrt{1 + t^4}\, dt$
Ans: (a) $4((x^2 + 1)^3 + 1)^3 \cdot 3 \cdot (x^2 + 1)^2 \cdot (2x)$ (b)$2x \cdot \sqrt{1 + x^8}$

3. Let f be continuous and define F by

$$F(x) = \int_0^x [t^2 \int_1^t f(u)\, du]\, dt.$$

Find $F'(x)$ and $F''(x)$. Ans: $F'(x) = x^2 \int_1^x f(u)\, du$, $F''(x) = 2x \int_1^x f(u)\, du + x^2 f(x)$

4. Evaluate the given integral
(i) $\int x(x + 1)^9\, dx$, Hint:$u = x + 1$
(ii)$\int \frac{\cos\theta}{\sin^2\theta - 2\sin\theta - 8}\, d\theta$. Hint $u = \cos\theta$
(iii) $\int \frac{dx}{e^x\sqrt{4+e^{2x}}}$. Hint: $u = e^x$, $u = 2\tan\theta$
(iv) $\int \frac{\ln x}{x\sqrt{1+\ln x}}\, dx$, Hint:$u = \ln x$
(v) $\int \frac{x^3}{\sqrt{x^2+1}}\, dx$. Hint: $x = \tan\theta$
(vi) $\int_{-\infty}^{\infty} \frac{e^{-x}}{(1+e^{-x})^2}\, dx$ Ans: 1

5. Evaluate the given integral
   (i) $\int \sec^3 t\,dt$
   (ii) $\int \sec t\,dt$
   Ans:(i) $(-\ln(\cos t/2 - \sin t/2) + \ln(\cos t/2 + \sin t/2) + \sec t \tan t)/2 + C$ (ii) $\ln|\sec t + \tan t| + C$

6. (a) $\int_{-\infty}^{\infty} x^3\,dx$ (b) $\lim_{R\to\infty} \int_{-R}^{R} x^3\,dx$ Ans: (a) DIV (b) 0

7. Determine whether the integral converges or diverges:
   (i) $\int_0^1 x^{-1/3}\,dx$
   (ii) $\int_0^1 x^{-4/3}\,dx$
   (iii) $\int_1^{\infty} x^{-1/3}\,dx$
   (iv) $\int_{-1}^{1} x^{-1/3}\,dx$ Ans: (i) 3/2 (ii) DIV (iii) DIV (iv) 0

Chap 5: Sec. 5.1-Sec. 5.6.

1. Find the region bounded by the parabola $x = 2 - y^2$ and the line $y = x$.
   Ans: $\int_{-2}^{1} (2 - y^2) - y\,dy = ...$

2. A solid is formed by revolving the circular disk $(x-5)^2 + y^2 = 4$ about the y-axis. Set up, **but do not evaluate**, a definite integral which give the volume of the solid.
   Ans: $\int_{-2}^{2} \pi[(5+\sqrt{4-y^2})^2 - (5-\sqrt{4-y^2})^2]\,dy$ or $\int_3^7 2\pi x[\sqrt{4-(x-5)^2} - (-\sqrt{4-(x-5)^2})]\,dx$

3. Let Ω be the region bounded by $y = \sec x$, $x = 0$, $x = \frac{\pi}{4}$ and $y = 0$. Find integrals represent the volume of the solids generated by Ω about (a) x-axis, (b) y-axis, (c) $y = -1$, (d) $x = -1$. (**Don't evaluate the integrals**) Ans: Midterm 2

4. Set up a definite integral for the arc length of an ellipse $x^2 + 4y^2 = 4$. Ans: $4\int_0^2 \sqrt{1 + (-\frac{x}{4\sqrt{1-x^2/4}})^2}\,dx$ or $4\int_0^{\pi/2} \sqrt{(-2\sin\theta)^2 + (\cos\theta)^2}\,d\theta$

5. Set up the integral for the surface area of the surface of revolution. $y = e^x, 0 \le x \le 1$, revolved about x-axis.
   Ans: $S = \int_0^1 2\pi e^x \sqrt{1 + (e^x)^2}\,dx$

6. (i) At time t, a particle has position $x(t) = 1 - \cos t, \quad y(t) = t - \sin t$ Find the total distance traveled from $t = 0$ to $t = 2\pi$. Find the speed of the particle at $t = \pi$.
   Ans: $\int_0^{2\pi} \sqrt{(\sin t)^2 + (1-\cos t)^2}\,dt$, speed at $t = \pi$ is 2
   (ii) Find the area of the surface generated by revolving the curve $y = \cosh x$, $x \in [0, \ln 2]$ about the x-axis.
   Ans: $S = \int_0^{\ln 2} 2\pi \cosh x \sqrt{1 + (\sinh x)^2}\,dx$

Chap 6: Sec. 6.1-Sec. 6.5.

1. Two years ago, there were 4 grams of a radioactive substance . Now there are 3 grams. How much was there 10 years ago? Ans: $\frac{4^5}{3^4}$

2. Find the size of permanent endowment needed to generate an annual \$2,000 forever at 10% (annual) interest compounded continuously. Ans: $2000 \cdot e^{-0.1}$

3. Solve the IVP, explicitly, if possible $y' = \frac{x-1}{y^2}, y(0) = 2$. Ans: $y = (\frac{3}{2}x^2 - 3x + 8)^{1/3}$