

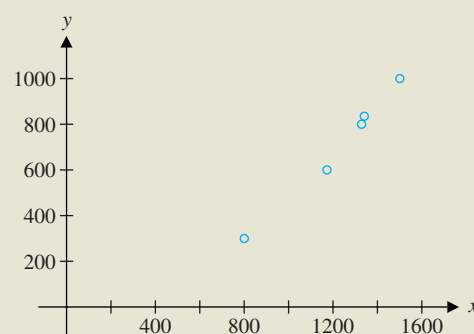
PRELIMINARIES



One of the most famous of all scientific legends has the 16th-century scientist Galileo dropping a wood ball and a lead ball simultaneously from the Leaning Tower of Pisa. In this story, the balls hit the ground at the same time, proving that the acceleration due to gravity is the same for all objects. As we now know, if Galileo had dropped the balls in a vacuum they *would* have hit the ground at the same time. In fact, *Apollo 15* Commander Dave Scott dropped a feather and a hammer in the vacuum of the moon. Just as the theory predicts, the feather and hammer fell at exactly the same rate.

Back in Italy, Galileo had to cope with the reality of air resistance. In fact, one reason that many historians don't believe that the Leaning Tower experiment ever occurred is that air resistance would cause the heavier lead ball to reach the ground significantly before the lighter wood ball. Whether or not the legend is true, Galileo made several vital contributions to our understanding of how objects move through the air. His most important experiments in this area were precise measurements of balls rolling down inclined planes. The slight slope of his inclined planes meant that the balls rolled slowly, reducing the effect of air resistance. This allowed Galileo to compute time accurately (remember, this was before clocks were invented). In one set of experiments, he set up his inclined plane on a platform a specified height above the floor, rolled a ball down the inclined plane and off the edge of the platform. Having dipped the balls in ink, Galileo could see where the balls landed and then measure the horizontal distance traveled. His data (the units are *punti*) and a plot of the data are shown below.

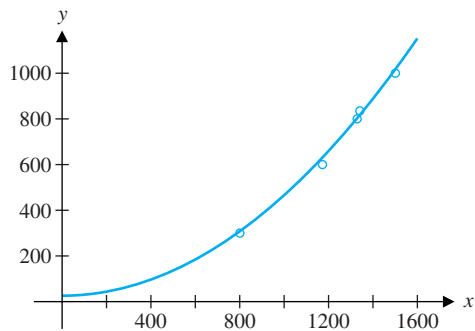
Distance (x)	Height (y)
800	300
1172	600
1328	800
1340	828
1500	1000



Throughout your study of calculus, you will look at relationships between variables in different forms. Here, we have both a numerical representation and a graphical representation of the same experiment. Galileo was also interested in an algebraic representation. That is, he wanted a formula by which he could compute the y -value given any x -value. In this chapter, we will examine a variety of (mostly familiar) functions and their graphs. Knowing these graphs, we can say that Galileo's data look like points

CHAPTER

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on the graph of a quadratic function. However, it also looks like data from a cubic function or an exponential function. We will consider techniques to help determine which of these functions fits the data best. One simple function that provides a nice fit is $y = 0.00044x^2 + 26$. (See the graph above.) Ultimately, Galileo wanted a theoretical explanation of why such a formula would work. It will take several chapters for us to get there, but the theory of moving objects is one of the great success stories of calculus. Moreover, the study of calculus provides us with numerous explanations of the way things in our world work.

0.1 POLYNOMIALS AND RATIONAL FUNCTIONS

The Real Number System

Although mathematics is far more than just a study of numbers, our journey into calculus begins with the real number system. You might think that this is a fairly mundane starting place. After all, you probably mastered the properties of the real numbers a long time ago. Nevertheless, we want to give you the opportunity to brush up on those properties that are of particular interest for calculus. A solid foundation here will help you to grasp the many subtleties you will encounter in your study of the calculus.

The most familiar set of numbers is the set of **integers**. The integers consist of the whole numbers and their additive inverses: $0, \pm 1, \pm 2, \pm 3, \dots$. A **rational number** is any number that can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. For example, $\frac{2}{3}$, $-\frac{7}{3}$ and $\frac{27}{125}$ are all rational numbers. Notice that every integer n is also a rational number, since we can write it as the quotient of two integers: $n = \frac{n}{1}$.

The **irrational numbers** are all those real numbers that cannot be written in the form $\frac{p}{q}$, where p and q are integers. The irrationals include the very important numbers $\sqrt{2}$, π and e . You may recall that rational numbers have a decimal expansion that either terminates or repeats. For instance, $\frac{1}{2} = 0.5$, $\frac{1}{3} = 0.3333\bar{3}$, $\frac{1}{8} = 0.125$ and $\frac{1}{6} = 0.1666\bar{6}$ are all rational numbers. By contrast, irrational numbers have decimal expansions that do not repeat or terminate. For instance, we have the decimal expansions

$$\sqrt{2} = 1.4142135623\dots,$$

$$\pi = 3.1415926535\dots$$

and

$$e = 2.7182818284\dots$$

for the irrational numbers $\sqrt{2}$, π and e .

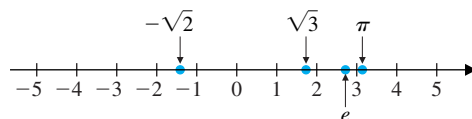


FIGURE 0.1
The real line.

We usually picture the real numbers arranged along a number line as displayed in Figure 0.1 (the **real line**), with center at 0 and extending off forever in both the positive and negative directions. We denote the set of real numbers by the symbol \mathbb{R} . The notation $x \in \mathbb{R}$ indicates that x is an element of the set of real numbers. (In other words, x is a real number.)

Recall that for real numbers a and b , where $a < b$, we define the **closed interval** $[a, b]$ to be the set of numbers between a and b , including a and b (the **endpoints**), that is,

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\},$$

as illustrated in Figure 0.2, where we have used the solid circles to indicate that a and b are included in $[a, b]$.

Similarly, we define the **open interval** (a, b) to be the set of numbers between a and b , but **not** including the endpoints a and b , that is,

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\},$$

as illustrated in Figure 0.3, where we have used the open circles to indicate that a and b are not included in (a, b) .



FIGURE 0.2
A closed interval.



FIGURE 0.3
An open interval.

□ Equations of Lines

By law, the federal government must conduct a nationwide census every 10 years to determine the population. This is done for a variety of reasons, notably so that the government can track the growth of various segments of the population for the purposes of planning. Population data for several recent decades are shown in the accompanying table.

Year	U.S. population	x	y
1960	179,323,175	0	179
1970	203,302,031	10	203
1980	226,542,203	20	227
1990	248,709,873	30	249

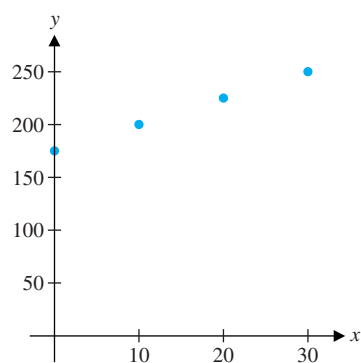


FIGURE 0.4
Population data.

One difficulty with analyzing this data is that the numbers are so large. This problem is remedied by **transforming** the data. We can simplify the year data by defining x to be the number of years since 1960. Then, 1960 corresponds to $x = 0$, 1970 corresponds to $x = 10$, and so on. The population data can be simplified by rounding the numbers off to the nearest million. The simplified data are shown in the accompanying table and a scatter plot of these data points is shown in Figure 0.4.

Asked to describe the pattern in Figure 0.4, most people would say that the points appear to form a straight line. (Use a ruler to see if you agree.) To determine whether the points are, in fact, on the same line (such points are called **colinear**), we might consider the population growth in each of the indicated decades. From 1960 to 1970, the growth was 24 million. (That is, to move from the first point to the second, you increase x by 10 and increase y by 24.) Likewise, from 1970 to 1980, the growth was 24 million. (So, to move

from the second point to the third point, you again increase x by 10 and y by 24.) However, from 1980 to 1990, the growth was only 22 million. (That is, to move from the third point to the fourth point, while x again increases by 10, y increases by only 22.) Since the rate of growth is not constant, this says that the data points do not fall on a line. Notice that to stay on the same line, y would had to have increased by 24 again. (Why is that?) The preceding argument involves the familiar concept of *slope*.

DEFINITION 1.1

For $x_1 \neq x_2$, the **slope** of the straight line through the points (x_1, y_1) and (x_2, y_2) is the number

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \quad (1.1)$$

When $x_1 = x_2$, the line through (x_1, y_1) and (x_2, y_2) is **vertical** and the slope is undefined.

We often describe slope as “the change in y divided by the change in x ,” written $\frac{\Delta y}{\Delta x}$, or more simply as $\frac{\text{Rise}}{\text{Run}}$ (see Figure 0.5a).

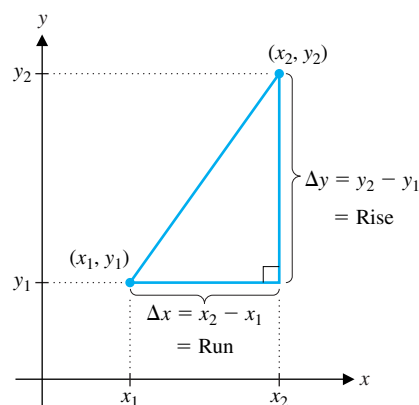


FIGURE 0.5a
Slope.

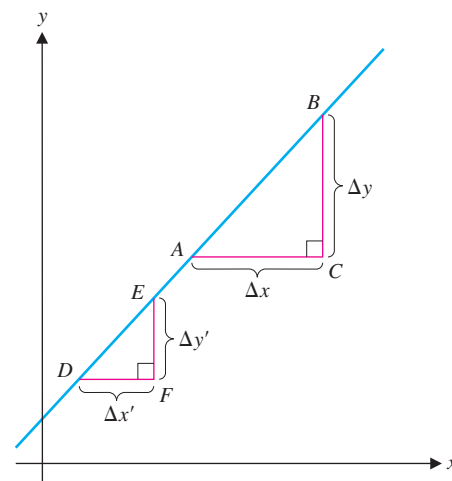


FIGURE 0.5b
Similar triangles and slope.

You should observe that the slope of a straight line is the same no matter which two points on the line you select. Referring to Figure 0.5b (where the line has positive slope), notice that for any four points A, B, D and E on the line, the two right triangles ΔABC and ΔDEF are similar. Recall that for similar triangles, the ratios of corresponding sides must be the same. In this case, this says that

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y'}{\Delta x'}$$

and so, the slope is the same no matter which two points on the line are selected. Furthermore, a line is the only curve with constant slope. Notice that a line is **horizontal** if and only if its slope is zero.

EXAMPLE 1.1 Finding the Slope of a Line

Find the slope of the line through the points (4, 3) and (2, 5).

Solution From (1.1), we get

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 3}{2 - 4} = \frac{2}{-2} = -1.$$

EXAMPLE 1.2 Using the Slope to Determine if Points Are Colinear

Use slopes to determine whether or not the points (1, 2), (3, 10), and (4, 14) are colinear.

Solution First, notice that the slope of the line joining (1, 2) and (3, 10) is

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10 - 2}{3 - 1} = \frac{8}{2} = 4.$$

Similarly, the slope through the line joining (3, 10) and (4, 14) is

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{14 - 10}{4 - 3} = 4.$$

Since the slopes are the same, the points must be colinear.

Recall that if you know the slope and a point through which the line must pass, you have enough information to graph the line. The easiest way to graph a line is to plot two points and then draw the line through them. In this case, you only need to find a second point.

EXAMPLE 1.3 Graphing a Line

If a line passes through the point (2, 1) with slope $\frac{2}{3}$, find a second point on the line and then graph the line.

Solution Since slope is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$, we take $m = \frac{2}{3}$, $y_1 = 1$ and $x_1 = 2$, to obtain

$$\frac{2}{3} = \frac{y_2 - 1}{x_2 - 2}.$$

You are free to choose the x -coordinate of the second point. For instance, to find the point at $x_2 = 5$, substitute this into the equation, and solve. From

$$\frac{2}{3} = \frac{y_2 - 1}{5 - 2} = \frac{y_2 - 1}{3},$$

we get $2 = y_2 - 1$ or $y_2 = 3$. A second point is then (5, 3). The graph of the line is shown in Figure 0.6a. An alternative method for finding a second point is to use the slope

$$m = \frac{2}{3} = \frac{\Delta y}{\Delta x}.$$

The slope of $\frac{2}{3}$ says that if we move three units to the right, we must move two units up to stay on the line, as illustrated in Figure 0.6b.

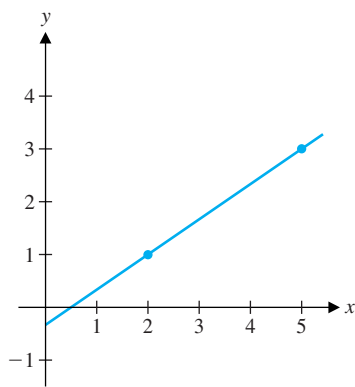


FIGURE 0.6a
Graph of a straight line.

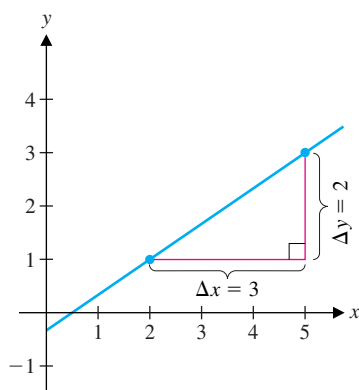


FIGURE 0.6b
Using slope to find a second point.

In example 1.3, note that the choice of $x = 5$ was entirely arbitrary; you can choose any x -value you want to find a second point. Further, since x can be any real number, you can leave x as a variable and write out an equation satisfied by any point (x, y) on the line.

In the general case of the line through the point (x_0, y_0) with slope m , we have from (1.1) that

$$m = \frac{y - y_0}{x - x_0}. \quad (1.2)$$

Multiplying both sides of (1.2) by $(x - x_0)$, we get

$$y - y_0 = m(x - x_0)$$

or

$$y = m(x - x_0) + y_0. \quad (1.3)$$

Point-slope form of a line

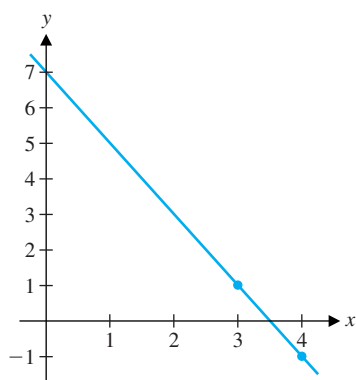


FIGURE 0.7
 $y = -2(x - 3) + 1.$

Equation (1.3) is called the **point-slope form** of the line. With this, you can find an equation of any line.

EXAMPLE 1.4 Finding the Equation of a Line Given Two Points

Find an equation of the line through the points $(3, 1)$ and $(4, -1)$ and graph the line.

Solution From (1.1), the slope is

$$m = \frac{-1 - 1}{4 - 3} = \frac{-2}{1} = -2.$$

Using (1.3) with slope $m = -2$, x -coordinate $x_0 = 3$ and y -coordinate $y_0 = 1$, we get the equation of the line:

$$y = -2(x - 3) + 1. \quad (1.4)$$

To graph the line, plot the points $(3, 1)$ and $(4, -1)$, and you can easily draw the line seen in Figure 0.7. ■

In example 1.4, you may be tempted to simplify the expression for y given in (1.4). As it turns out, the point-slope form of the equation is often the most convenient to work with. So, we will typically not ask you to rewrite this expression in other forms. At times, a form of the equation called the **slope-intercept form** is more convenient. This is an equation of the form

$$y = mx + b,$$

where m is the slope and b is the y -intercept (i.e., the place where the graph crosses the y -axis). In example 1.4, you simply multiply out (1.4) to get $y = -2x + 6 + 1$ or

$$y = -2x + 7.$$

As you can see from Figure 0.7, the graph crosses the y -axis at $y = 7$.

Theorem 1.1 presents a familiar result on parallel and perpendicular lines.

THEOREM 1.1

Two (nonvertical) lines are **parallel** if they have the same slope. Further, any two vertical lines are parallel. Two (nonvertical) lines of slope m_1 and m_2 are **perpendicular** whenever the product of their slopes is -1 (i.e., $m_1 \cdot m_2 = -1$). Also, any vertical line and any horizontal line are perpendicular.

Since we can read the slope from the equation of a line, it's a simple matter to determine when two lines are parallel or perpendicular. We illustrate this in examples 1.5 and 1.6.

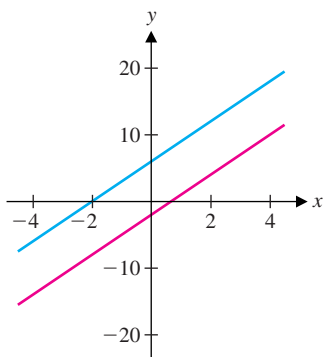


FIGURE 0.8
Parallel lines.

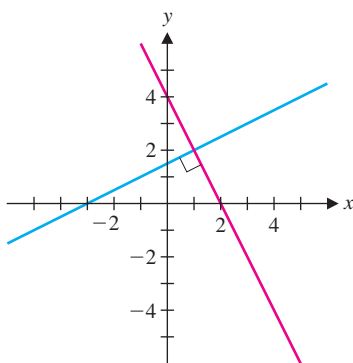


FIGURE 0.9
Perpendicular lines.

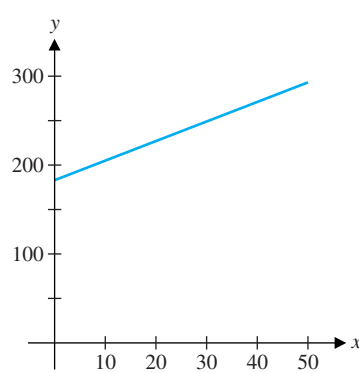


FIGURE 0.10
Population.

EXAMPLE 1.5 Finding the Equation of a Parallel Line

Find an equation of the line parallel to $y = 3x - 2$ and through the point $(-1, 3)$.

Solution It's easy to read the slope from the equation: $m = 3$. The equation of the parallel line is then

$$y = 3[x - (-1)] + 3$$

or simply $y = 3(x + 1) + 3$. We show a graph of both lines in Figure 0.8. ■

EXAMPLE 1.6 Finding the Equation of a Perpendicular Line

Find an equation of the line perpendicular to $y = -2x + 4$ and intersecting the line at the point $(1, 2)$.

Solution The slope of $y = -2x + 4$ is -2 . The slope of the perpendicular line is then $-1/(-2) = \frac{1}{2}$. Since the line must pass through the point $(1, 2)$, the equation of the perpendicular line is now

$$y = \frac{1}{2}(x - 1) + 2.$$

We show a graph of the two lines in Figure 0.9. ■

We now return to the section's introductory example to use the equation of a line to estimate the population in the year 2000.

EXAMPLE 1.7 Using a Line to Estimate Population

Given the population data for the census years 1960, 1970, 1980 and 1990, estimate the population for the year 2000.

Solution First, recall that we began this section by showing that the points in the corresponding table were not colinear. Nonetheless, they were *nearly* colinear. So, why not use the straight line connecting the last two points $(20, 227)$ and $(30, 249)$ (recall that these correspond to the population in the years 1980 and 1990) to estimate the population in 2000? (This is a simple example of a more general procedure called **extrapolation**.) The slope of the line joining the two data points is

$$m = \frac{249 - 227}{30 - 20} = \frac{22}{10} = \frac{11}{5}.$$

The equation of the line is then

$$y = \frac{11}{5}(x - 30) + 249.$$

See Figure 0.10 for a graph of the line. If we follow this line to the point corresponding to $x = 40$ (the year 2000), we have the estimated population

$$\frac{11}{5}(40 - 30) + 249 = 271.$$

That is, the estimated population is 271 million people. The actual census figure for 2000 is 281 million, so that our estimate of 271 million is off by 10 million. This says that the U.S. population has not continued to grow at a linear rate. So, our linear model is not particularly accurate. ■

In this section, we have so far drawn connections between the equations and graphs of lines. This process can be extended to a variety of other curves. In fact, most of this introductory chapter is devoted to exploring such connections. Before doing so, we need some general definitions.

□ Functions

For any two subsets A and B of the real line, we make the following familiar definition.

DEFINITION 1.2

A **function** f is a rule that assigns *exactly one* element y in a set B to each element x in a set A . In this case, we write $y = f(x)$.

We call the set A the **domain** of f . The set of all values $f(x)$ in B is called the **range** of f . That is, the range of f is $\{f(x) \mid x \in A\}$. Unless explicitly stated otherwise, the domain of a function f is the largest set of real numbers for which the function is defined. We refer to x as the **independent variable** and to y as the **dependent variable**.

REMARK 1.1

Functions are often defined by simple formulas, such as $f(x) = 3x + 2$, but in general, any correspondence meeting the requirement of matching exactly *one* y to each x defines a function.

By the **graph** of a function f , we mean the graph of the equation $y = f(x)$. That is, the graph consists of all points (x, y) , where x is in the domain of f and where $y = f(x)$.

Notice that not every curve is the graph of a function, since for a function, there is only one y -value that can correspond to a given value of x . Recall that you can graphically determine whether a curve is the graph of a function by using the **vertical line test**: if any vertical line intersects the graph in more than one point, the curve is not the graph of a function.

EXAMPLE 1.8 Using the Vertical Line Test

Determine which curves correspond to functions.

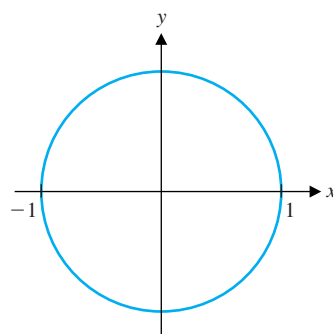


FIGURE 0.11a

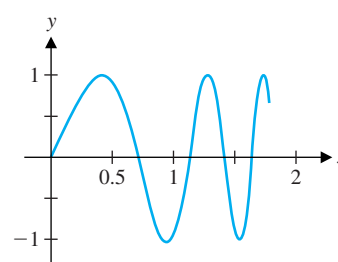


FIGURE 0.11b

Solution Notice that the circle in Figure 0.11a is not the graph of a function, since a vertical line at $x = 0.5$ intersects the circle twice (see Figure 0.12a). The graph in

Figure 0.11b is the graph of a function, even though it swings up and down repeatedly. Although horizontal lines intersect the graph repeatedly, vertical lines, such as the one at $x = 1.2$, intersect only once (see Figure 0.12b).

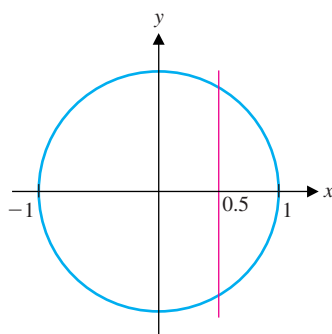


FIGURE 0.12a
Curve fails vertical line test.

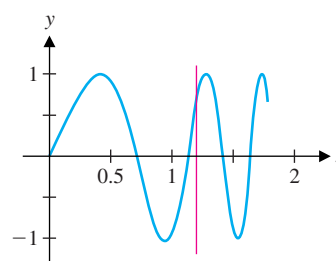


FIGURE 0.12b
Curve passes vertical line test.

You are already familiar with a number of different types of functions and we will only briefly review these here and in sections 0.4 and 0.5. The functions that you are probably most familiar with are **polynomials**. These are the simplest functions to work with because they are defined entirely in terms of arithmetic. ■

DEFINITION 1.3

A **polynomial** is any function that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_0, a_1, a_2, \dots, a_n$ are real numbers (the **coefficients** of the polynomial) with $a_n \neq 0$ and $n \geq 0$ is an integer (the **degree** of the polynomial).

Note that the domain of every polynomial function is the entire real line. Further, recognize that the graph of the linear (degree 1) polynomial $f(x) = ax + b$ is a straight line.

EXAMPLE 1.9 Sample Polynomials

The following are all examples of polynomials:

$$f(x) = 2 \text{ (polynomial of degree 0 or **constant**)},$$

$$f(x) = 3x + 2 \text{ (polynomial of degree 1 or **linear** polynomial)},$$

$$f(x) = 5x^2 - 2x + 1 \text{ (polynomial of degree 2 or **quadratic** polynomial)},$$

$$f(x) = x^3 - 2x + 1 \text{ (polynomial of degree 3 or **cubic** polynomial)},$$

$$f(x) = -6x^4 + 12x^2 - 3x + 13 \text{ (polynomial of degree 4 or **quartic** polynomial)},$$

and

$$f(x) = 2x^5 + 6x^4 - 8x^2 + x - 3 \text{ (polynomial of degree 5 or **quintic** polynomial)}.$$

We show graphs of these six functions in Figures 0.13a–0.13f (shown on the next page).

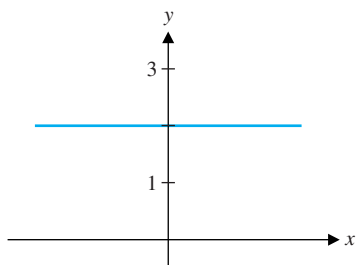


FIGURE 0.13a
 $f(x) = 2$.

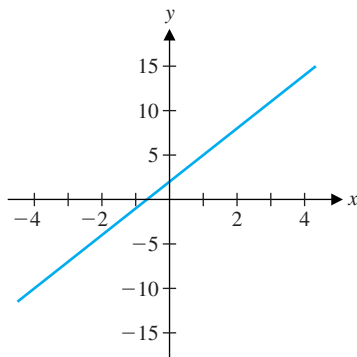


FIGURE 0.13b
 $f(x) = 3x + 2$.

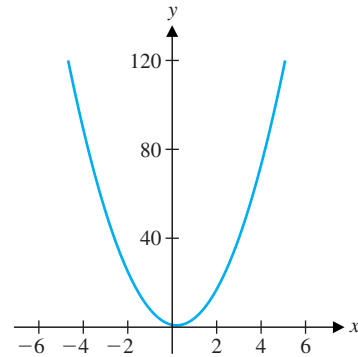


FIGURE 0.13c
 $f(x) = 5x^2 - 2x + 1$.

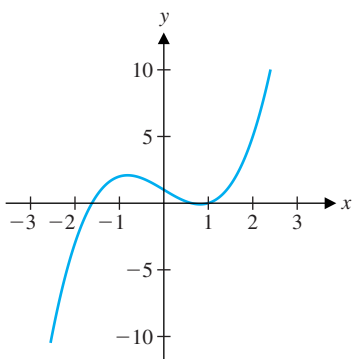


FIGURE 0.13d
 $f(x) = x^3 - 2x + 1$.

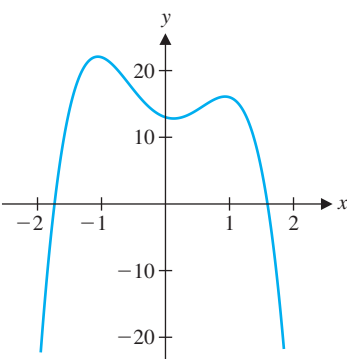


FIGURE 0.13e
 $f(x) = -6x^4 + 12x^2 - 3x + 13$.

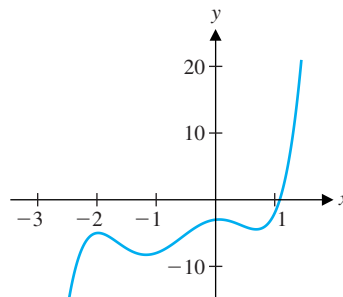


FIGURE 0.13f
 $f(x) = 2x^5 + 6x^4 - 8x^2 + x - 3$.

DEFINITION 1.4

Any function that can be written in the form

$$f(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomials, with $q(x) \neq 0$, is called a **rational function**.

Notice that since $p(x)$ and $q(x)$ are polynomials, they are both defined for all x , and so, the rational function $f(x) = \frac{p(x)}{q(x)}$ is defined for all x for which $q(x) \neq 0$.

EXAMPLE 1.10 A Sample Rational Function

Find the domain of the function

$$f(x) = \frac{x^2 + 7x - 11}{x^2 - 4}.$$

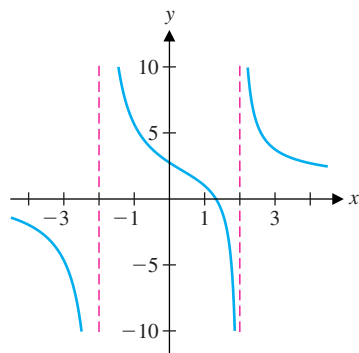


FIGURE 0.14
 $f(x) = \frac{x^2 + 7x - 11}{x^2 - 4}$.

Solution Here, $f(x)$ is a rational function. We show a graph in Figure 0.14. Further, its domain consists of those values of x for which the denominator is nonzero. Notice that

$$x^2 - 4 = (x - 2)(x + 2)$$

and so, the denominator is zero if and only if $x = \pm 2$. This says that the domain of f is

$$\{x \in \mathbb{R} \mid x \neq \pm 2\} = (-\infty, -2) \cup (-2, 2) \cup (2, \infty).$$

No doubt, you will recall the following standard definition.

DEFINITION 1.5

The **absolute value** of a real number x is $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$.

Make certain that you read Definition 1.5 correctly. If x is negative, then $-x$ is positive. This says that $|x| \geq 0$ for all real numbers x . For instance, using the definition,

$$|-4| = -(-4) = 4.$$

Notice that for any real numbers a and b ,

$$|a \cdot b| = |a| \cdot |b|.$$

However,

$$|a + b| \neq |a| + |b|,$$

in general. (To verify this, simply take $a = 5$ and $b = -2$ and compute both quantities.)

However, it is always true that

$$|a + b| \leq |a| + |b|. \tag{1.5}$$

The inequality (1.5) is referred to as the **triangle inequality**. You will find occasional uses for it as you progress through the calculus.

The **square root** function is defined in the usual way. When we write $y = \sqrt{x}$, we mean that y is that number for which $y \geq 0$ and $y^2 = x$. In particular, $\sqrt{4} = 2$. Be careful not to write erroneous statements like $\sqrt{4} = \pm 2$. This is a common misconception. While it's true that $2^2 = 4$ and $(-2)^2 = 4$, when we write $\sqrt{4}$, we are looking for the **positive** number whose square is 4. In this way, \sqrt{x} defines a function with domain $[0, \infty)$. If we allowed \sqrt{x} to take on both positive and negative values, this would not define a function. In particular, be careful to write

$$\sqrt{x^2} = |x|.$$

Since $\sqrt{x^2}$ is asking for the *nonnegative* number whose square is x^2 , we are looking for $|x|$ and not x . We can say

$$\sqrt{x^2} = x, \text{ only for } x \geq 0.$$

EXAMPLE 1.11 Finding the Domain of a Function Involving a Square Root or a Cube Root

Find the domains of $f(x) = \sqrt{x^2 - 4}$ and $g(x) = \sqrt[3]{x^2 - 4}$.

Solution Since even roots are defined only for nonnegative values, $f(x)$ is defined only for $x^2 - 4 \geq 0$. Notice that this is equivalent to having $x^2 \geq 4$, which occurs when $x \geq 2$ or $x \leq -2$. The domain of f is then $(-\infty, -2] \cup [2, \infty)$. On the other hand, odd roots are defined for both positive and negative values. Consequently, the domain of $g(x)$ is the entire real line, $(-\infty, \infty)$. ■

In our study of calculus, we often find it useful to label intercepts and other significant points on a graph. Finding these points often involves solving equations. A solution of the equation $f(x) = 0$ is called a **zero** of the function f or a **root** of the equation $f(x) = 0$. Notice that a zero of the function f corresponds to an x -intercept of the graph of $y = f(x)$.

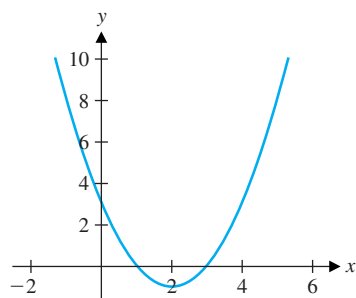


FIGURE 0.15
 $y = x^2 - 4x + 3$.

EXAMPLE 1.12 Finding Intercepts of a Graph

Find all x - and y -intercepts of $y = x^2 - 4x + 3$.

Solution To find the y -intercept, recall that we set $x = 0$ to obtain

$$y = 0 - 0 + 3 = 3.$$

Notice that in Figure 0.15, the graph crosses the y -axis at $y = 3$.

Similarly, to find the x -intercepts, we solve the equation $f(x) = 0$. In this case, we can factor to get

$$0 = x^2 - 4x + 3 = (x - 1)(x - 3),$$

so that the x -intercepts are $x = 1$ and $x = 3$, as indicated in Figure 0.15. ■

We briefly discuss several ideas for finding zeros here.

Unfortunately, factoring is not always so easy. Of course, for the quadratic equation

$$ax^2 + bx + c = 0$$

(for a, b and c all real numbers and $a \neq 0$), the solution(s) are given by the familiar **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

EXAMPLE 1.13 Finding Zeros Using the Quadratic Formula

Find the zeros of $f(x) = x^2 - 5x - 12$.

Solution You might try to factor this one, but you probably won't have much luck. However, from the quadratic formula, we have

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 1 \cdot (-12)}}{2 \cdot 1} = \frac{5 \pm \sqrt{25 + 48}}{2} = \frac{5 \pm \sqrt{73}}{2}.$$

So, the two solutions are given by $x = \frac{5}{2} + \frac{\sqrt{73}}{2} \approx 6.772$ and $x = \frac{5}{2} - \frac{\sqrt{73}}{2} \approx -1.772$. (No wonder you couldn't factor the polynomial!) ■

Finding zeros of polynomials of degree higher than 2 and other functions is usually quite a bit trickier and is sometimes impossible. At the least, you can always find an approximation of any zeros by using a graph to zoom in closer and closer to any point where the graph crosses the x -axis, as we'll illustrate shortly. (Keep in mind that the zeros of a function correspond to the x -intercepts of its graph.) A more basic question, though, is to determine *how many* zeros a given function has. In general, there is no way to answer this

question without the use of calculus. For the case of polynomials, however, the following theorem (a consequence of the Fundamental Theorem of Algebra) provides a clue.

THEOREM 1.2

A polynomial of degree n has *at most* n distinct zeros.

Notice that this theorem does not say how many zeros a given polynomial has, but rather, that the *maximum* number of distinct (i.e., different) zeros is the same as the degree. A polynomial of degree n may have anywhere from 0 to n distinct real zeros. [Recall that polynomials may also have complex zeros. For instance, $f(x) = x^2 + 1$ has only the complex zeros $x = \pm i$, where i is the imaginary number defined by $i = \sqrt{-1}$.] However, polynomials of odd degree must have *at least one* real zero. (Why is that?) For instance, for the case of a cubic polynomial, we have one of the three possibilities illustrated in Figures 0.16a, 0.16b and 0.16c.

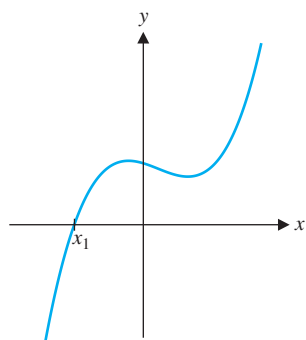


FIGURE 0.16a
One zero.

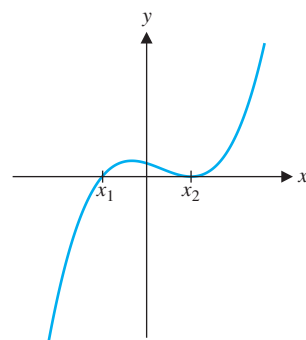


FIGURE 0.16b
Two zeros.

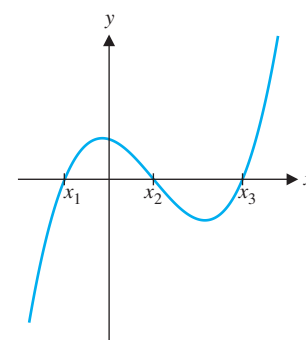


FIGURE 0.16c
Three zeros.

In these three figures, we show the graphs of cubic polynomials with 1, 2 and 3 distinct, real zeros, respectively. These are the graphs of the functions

$$f(x) = x^3 - 2x^2 + 3 = (x + 1)(x^2 - 3x + 3),$$

$$g(x) = x^3 - x^2 - x + 1 = (x + 1)(x - 1)^2$$

and

$$h(x) = x^3 - 3x^2 - x + 3 = (x + 1)(x - 1)(x - 3),$$

respectively. Note that you can see from the factored form where the zeros are (and how many there are).

The result in Theorem 1.3 provides an important connection between factors and zeros of polynomials.

THEOREM 1.3 (Factor Theorem)

For any polynomial f , $f(a) = 0$ if and only if $(x - a)$ is a factor of $f(x)$.

EXAMPLE 1.14 Finding the Zeros of a Cubic Polynomial

Find the zeros of $f(x) = x^3 - x^2 - 2x + 2$.

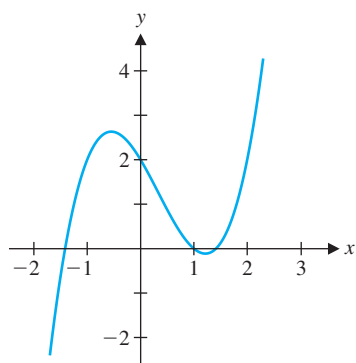


FIGURE 0.17a
 $y = x^3 - x^2 - 2x + 2$.

Solution You might quickly recognize that one zero of this function is $x = 1$. [Just calculate $f(1)$ to verify this.] But how many other zeros are there? A graph of the function may be of some value here (see Figure 0.17a). You can see from the graph that the other two zeros of f are near $x = -1.5$ and near $x = 1.5$. You can find these zeros more precisely by using your graphing calculator or computer algebra system to zoom in on the locations of these zeros (one at a time, as in Figures 0.17b and 0.17c). From these zoomed graphs, it is clear that the two remaining zeros of f are near $x = 1.414$ and $x = -1.414$. Of course, you can make these estimates more precise by zooming in even more closely. Most graphing calculators and computer algebra systems can also find approximate zeros for you, using a built-in “solve” program. In Chapter 3, we present a versatile method (called Newton’s method) for obtaining accurate approximations to zeros. The only way to find the exact solutions is to factor the expression (using either long division or synthetic division). Here, we have

$$f(x) = x^3 - x^2 - 2x + 2 = (x - 1)(x^2 - 2) = (x - 1)(x - \sqrt{2})(x + \sqrt{2}),$$

from which you can see that the zeros are $x = 1$, $x = \sqrt{2}$ and $x = -\sqrt{2}$.

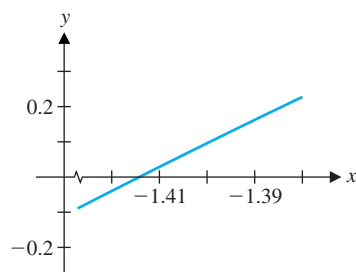


FIGURE 0.17b
 Zoomed in on zero near $x = -1.4$.

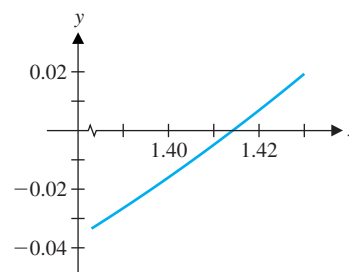


FIGURE 0.17c
 Zoomed in on zero near $x = 1.4$.

Recall that to find the points of intersection of two curves defined by $y = f(x)$ and $y = g(x)$, we set $f(x) = g(x)$ and solve for any x ’s satisfying the equation (the x -coordinates of the points of intersection).

EXAMPLE 1.15 Finding the Intersections of a Line and a Parabola

Find the points of intersection of the parabola $y = x^2 - x - 5$ and the line $y = x + 3$.

Solution First, we draw a sketch of the two curves, as seen in Figure 0.18. Notice from the graph that there are two intersections indicated, one near $x = -2$ and the other near $x = 4$. To determine these precisely, we set the two functions equal and solve for x :

$$x^2 - x - 5 = x + 3.$$

Subtracting $(x + 3)$ from both sides leaves us with

$$0 = x^2 - 2x - 8 = (x - 4)(x + 2).$$

This says that the solutions are exactly $x = -2$ and $x = 4$. We compute the corresponding y -values from the equation of the line $y = x + 3$ (or the equation of the parabola). The points of intersection are then $(-2, 1)$ and $(4, 7)$. Notice that these are consistent with the intersections seen in Figure 0.18. ■

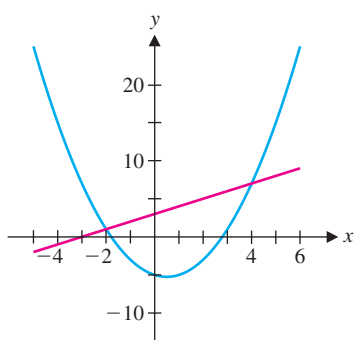


FIGURE 0.18
 $y = x + 3$ and $y = x^2 - x - 5$.

Unfortunately, you won't always be able to solve equations exactly, as we did in examples 1.12–1.15. We explore some options for dealing with more difficult equations in section 0.2.

EXERCISES 0.1

WRITING EXERCISES

- If the slope between points A and B equals the slope between points B and C , explain why the points A , B and C are colinear.
- If a graph fails the vertical line test, it is not the graph of a function. Explain this result in terms of the definition of a function.
- You should not automatically write the equation of a line in slope-intercept form. Compare the following forms of the same line: $y = 2.4(x - 1.8) + 0.4$ and $y = 2.4x - 3.92$. Given $x = 1.8$, which equation would you rather use to compute y ? How about if you are given $x = 0$? For $x = 8$, is there any advantage to one equation over the other? Can you quickly read off the slope from either equation? Explain why neither form of the equation is “better.”
- To understand Definition 1.5, you must believe that $|x| = -x$ for negative x 's. Using $x = -3$ as an example, explain in words why multiplying x by -1 produces the same result as taking the absolute value of x .

In exercises 1–4, determine if the points are colinear.

- $(2, 1), (0, 2), (4, 0)$
- $(3, 1), (4, 4), (5, 8)$
- $(4, 1), (3, 2), (1, 3)$
- $(1, 2), (2, 5), (4, 8)$

In exercises 5–10, find the slope of the line through the given points.

- $(1, 2), (3, 6)$
- $(1, 2), (3, 3)$
- $(3, -6), (1, -1)$
- $(1, -2), (-1, -3)$
- $(0.3, -1.4), (-1.1, -0.4)$
- $(1.2, 2.1), (3.1, 2.4)$

In exercises 11–16, find a second point on the line with slope m and point P , graph the line and find an equation of the line.

- $m = 2, P = (1, 3)$
- $m = -2, P = (1, 4)$
- $m = 0, P = (-1, 1)$
- $m = \frac{1}{2}, P = (2, 1)$
- $m = 1.2, P = (2.3, 1.1)$
- $m = -\frac{1}{4}, P = (-2, 1)$

In exercises 17–22, determine if the lines are parallel, perpendicular, or neither.

- $y = 3(x - 1) + 2$ and $y = 3(x + 4) - 1$
- $y = 2(x - 3) + 1$ and $y = 4(x - 3) + 1$

19. $y = -2(x + 1) - 1$ and $y = \frac{1}{2}(x - 2) + 3$

20. $y = 2x - 1$ and $y = -2x + 2$

21. $y = 3x + 1$ and $y = -\frac{1}{3}x + 2$

22. $x + 2y = 1$ and $2x + 4y = 3$

In exercises 23–26, find an equation of a line through the given point and (a) parallel to and (b) perpendicular to the given line.

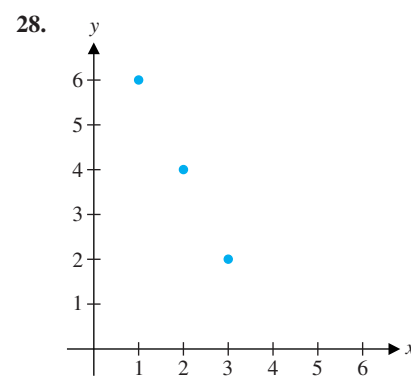
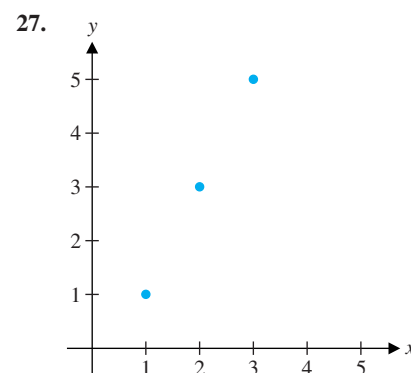
23. $y = 2(x + 1) - 2$ at $(2, 1)$

24. $y = 3(x - 2) + 1$ at $(0, 3)$

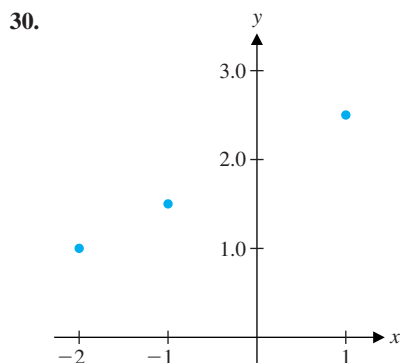
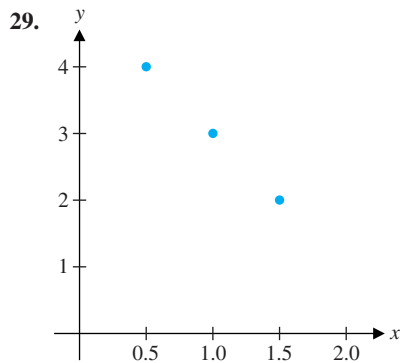
25. $y = 2x + 1$ at $(3, 1)$

26. $y = 1$ at $(0, -1)$

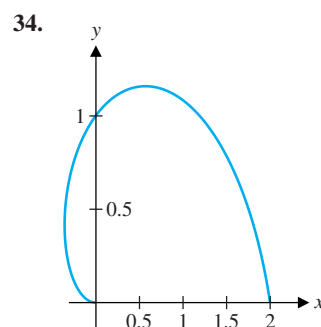
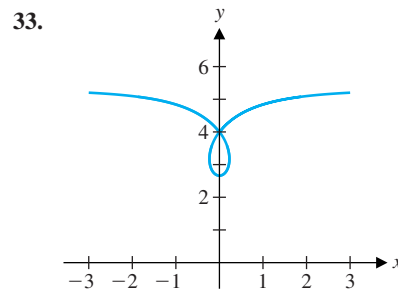
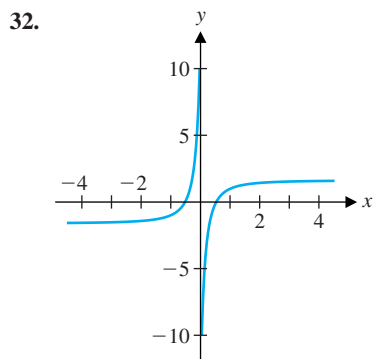
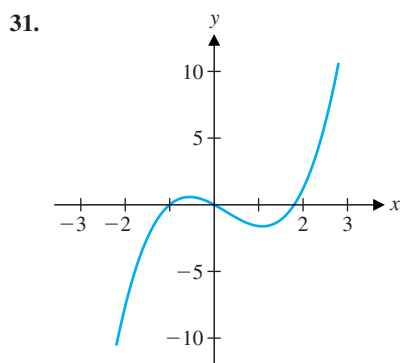
In exercises 27–30, find an equation of the line through the given points and compute the y -coordinate of the point on the line corresponding to $x = 4$.



16 CHAPTER 0 Preliminaries



In exercises 31–34, use the vertical line test to determine whether or not the curve is the graph of a function.



In exercises 35–40, identify the given function as polynomial, rational, both or neither.

35. $f(x) = x^3 - 4x + 1$ 36. $f(x) = 3 - 2x + x^4$
 37. $f(x) = \frac{x^2 + 2x - 1}{x + 1}$ 38. $f(x) = \frac{x^3 + 4x - 1}{x^4 - 1}$
 39. $f(x) = \sqrt{x^2 + 1}$ 40. $f(x) = 2x - x^{2/3} - 6$

In exercises 41–46, find the domain of the function.

41. $f(x) = \sqrt{x + 2}$ 42. $f(x) = \sqrt{2x + 1}$
 43. $f(x) = \sqrt[3]{x - 1}$ 44. $f(x) = \sqrt{x^2 - 4}$
 45. $f(x) = \frac{4}{x^2 - 1}$ 46. $f(x) = \frac{4x}{x^2 + 2x - 6}$

In exercises 47–50, find the indicated function values.

47. $f(x) = x^2 - x - 1$; $f(0)$, $f(2)$, $f(-3)$, $f(1/2)$
 48. $f(x) = \frac{x + 1}{x - 1}$; $f(0)$, $f(2)$, $f(-2)$, $f(1/2)$
 49. $f(x) = \sqrt{x + 1}$; $f(0)$, $f(3)$, $f(-1)$, $f(1/2)$
 50. $f(x) = \frac{3}{x}$; $f(1)$, $f(10)$, $f(100)$, $f(1/3)$

In exercises 51–54, a brief description is given of a physical situation. For the indicated variable, state a reasonable domain.

51. A parking deck is to be built; x = width of deck (in feet).
 52. A parking deck is to be built on a 200'-by-200' lot; x = width of deck (in feet).
 53. A new candy bar is to be sold; x = number of candy bars sold in first month.

54. A new candy bar is to be sold; x = cost of candy bar (in cents).

In exercises 55–58, discuss whether or not you think y would be a function of x .

55. y = grade you get on an exam, x = number of hours you study
 56. y = probability of getting lung cancer, x = number of cigarettes smoked per day
 57. y = a person's weight, x = number of minutes exercising per day
 58. y = speed at which an object falls, x = weight of object
 59. Figure A shows the speed of a bicyclist as a function of time. For the portions of this graph that are flat, what is happening to the bicyclist's speed? What is happening to the bicyclist's speed when the graph goes up? down? Identify the portions of the graph that correspond to the bicyclist going uphill; downhill.

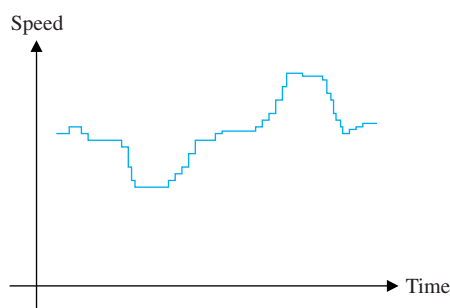


FIGURE A
Bicycle speed.

60. Figure B shows the population of a small country as a function of time. During the time period shown, the country experienced two influxes of immigrants, a war and a plague. Identify these important events.

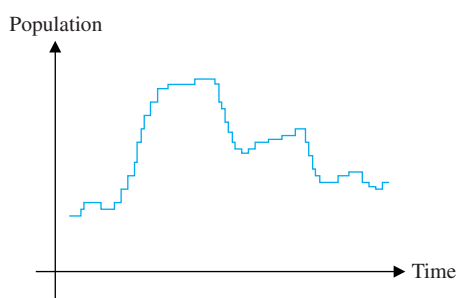


FIGURE B
Population.

In exercises 61–66, find all intercepts of the given graph.

61. $y = x^2 - 2x - 8$ 62. $y = x^2 + 4x + 4$
 63. $y = x^3 - 8$ 64. $y = x^3 - 3x^2 + 3x - 1$
 65. $y = \frac{x^2 - 4}{x + 1}$ 66. $y = \frac{2x - 1}{x^2 - 4}$

In exercises 67–74, factor and/or use the quadratic formula to find all zeros of the given function.

67. $f(x) = x^2 - 4x + 3$ 68. $f(x) = x^2 + x - 12$
 69. $f(x) = x^2 - 4x + 2$ 70. $f(x) = 2x^2 + 4x - 1$
 71. $f(x) = x^3 - 3x^2 + 2x$ 72. $f(x) = x^3 - 2x^2 - x + 2$
 73. $f(x) = x^6 + x^3 - 2$ 74. $f(x) = x^3 + x^2 - 4x - 4$
 75. The boiling point of water (in degrees Fahrenheit) at elevation h (in thousands of feet above sea level) is given by $B(h) = -1.8h + 212$. Find h such that water boils at 98.6° . Why would this altitude be dangerous to humans?
 76. The spin rate of a golf ball hit with a 9 iron has been measured at 9100 rpm for a 120-compression ball and at 10,000 rpm for a 60-compression ball. Most golfers use 90-compression balls. If the spin rate is a linear function of compression, find the spin rate for a 90-compression ball. Professional golfers often use 100-compression balls. Estimate the spin rate of a 100-compression ball.
 77. The chirping rate of a cricket depends on the temperature. A species of tree cricket chirps 160 times per minute at 79°F and 100 times per minute at 64°F . Find a linear function relating temperature to chirping rate.
 78. When describing how to measure temperature by counting cricket chirps, most guides suggest that you count the number of chirps in a 15-second time period. Use exercise 77 to explain why this is a convenient period of time.

EXPLORATORY EXERCISES

- Suppose you have a machine that will proportionally enlarge a photograph. For example, it could enlarge a 4×6 photograph to 8×12 by doubling the width and height. You could make an 8×10 picture by cropping 1 inch off each side. Explain how you would enlarge a $3\frac{1}{2} \times 5$ picture to an 8×10 . A friend returns from Scotland with a $3\frac{1}{2} \times 5$ picture showing the Loch Ness monster in the outer $\frac{1}{4}$ " on the right. If you use your procedure to make an 8×10 enlargement, does Nessie make the cut?
- Solve the equation $|x - 2| + |x - 3| = 1$. (Hint: It's an unusual solution, in that it's more than just a couple of numbers.) Then, solve the equation $\sqrt{x + 3} - 4\sqrt{x - 1} + \sqrt{x + 8} - 6\sqrt{x - 1} = 1$. (Hint: If you make the correct substitution, you can use your solution to the previous equation.)

0.2 GRAPHING CALCULATORS AND COMPUTER ALGEBRA SYSTEMS

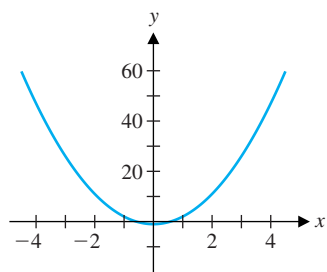


FIGURE 0.19a
 $y = 3x^2 - 1$.

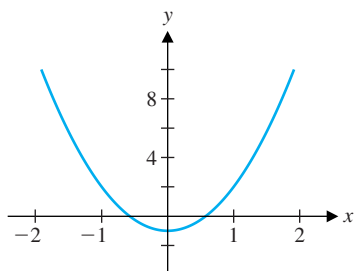


FIGURE 0.19b
 $y = 3x^2 - 1$.

The relationship between a function and its graph is one of the central topics in calculus. Graphing calculators and user-friendly computer software allow you to explore these relationships for a much wider variety of functions than you could with pencil and paper alone. This section presents a general framework for using technology to explore the graphs of functions.

As we observed in section 0.1, the graphs of linear functions are straight lines. You probably also remember that the graphs of quadratic polynomials are **parabolas**. One of the goals of this section is for you to become more familiar with the graphs of other functions. The best way to become familiar is by experience, working example after example.

EXAMPLE 2.1 Generating a Calculator Graph

Use your calculator or computer to sketch a graph of $f(x) = 3x^2 - 1$.

Solution You should get an initial graph that looks something like that in Figure 0.19a. This is simply a parabola opening upward. A graph is often used to search for important points, such as x -intercepts, y -intercepts, or peaks and troughs. In this case, we could see these points better if we **zoom in**; that is, display a smaller range of x - and y -values than the technology has initially chosen for us. The graph in Figure 0.19b shows x -values from $x = -2$ to $x = 2$ and y -values from $y = -2$ to $y = 10$.

You can see more clearly in Figure 0.19b that the parabola bottoms out roughly at the point $(0, -1)$ and crosses the x -axis at approximately $x = -0.5$ and $x = 0.5$. You can make this more precise by doing some algebra. Recall that an x -intercept is a point where $y = 0$ or $f(x) = 0$. Solving $3x^2 - 1 = 0$ gives $3x^2 = 1$ or $x^2 = \frac{1}{3}$, so that $x = \pm\sqrt{\frac{1}{3}} \approx \pm 0.57735$.

Notice the interplay in example 2.1 between the graphics and the algebra. The graph suggested approximate values for the two x -intercepts, but you needed the algebra to find the values exactly. (It's worth noting that since $\sqrt{\frac{1}{3}} \approx 0.577$, our visual guess of 0.5 was not especially accurate.) Also note that by zooming in, we used the technology to show us a view of the graph that highlighted the features of the graph that we wanted. But, in general, how do you know what the best view is? For example, the graphs in Figures 0.19a and 0.19b indicate a simple parabola, but how do you know that there are no interesting features of the graph lurking just off the screen? In short, you don't, without some calculus.

Before investigating other graphs, we should say a few words about what a computer- or calculator-generated graph really is. Actually, computers and calculators do not draw graphs. Yes, we usually call them graphs, but what the computer actually does is light up some tiny screen elements called **pixels**. If the pixels are small enough, the image appears to be a continuous curve or graph.



By **graphing window**, we mean the rectangle defined by the range of x - and y -values displayed. The graphing window can dramatically affect the look of a graph. For example, suppose the x 's run from $x = -2$ to $x = 2$. If the computer or calculator screen is wide enough for 400 columns of pixels from left to right, then points will be displayed for $x = -2, x = -1.99, x = -1.98, \dots$. If there is an interesting feature of this function located between $x = -1.99$ and $x = -1.98$, you will not see it unless you zoom in some. In this case, zooming in would reduce the difference between adjacent x 's. Similarly, suppose that the y 's run from $y = 0$ to $y = 3$ and that there are 600 rows of pixels from top to bottom.

Then, there will be pixels corresponding to $y = 0$, $y = 0.005$, $y = 0.01$, \dots . Now, suppose that $f(-2) = 0.0049$ and $f(-1.99) = 0.0051$. Before points are plotted, function values are rounded to the nearest y -value, in this case 0.005 . You won't be able to see any difference in the y -values of these points. If the actual difference is important, you will need to zoom in some to see it.

REMARK 2.1

Most calculators and computer drawing packages follow one of the following two schemes for defining the graphing window for a given function.

- **Fixed graphing window:** Most calculators follow this method. Graphs are plotted in a preselected range of x - and y -values, unless you specify otherwise. For example, the very popular Texas Instruments graphing calculators will automatically plot points in the rectangle defined by $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$.
- **Automatic graphing window:** Most computer drawing packages and some calculators follow this method. Graphs are plotted for a preselected range of x -values and the computer calculates the range of y -values so that all of the calculated points will fit in the screen's range.

Get to know how your calculator or computer software operates and use it routinely as you progress through this course. Whether you are using a graphing calculator or a computer, you should always be able to reproduce the computer-generated graphs used in this text by adjusting your graphing window appropriately.

Graphs are drawn to provide visual displays of the significant features of a function. What qualifies as *significant* will vary from problem to problem, but often the x - and y -intercepts and points known as **extrema** are of interest. The function value $f(M)$ is called a **local maximum** of the function f if $f(M) \geq f(x)$ for all x 's "nearby" $x = M$ [more precisely, if there exist numbers a and b with $a < M < b$ such that $f(M) \geq f(x)$ for all x such that $a < x < b$]. Similarly, the function value $f(m)$ is a **local minimum** of the function f if $f(m) \leq f(x)$ for all x 's "nearby" $x = m$. A **local extremum** is a function value that is either a local maximum or local minimum. Whenever possible, you should produce graphs that show all intercepts and extrema.

EXAMPLE 2.2 Sketching a Graph

Sketch a graph of $f(x) = x^3 + 4x^2 - 5x - 1$ showing all intercepts and extrema.

Solution Depending on your calculator or computer software, you may initially get a graph that looks like one of those in Figures 0.20a or 0.20b.

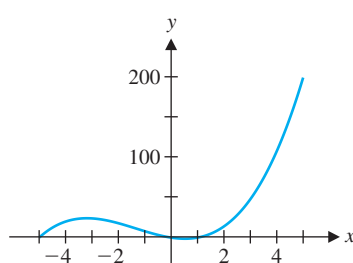


FIGURE 0.20a
 $y = x^3 + 4x^2 - 5x - 1$.

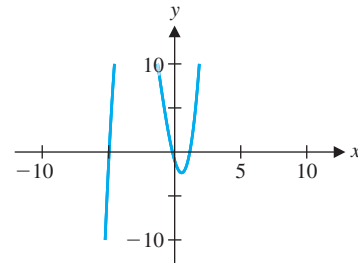


FIGURE 0.20b
 $y = x^3 + 4x^2 - 5x - 1$.

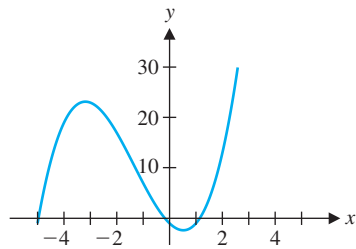


FIGURE 0.21
 $y = x^3 + 4x^2 - 5x - 1$.

Neither graph is completely satisfactory, although both should give you the idea of a graph that (reading left to right) rises to a local maximum near $x = -3$, drops to a local minimum near $x = 1$, and then rises again. To get a better graph, notice the scales on the x - and y -axes. The graphing window for Figure 0.20a is the rectangle defined by $-5 \leq x \leq 5$ and $-6 \leq y \leq 203$. The graphing window for Figure 0.20b is defined by the rectangle $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$. From either graph, it appears that we need to show y -values larger than 10, but not nearly as large as 203, to see the local maximum. Since all of the significant features appear to lie between $x = -6$ and $x = 6$, one choice for a better window is $-6 \leq x \leq 6$ and $-6 \leq y \leq 30$, as seen in Figure 0.21. There, you can clearly see the three x -intercepts, the local maximum and the local minimum. ■

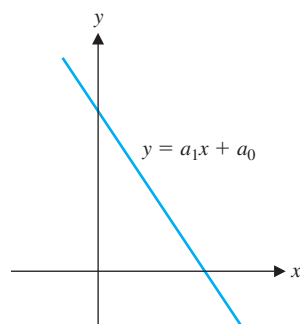


FIGURE 0.22a
 Line, $a_1 < 0$.

Note that the graph in example 2.2 was produced by a process of trial and error with thoughtful corrections. You are unlikely to get a perfect picture on your first try. But, from this starting place, you can enlarge the graphing window (i.e., **zoom out**) if you need to see more or shrink the graphing window (i.e., **zoom in**) if the details are hard to see. You should get comfortable enough with your technology that this revision process is routine (and even fun!).

In the exercises, you will be asked to graph a variety of functions and discuss the shapes of the graphs of polynomials of different degrees. Having some knowledge of the general shapes will help you decide whether or not you have found an acceptable graph. To get you started, we summarize the different shapes of linear, quadratic and cubic polynomials below. Of course, the graphs of linear functions of the form $f(x) = a_1x + a_0$ are simply straight lines of slope a_1 . Two possibilities are shown in Figures 0.22a and 0.22b. Whenever function values get smaller, as you look from left to right, as in Figure 0.22a, we say that the function f is **decreasing**. If function values get larger, as you look from left to right, as in Figure 0.22b, we say that the function is **increasing**.

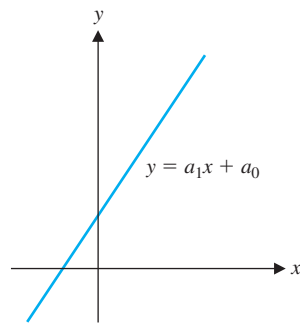


FIGURE 0.22b
 Line, $a_1 > 0$.

You will also recall that the graphs of quadratic polynomials of the form $f(x) = a_2x^2 + a_1x + a_0$ are parabolas. The parabola opens upward if $a_2 > 0$ and opens downward if $a_2 < 0$. We show typical parabolas in Figures 0.23a and 0.23b.

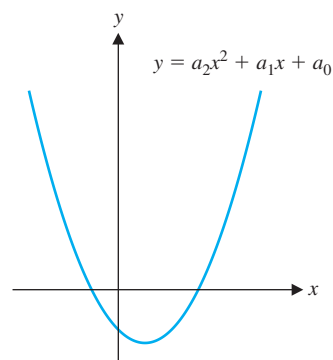


FIGURE 0.23a
 Parabola, $a_2 > 0$.

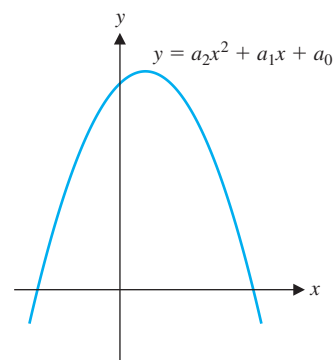


FIGURE 0.23b
 Parabola, $a_2 < 0$.

Notice that the parabolas indicated in Figures 0.23a and 0.23b are both increasing and decreasing on different intervals of the x -axis. One of our goals for later in the text is to learn how to precisely determine the intervals on which a given function is increasing and decreasing. The graphs of cubic functions of the form $a_3x^3 + a_2x^2 + a_1x + a_0$ are somewhat S-shaped. Reading from left to right, the function begins negative and ends positive

if $a_3 > 0$ and begins positive and ends negative if $a_3 < 0$, as indicated in Figures 0.24a and 0.24b.

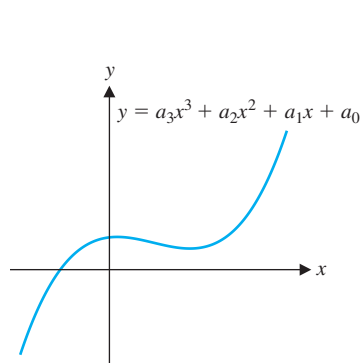


FIGURE 0.24a
Cubic: one max, min, $a_3 > 0$.

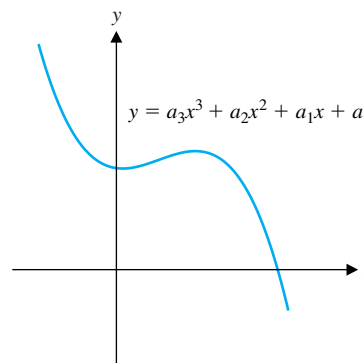


FIGURE 0.24b
Cubic: one max, min, $a_3 < 0$.

Cubics often have one local maximum and one local minimum, as do those in Figures 0.24a and 0.24b. Many cubics, such as those in Figures 0.25a and 0.25b, temporarily flatten out without turning around to create a local maximum or minimum. The point where the curve changes its shape (from being bent upward, to being bent downward, or vice versa), is called an **inflection point**.

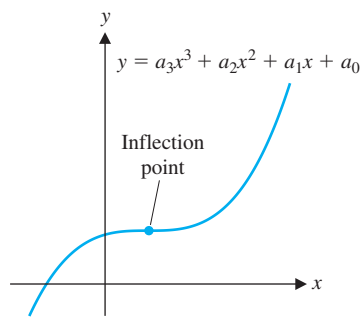


FIGURE 0.25a
Cubic: no max or min, $a_3 > 0$.

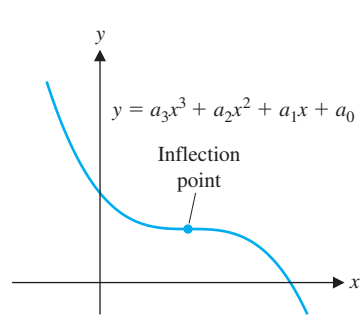


FIGURE 0.25b
Cubic: no max or min, $a_3 < 0$.

We will use the calculus developed over the next several chapters to see how to adjust the graphing window so that all of the significant features of a graph are shown. You can already use your knowledge of the general shapes of certain functions to see how to adjust the graphing window, as in example 2.3.

EXAMPLE 2.3 Sketching the Graph of a Cubic Polynomial

Sketch a graph of the cubic polynomial $f(x) = x^3 - 20x^2 - x + 20$.

Solution Your initial graph probably looks like one of those in Figures 0.26a or 0.26b (shown on the next page).

From the preceding discussion on the general shapes of cubics, you should recognize that neither of these graphs looks like a cubic. More than anything else, they look like parabolas. To see the S-shape behavior in the graph, we need to look on a larger range of

22 CHAPTER 0 Preliminaries

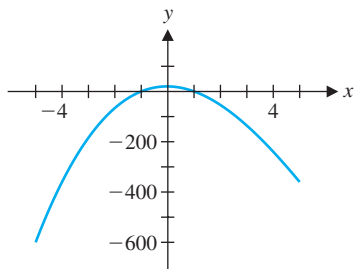


FIGURE 0.26a
 $f(x) = x^3 - 20x^2 - x + 20$.

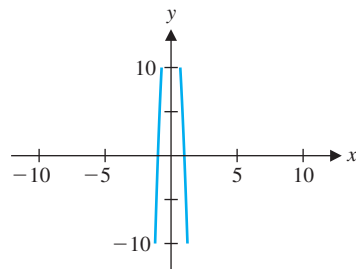


FIGURE 0.26b
 $f(x) = x^3 - 20x^2 - x + 20$.

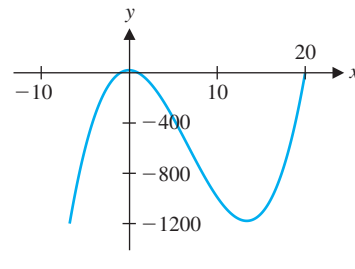


FIGURE 0.26c
 $f(x) = x^3 - 20x^2 - x + 20$.

x -values. But how much larger? Unfortunately, we can't answer that question in general, before we develop some of the concepts of calculus. For the moment, the answer is to use trial and error, until the graph resembles the shape of a cubic. You should recognize the characteristic shape of a cubic in Figure 0.26c. Notice from Figure 0.26c, that although we now see more of the big picture (often referred to as the **global** behavior of the function), we have lost some of the details (such as the x -intercepts), which we could clearly see in Figures 0.26a and 0.26b (often referred to as the **local** behavior of the function).

Rational functions have some properties not seen in polynomials, as we see in examples 2.4 to 2.6.

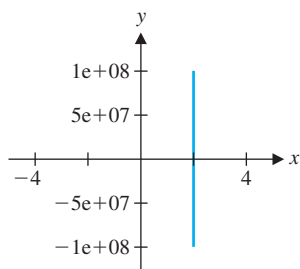


FIGURE 0.27a
 $y = \frac{x-1}{x-2}$.

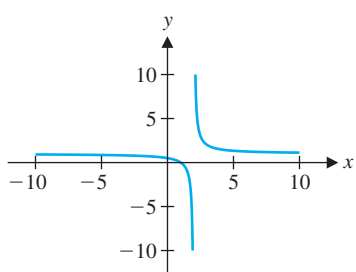


FIGURE 0.27b
 $y = \frac{x-1}{x-2}$.

EXAMPLE 2.4 Sketching the Graph of a Rational Function

Sketch a graph of $f(x) = \frac{x-1}{x-2}$ and describe the behavior of the graph near $x = 2$.

Solution Your initial graph should look something like one of Figures 0.27a or 0.27b. From either graph, it should be clear that something unusual is happening near $x = 2$. Zooming in closer to $x = 2$ should yield a graph like that in Figures 0.28a or 0.28b.

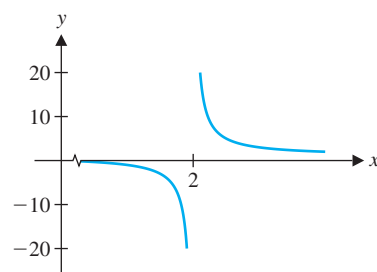


FIGURE 0.28a
 $y = \frac{x-1}{x-2}$.

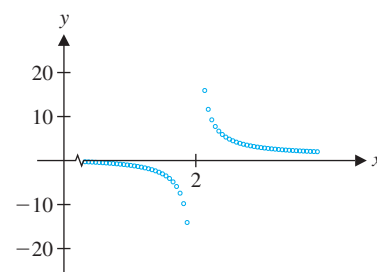


FIGURE 0.28b
 $y = \frac{x-1}{x-2}$.

Figure 0.28a shows a *connected* (or line) plot where along with computing one point per x -value, the computer highlights extra pixels to try to smooth out the graph. The *disconnected* (or dot) plot in Figure 0.28b simply displays the computed points. In both graphs, it appears that as x increases up to 2, the function values get more and more negative, while as x decreases down to 2, the function values get more and more positive. This is also observed in the following table of function values.

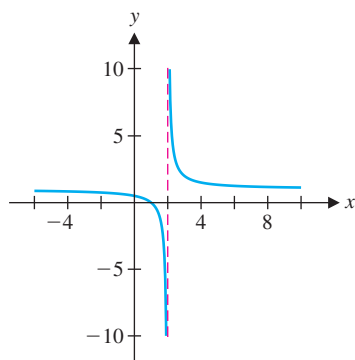


FIGURE 0.29
Vertical asymptote.

x	$f(x)$
1.8	-4
1.9	-9
1.99	-99
1.999	-999
1.9999	-9999

x	$f(x)$
2.2	6
2.1	11
2.01	101
2.001	1001
2.0001	10,001

Note that at $x = 2$, $f(x)$ is undefined. However, as x approaches 2 from the left, the graph veers down sharply. In this case, we say that $f(x)$ tends to $-\infty$. Likewise, as x approaches 2 from the right, the graph rises sharply. Here, we say that $f(x)$ tends to ∞ and there is a **vertical asymptote** at $x = 2$. (We'll define this more carefully in Chapter 1.) It is common to draw a vertical dashed line at $x = 2$ to indicate this (see Figure 0.29). Since $f(2)$ is undefined, there is no point plotted at $x = 2$. ■

Many rational functions have vertical asymptotes. Notice that there is no point plotted on the vertical asymptote since the function is undefined at such an x -value (due to the division by zero when that value of x is substituted in). Given a rational function, you can locate possible vertical asymptotes by finding x -values where the denominator is zero. It turns out that if the numerator is not zero for such an x , there is a vertical asymptote at this value of x .

EXAMPLE 2.5 A Graph with Several Vertical Asymptotes

Find all vertical asymptotes for $f(x) = \frac{x - 1}{x^2 - 5x + 6}$.

Solution Note that the denominator is

$$x^2 - 5x + 6 = (x - 2)(x - 3),$$

so that the only possible locations for vertical asymptotes are $x = 2$ and $x = 3$. Since neither x -value makes the numerator $(x - 1)$ equal to zero, there are vertical asymptotes at both $x = 2$ and $x = 3$. A quick computer-generated graph gives little indication of how the function behaves near the asymptotes (see Figure 0.30).

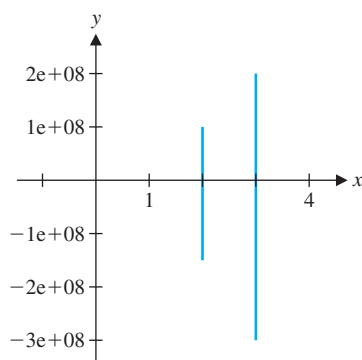


FIGURE 0.30
 $y = \frac{x - 1}{x^2 - 5x + 6}$

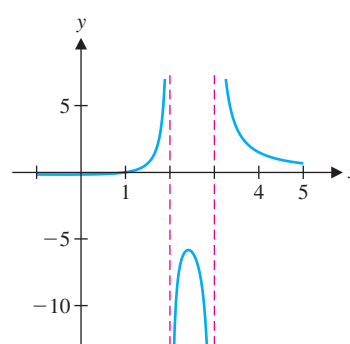


FIGURE 0.31
 $y = \frac{x - 1}{x^2 - 5x + 6}$

We can improve the graph by zooming in in both the x - and y -directions. Figure 0.31 shows a graph of the same function using the graphing window defined by the rectangle

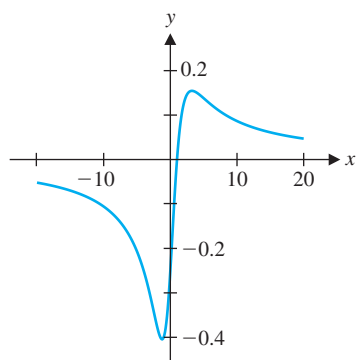


FIGURE 0.32

$$y = \frac{x-1}{x^2+4}$$

$-1 \leq x \leq 5$ and $-13 \leq y \leq 7$. While this graph can still stand some improvement (we'll see how to use the calculus to do this in Chapter 3), the graph is adequate and clearly shows the vertical asymptotes at $x = 2$ and $x = 3$. ■

As we see in example 2.6, not all rational functions have vertical asymptotes.

EXAMPLE 2.6 A Rational Function with No Vertical Asymptotes

Find all vertical asymptotes of $\frac{x-1}{x^2+4}$.

Solution Notice that $x^2 + 4 = 0$ has no (real) solutions, since $x^2 + 4 > 0$ for all real numbers, x . So, there are no vertical asymptotes. The graph in Figure 0.32 is consistent with this observation. ■

Graphs are useful for finding approximate solutions of difficult equations. We explore this in examples 2.7 and 2.8.

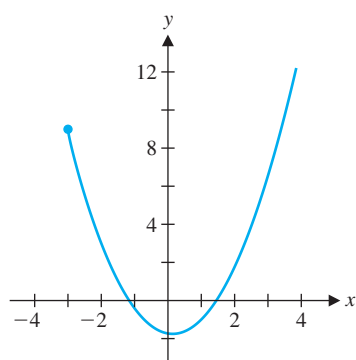


FIGURE 0.33

$$y = x^2 - \sqrt{x+3}$$

EXAMPLE 2.7 Finding Zeros Approximately

Find approximate solutions of the equation $x^2 = \sqrt{x+3}$.

Solution First, note that you can rewrite this equation as $x^2 - \sqrt{x+3} = 0$. You can then look for zeros in the graph of $f(x) = x^2 - \sqrt{x+3}$, seen in Figure 0.33. Note that two zeros are clearly indicated: one near -1 , the other near 1.5 . However, since you know very little of the nature of the function $x^2 - \sqrt{x+3}$, you cannot say whether or not there are any other zeros, ones that don't show up in the window seen in Figure 0.33. On the other hand, if you graph the two functions on either side of the equation on the same set of axes, as in Figure 0.34, you can clearly see two points where the graphs intersect (corresponding to the two zeros seen in Figure 0.33). Further, since you know the general shapes of both of the graphs, you can infer from Figure 0.34 that there are no other intersections (i.e., there are no other zeros of f). This is important information that you cannot obtain from Figure 0.33. Keep this useful alternative in mind when you are looking for zeros. Now that you know how many solutions there are, you need to estimate their values. As we have already illustrated, one method is to zoom in on the zeros graphically. This method is reliable but rather tedious if you want more than two or three digits of accuracy. We leave it as an exercise to verify that the zeros are approximately $x = 1.4$ and $x = -1.2$. If your calculator or computer algebra system has a solve command, you can use it to quickly obtain an accurate approximation. In this case, we get $x \approx 1.452626878$ and $x \approx -1.164035140$. ■

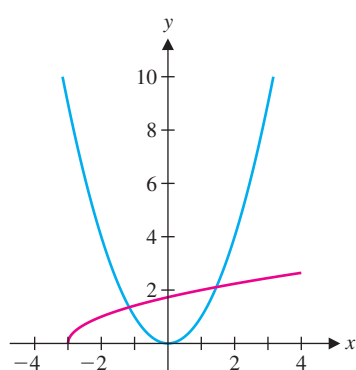


FIGURE 0.34

$$y = x^2 \text{ and } y = \sqrt{x+3}$$

When using the solve command on your calculator or computer algebra system, be sure to check that the solutions make sense. If the results don't match what you've seen in your preliminary sketches and zooms, beware! Even high-tech equation solvers make mistakes occasionally.

EXAMPLE 2.8 Finding Intersections by Calculator: An Oversight

Find all points of intersection of the graphs of $y = 2 \cos x$ and $y = 2 - x$.

Solution Notice that the intersections correspond to solutions of the equation $2 \cos x = 2 - x$. Using the solve command on the TI-92 graphing calculator, we found intersections at $x \approx 3.69815$ and $x = 0$. So, what's the problem? A quick sketch of the graphs of

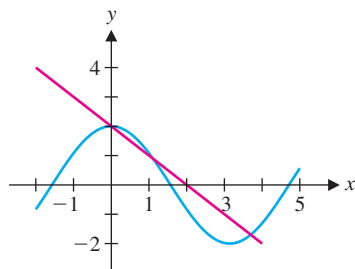


FIGURE 0.35
 $y = 2 \cos x$ and $y = 2 - x$.

$y = 2 - x$ and $y = 2 \cos x$ (we discuss this function in the next section) clearly shows *three* intersections (see Figure 0.35).

The middle solution, $x \approx 1.10914$, was somehow passed over by the calculator's solve routine. The lesson here is to use graphical evidence to support your solutions, especially when using software and/or functions with which you are less than completely familiar. ■

You need to look skeptically at the answers provided by your calculator's solver program. While such solvers provide a quick and easy means of approximating solutions of equations, these programs will sometimes return totally incorrect answers. So, how do you know if your solver is giving you an accurate answer or one that's totally incorrect? The only answer to this is that you must carefully test your calculator's solution, by separately calculating both sides of the equation (*by hand*) at the calculated solution. Sure, it's much easier to simply accept the calculator's answer, but doing so blindly is fraught with peril, as we illustrate with example 2.9.



**EXAMPLE 2.9 Solving an Equation by Calculator:
 An Erroneous Answer**

Use your calculator's solver program to solve the equation $x + \frac{1}{x} = \frac{1}{x}$.

Solution Certainly, you don't need a calculator to solve this equation, but consider what happens when you use one. Most calculators report a solution that is very close to zero, while others, like the TI-89 (and the author's computer algebra system), report that the solution is $x = 0$. Not only are these answers incorrect, but the given equation has *no* solution, as follows. First, notice that the equation only makes sense when $x \neq 0$. Then, subtracting $\frac{1}{x}$ from both sides of the equation leaves us with $x = 0$, which can't possibly be a solution, since it does not satisfy the original equation. Notice further, that if your calculator returns the approximate solution $x = 1 \times 10^{-7}$ and you use your calculator to compute the values on both sides of the equation, the calculator will compute

$$x + \frac{1}{x} = 1 \times 10^{-7} + 1 \times 10^7,$$

which it approximates as $1 \times 10^7 = \frac{1}{x}$, since calculators carry only a finite number of digits. In other words, although

$$1 \times 10^{-7} + 1 \times 10^7 \neq 1 \times 10^7,$$

your calculator treats these numbers as the same and so, reports that the equation is satisfied, when in fact, it's not. The moral of this story is to be an intelligent user of technology and don't blindly accept everything a calculator tells you. ■

We want to emphasize again that graphing should be the first step in the equation-solving process. A good graph will show you how many solutions to expect, as well as give their approximate locations. Whenever possible, you should factor or use the quadratic formula to get exact solutions. When this is impossible, approximate the solutions by zooming in on them graphically or by using your calculator's solve command. *Always* compare your results to a graph to see if there's anything you've missed.

EXERCISES 0.2

WRITING EXERCISES

1. Explain why there is a significant difference between Figures 0.28a and 0.28b.
2. In Figure 0.28a, the graph approaches the lower portion of the vertical asymptote from the left, whereas the graph approaches the upper portion of the vertical asymptote from the right. Use the table of function values found on page 23 to explain how to determine whether a graph approaches a vertical asymptote by dropping down or rising up.
3. In the text, we discussed the difference between graphing with a fixed window versus an automatic window. Discuss the advantages and disadvantages of each. (Hint: Consider the case of a first graph of a function you know nothing about and the case of hoping to see the important details of a graph for which you know the general shape.)
4. Examine the graph of $y = \frac{x^3 + 1}{x}$ with each of the following graphing windows: (a) $-10 \leq x \leq 10$, (b) $-1000 \leq x \leq 1000$. Explain why the graph in (b) doesn't show the details that the graph in (a) does.

T In exercises 1–30, sketch a graph of the function showing all extrema, intercepts and asymptotes.

1. $f(x) = x^2 - 1$
2. $f(x) = 3 - x^2$
3. $f(x) = x^2 + 2x + 8$
4. $f(x) = x^2 - 20x + 11$
5. $f(x) = x^3 + 1$
6. $f(x) = 10 - x^3$
7. $f(x) = x^3 + 2x - 1$
8. $f(x) = x^3 - 3x + 1$
9. $f(x) = x^4 - 1$
10. $f(x) = 2 - x^4$
11. $f(x) = x^4 + 2x - 1$
12. $f(x) = x^4 - 6x^2 + 3$
13. $f(x) = x^5 + 2$
14. $f(x) = 12 - x^5$
15. $f(x) = x^5 - 8x^3 + 20x - 1$
16. $f(x) = x^5 + 5x^4 + 2x^3 + 1$
17. $f(x) = \frac{3}{x-1}$
18. $f(x) = \frac{4}{x+2}$
19. $f(x) = \frac{3x}{x-1}$
20. $f(x) = \frac{4x}{x+2}$
21. $f(x) = \frac{3x^2}{x-1}$
22. $f(x) = \frac{4x^2}{x+2}$
23. $f(x) = \frac{2}{x^2-4}$
24. $f(x) = \frac{6}{x^2-9}$
25. $f(x) = \frac{3}{x^2+4}$
26. $f(x) = \frac{6}{x^2+9}$
27. $f(x) = \frac{x+2}{x^2+x-6}$
28. $f(x) = \frac{x-1}{x^2+4x+3}$
29. $f(x) = \frac{3x}{\sqrt{x^2+4}}$
30. $f(x) = \frac{2x}{\sqrt{x^2+1}}$

In exercises 31–38, find all vertical asymptotes.

31. $f(x) = \frac{3x}{x^2-4}$
32. $f(x) = \frac{x+4}{x^2-9}$
33. $f(x) = \frac{4x}{x^2+3x-10}$
34. $f(x) = \frac{x+2}{x^2-2x-15}$
35. $f(x) = \frac{4x}{x^2+4}$
36. $f(x) = \frac{3x}{\sqrt{x^2-9}}$
37. $f(x) = \frac{x^2+1}{x^3+3x^2+2x}$
38. $f(x) = \frac{3x}{x^4-1}$

T In exercises 39–42, a standard graphing window will not reveal all of the important details of the graph. Adjust the graphing window to find the missing details.

39. $f(x) = \frac{1}{3}x^3 - \frac{1}{400}x$
40. $f(x) = x^4 - 11x^3 + 5x - 2$
41. $f(x) = x\sqrt{144-x^2}$
42. $f(x) = \frac{1}{5}x^5 - \frac{7}{8}x^4 + \frac{1}{3}x^3 + \frac{7}{2}x^2 - 6x$

T In exercises 43–48, adjust the graphing window to identify all vertical asymptotes.

43. $f(x) = \frac{3}{x-1}$
44. $f(x) = \frac{4x}{x^2-1}$
45. $f(x) = \frac{3x^2}{x^2-1}$
46. $f(x) = \frac{2x}{x+4}$
47. $f(x) = \frac{x^2-1}{\sqrt{x^4+x}}$
48. $f(x) = \frac{2x}{\sqrt{x^2+x}}$

T In exercises 49–56, determine the number of (real) solutions. Solve for the intersection points exactly if possible and estimate the points if necessary.

49. $\sqrt{x-1} = x^2 - 1$
50. $\sqrt{x^2+4} = x^2 + 2$
51. $x^3 - 3x^2 = 1 - 3x$
52. $x^3 + 1 = -3x^2 - 3x$
53. $(x^2 - 1)^{2/3} = 2x + 1$
54. $(x+1)^{2/3} = 2 - x$
55. $\cos x = x^2 - 1$
56. $\sin x = x^2 + 1$

T In exercises 57–62, use a graphing calculator or computer graphing utility to estimate all zeros.

57. $f(x) = x^3 - 3x + 1$
58. $f(x) = x^3 - 4x^2 + 2$
59. $f(x) = x^4 - 3x^3 - x + 1$
60. $f(x) = x^4 - 2x + 1$
61. $f(x) = x^4 - 7x^3 - 15x^2 - 10x - 1410$
62. $f(x) = x^6 - 4x^4 + 2x^3 - 8x - 2$

T 63. Graph $y = x^2$ with the graphing window $-10 \leq x \leq 10$, $-10 \leq y \leq 10$, without drawing the x - and y -axes. Adjust the graphing window for $y = 2(x-1)^2 + 3$ so that (without the axes showing) the graph looks identical to that of $y = x^2$.

T 64. Graph $y = x^2$ with the graphing window $-10 \leq x \leq 10$, $-10 \leq y \leq 10$. Separately graph $y = x^4$ with the same graphing window. Compare and contrast the graphs. Then graph the

two functions on the same axes and carefully examine the differences with $-1 < x < 1$ and $x > 1$.

65. In this exercise, you will find an equation describing all points equidistant from the x -axis and the point $(0, 2)$. First, see if you can sketch a picture of what the curve through these points ought to look like. For a point (x, y) that is on the curve, explain why $\sqrt{y^2} = \sqrt{x^2 + (y - 2)^2}$. Square both sides of this equation and solve for y . Identify the curve.
66. Find an equation describing all points equidistant from the x -axis and $(1, 4)$ (see exercise 65).

$(1.2, 1.4)$. (a) $f(0.4)$, (b) $f(0.39)$, (c) $f(1.17)$, (d) $f(1.20)$, (e) $f(1.8)$, (f) $f(1.81)$. Repeat (c)–(d) if the graphing window is zoomed in so that $x = 1.00, 1.01, \dots, 1.20$ and $y = 1.30, 1.31, \dots, 1.50$. Repeat (e)–(f) if the graphing window is zoomed in so that $x = 1.800, 1.801, \dots, 1.820$ and $y = 3.200, 3.205, \dots, 3.300$.

- T 2. Graph $y = x^2 - 1$, $y = x^2 + x - 1$, $y = x^2 + 2x - 1$, $y = x^2 - x - 1$ and $y = x^2 - 2x - 1$ and other functions of the form $y = x^2 + cx - 1$. Describe the effect(s) a change in c has on the graph.
- T 3. Figures 0.24 and 0.25 provide a catalog of the possible types of graphs of cubic polynomials. In this exercise, you will compile a catalog of graphs of fourth-order polynomials (i.e., $y = ax^4 + bx^3 + cx^2 + dx + e$). Start by using your calculator or computer to sketch graphs with different values of a, b, c, d and e . Try $y = x^4$, $y = 2x^4$, $y = -2x^4$, $y = x^4 + x^3$, $y = x^4 + 2x^3$, $y = x^4 - 2x^3$, $y = x^4 + x^2$, $y = x^4 - x^2$, $y = x^4 - 2x^2$, $y = x^4 + x$, $y = x^4 - x$ and so on. Try to determine what effect each constant has.

EXPLORATORY EXERCISES

- T 1. Suppose a graphing calculator is set up with pixels corresponding to $x = 0, 0.1, 0.2, 0.3, \dots, 2.0$ and $y = 0, 0.1, 0.2, 0.3, \dots, 4.0$. For the function $f(x) = x^2$, compute the indicated function values and round off to give pixel coordinates [e.g., the point $(1.19, 1.4161)$ has pixel coordinates

0.3 INVERSE FUNCTIONS

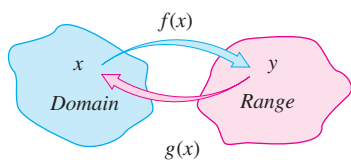


FIGURE 0.36
 $g(x) = f^{-1}(x)$.

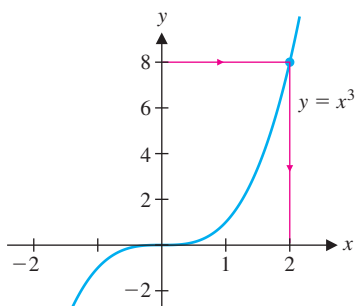


FIGURE 0.37
 Finding the x -value corresponding to $y = 8$.

The notion of an *inverse* relationship is basic to many areas of science, although the term is only infrequently used. The number of common inverse problems is immense. As only one example, take the case of the electrocardiogram (EKG). In an EKG, technicians connect a series of electrodes to a patient's chest and use measurements of electrical activity on the surface of the body to infer something about the electrical activity on the surface of the heart. This is referred to as an *inverse* problem, since physicians are attempting to determine what *inputs* (i.e., the electrical activity on the surface of the heart) cause an observed *output* (the measured electrical activity on the surface of the chest).

The mathematical notion of inverse is much the same as that described above. Given an output (i.e., a value in the range of a given function), we wish to find the input (the value in the domain) that produced the observed output. That is, given a y in the range of f , find the x in the domain of f for which $y = f(x)$. If we can do this, we say that $x = f^{-1}(y)$. (See the illustration of the inverse function $g(x)$ shown in Figure 0.36.)

For instance, suppose that $f(x) = x^3$ and $y = 8$. Can you find an x such that $x^3 = 8$? That is, can you find the x -value corresponding to $y = 8$ (see Figure 0.37)? Of course, you know the solution of this particular equation: $x = \sqrt[3]{8} = 2$. In fact, in general, if $x^3 = y$, then $x = \sqrt[3]{y}$. In light of this, we say that the cube root function is the **inverse** of $f(x) = x^3$. We use this first example of a pair of inverse functions to illustrate a property shared by all inverse functions.

EXAMPLE 3.1 Two Functions That Reverse the Action of Each Other

If $f(x) = x^3$ and $g(x) = x^{1/3}$, show that

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x,$$

for all x .

CAUTION

Pay close attention to the notation. Notice that $f^{-1}(x)$ does *not* mean $\frac{1}{f(x)}$. The reciprocal of $f(x)$ is written as:

$$\frac{1}{f(x)} = [f(x)]^{-1}.$$

Solution For all real numbers x , we have

$$f(g(x)) = f(x^{1/3}) = (x^{1/3})^3 = x$$

and

$$g(f(x)) = g(x^3) = (x^3)^{1/3} = x. \blacksquare$$

Notice in example 3.1 that the action of f undoes the action of g and vice versa. We may take this as the definition of an inverse function (again, think of Figure 0.36). More precisely, we have the following definition.

DEFINITION 3.1

Assume that f and g have domains A and B , respectively, and that $f(g(x))$ is defined for all $x \in B$ and $g(f(x))$ is defined for all $x \in A$. If

$$f(g(x)) = x, \quad \text{for all } x \in B \quad \text{and}$$

$$g(f(x)) = x, \quad \text{for all } x \in A,$$

we say that g is the **inverse** of f , written $g = f^{-1}$. Equivalently, f is the inverse of g , written $f = g^{-1}$.

Observe that many familiar functions have no inverse.

EXAMPLE 3.2 A Function with No Inverse

Show that $f(x) = x^2$ has no inverse on the interval $(-\infty, \infty)$.

Solution Notice that $f(4) = 16$ and $f(-4) = 16$. That is, there are *two* x -values that produce the same y -value. So, if we were to try to define an inverse of f , how would we define $f^{-1}(16)$? Look at the graph of $y = x^2$ (see Figure 0.38) to see what the problem is. For each $y > 0$, there are *two* x -values for which $y = x^2$. Because of this, the function f does not have an inverse. \blacksquare

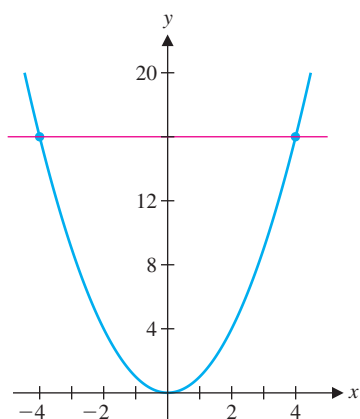


FIGURE 0.38
 $y = x^2$.

REMARK 3.1

For $f(x) = x^2$, it is tempting to jump to the conclusion that $g(x) = \sqrt{x}$ is the inverse of $f(x)$. Notice that although $f(g(x)) = (\sqrt{x})^2 = x$ for all $x \geq 0$ (i.e., for all x in the domain of $g(x)$), it is *not* generally true that $g(f(x)) = \sqrt{x^2} = x$. In fact, this last equality holds *only* for $x \geq 0$. (Recall that $\sqrt{x^2} = |x|$.) However, for $f(x) = x^2$ restricted to the domain $x \geq 0$, we do have that $f^{-1}(x) = \sqrt{x}$.

As we saw in example 3.2, a function that produces any given y -value for two or more different x -values does not have an inverse. That is, for a function to have an inverse, different x 's must correspond to different y 's.

DEFINITION 3.2

A function f is called **one-to-one** when for every $y \in \text{Range}\{f\}$, there is *exactly one* $x \in \text{Domain}\{f\}$ for which $y = f(x)$.

REMARK 3.2

Observe that an equivalent definition of one-to-one is the following. A function $f(x)$ is one-to-one if and only if the equality $f(a) = f(b)$ implies $a = b$. This version of the definition is often useful for proofs involving one-to-one functions.

It is also most helpful to think of the concept of one-to-one in graphical terms. Notice that a function f is one-to-one if and only if every horizontal line intersects the graph in at most one point. This is usually referred to as the **horizontal line test**. We illustrate this in Figures 0.39a and 0.39b. One can then prove the following result.

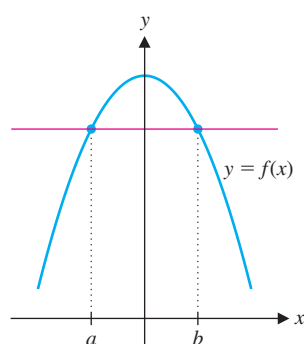


FIGURE 0.39a
 $f(a) = f(b)$, for $a \neq b$. So, f does not pass the horizontal line test and is not one-to-one.

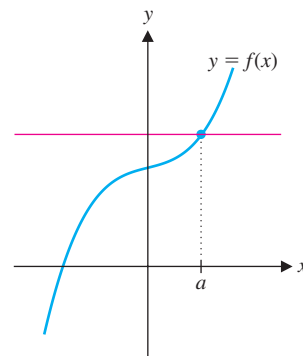


FIGURE 0.39b
 Every horizontal line intersects the curve in at most one point. So, f passes the horizontal line test and is one-to-one.

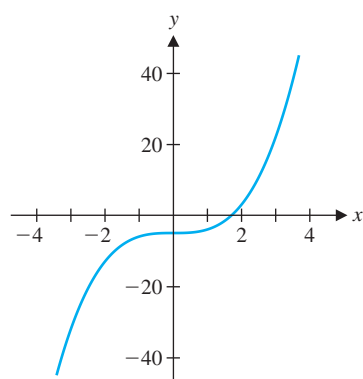


FIGURE 0.40
 $y = x^3 - 5$.

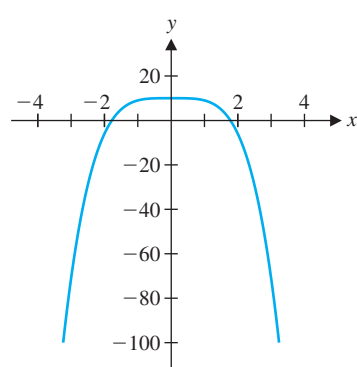


FIGURE 0.41
 $y = 10 - x^4$.

THEOREM 3.1

A function f has an inverse if and only if it is one-to-one.

This simply says that every one-to-one function has an inverse and every function that has an inverse is one-to-one. However, it says nothing about how to find an inverse. For very simple functions, we can find inverses by solving equations.

EXAMPLE 3.3 Finding an Inverse Function

Find the inverse of $f(x) = x^3 - 5$.

Solution Note that it is not entirely clear from the graph (see Figure 0.40) whether or not f passes the horizontal line test. To find the inverse function, write $y = f(x)$ and solve for x (i.e., solve for the input x that produced the observed output y). We have

$$y = x^3 - 5.$$

Adding 5 to both sides and taking the cube root gives us

$$(y + 5)^{1/3} = (x^3)^{1/3} = x.$$

So, we have that $x = f^{-1}(y) = (y + 5)^{1/3}$. Reversing the variables x and y (think about why this makes sense), we have

$$f^{-1}(x) = (x + 5)^{1/3}. \blacksquare$$

EXAMPLE 3.4 A Function That Is Not One-to-One

Show that $f(x) = 10 - x^4$ is not one-to-one.

Solution You can see from a graph (see Figure 0.41) that f is not one-to-one; for instance, $f(1) = f(-1) = 9$. Consequently, f does not have an inverse. \blacksquare

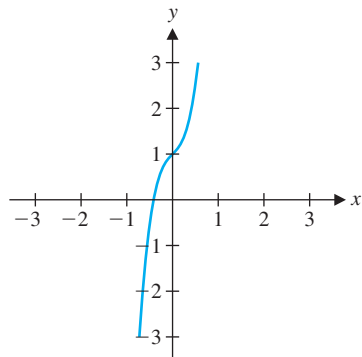


FIGURE 0.42
 $y = x^5 + 8x^3 + x + 1$.

Most often, we cannot find a formula for an inverse function and must be satisfied with simply knowing that the inverse function exists. Example 3.5 is typical of this situation.

EXAMPLE 3.5 Finding Function Values for an Inverse Function

Given that $f(x) = x^5 + 8x^3 + x + 1$ has an inverse, find $f^{-1}(1)$ and $f^{-1}(11)$.

Solution First, notice that from the graph shown in Figure 0.42, the function looks like it might be one-to-one, but how could you be certain of this? (Remember that graphs can be deceptive!) Until we develop some calculus, we will be unable to verify this. Ideally, we would show that f has an inverse by finding a formula for the inverse, as in example 3.3. However, in this case, we must solve the equation

$$y = x^5 + 8x^3 + x + 1$$

for x . Think about this for a moment; you should realize that you do not know how to solve for x in terms of y . (Don't worry; the authors don't know how to do this, either.) So, for the moment, we will need to assume that the inverse exists, as indicated in the instructions.

Turning to the problem of finding $f^{-1}(1)$ and $f^{-1}(11)$, you might wonder if this is possible, since we were unable to find a formula for $f^{-1}(x)$. While it's certainly true that we have no such formula, you might observe that $f(0) = 1$, so that $f^{-1}(1) = 0$. By trial and error, you might also discover that $f(1) = 11$ and so, $f^{-1}(11) = 1$. ■

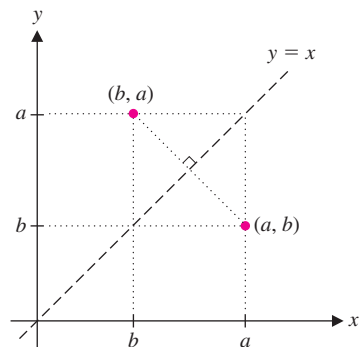


FIGURE 0.43
 Reflection through $y = x$.

In example 3.5, we examined a function that has an inverse, although we could not find that inverse algebraically. Even when we can't find an inverse function explicitly, we can say something graphically. Notice that if (a, b) is a point on the graph of $y = f(x)$ and f has an inverse, f^{-1} , then since

$$b = f(a),$$

we have that

$$f^{-1}(b) = f^{-1}(f(a)) = a.$$

That is, (b, a) is a point on the graph of $y = f^{-1}(x)$. This tells us a great deal about the inverse function. In particular, we can immediately obtain any number of points on the graph of $y = f^{-1}(x)$, simply by inspection. Further, notice that the point (b, a) is the reflection of the point (a, b) through the line $y = x$ (see Figure 0.43). It now follows that given the graph of any one-to-one function, you can draw the graph of its inverse simply by reflecting the entire graph through the line $y = x$. In example 3.6, we illustrate the symmetry of a function and its inverse.

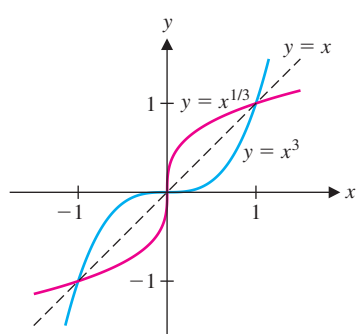


FIGURE 0.44
 $y = x^3$ and $y = x^{1/3}$.

EXAMPLE 3.6 The Graph of a Function and Its Inverse

Draw a graph of $f(x) = x^3$ and its inverse.

Solution From example 3.1, the inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$. Notice the symmetry of their graphs shown in Figure 0.44. ■

Observe that we can use this symmetry principle to draw the graph of an inverse function, even when we don't have a formula for that function (see Figure 0.45).

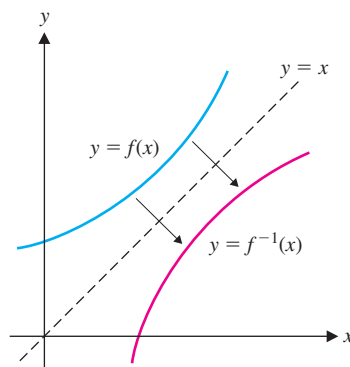


FIGURE 0.45
Graph of f and f^{-1} .

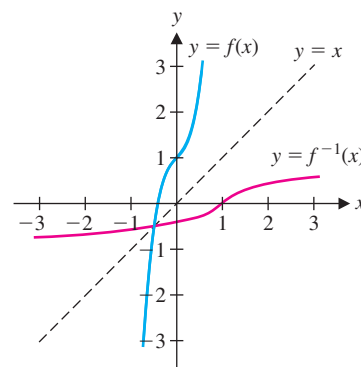


FIGURE 0.46
 $y = f(x)$ and $y = f^{-1}(x)$.

EXAMPLE 3.7 Drawing the Graph of an Unknown Inverse Function

Draw a graph of $f(x) = x^5 + 8x^3 + x + 1$ and its inverse.

Solution In example 3.5, we were unable to find a formula for the inverse function. Despite this, we can draw a graph of f^{-1} with ease. We simply take the graph of $y = f(x)$ seen in Figure 0.42 and reflect it across the line $y = x$. The result is shown in Figure 0.46. (When we introduce parametric equations in section 0.7, we will see a clever way to draw this graph with a graphing calculator.) ■

In example 3.8, we apply our theoretical knowledge of inverse functions in a medical setting.

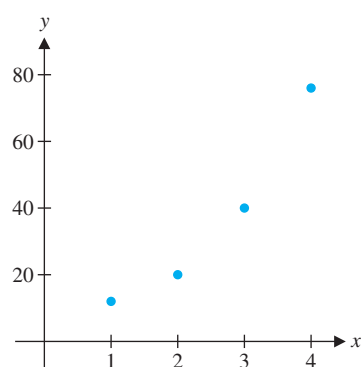


FIGURE 0.47a
Hormone data.

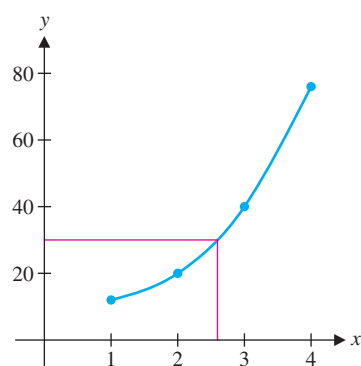


FIGURE 0.47b
Approximate curve.

EXAMPLE 3.8 Determining the Proper Dosage of a Drug

Suppose that the injection of a certain drug raises the level of a key hormone in the body. Physicians want to determine the dosage that produces a healthy hormone level. Dosages of 1, 2, 3 and 4 mg produce hormone levels of 12, 20, 40 and 76, respectively. If the desired hormone level is 30, what is the proper dosage?

Solution A plot of the points $(1, 12)$, $(2, 20)$, $(3, 40)$ and $(4, 76)$ summarizes the data (see Figure 0.47a). The problem is an inverse problem: given $y = 30$, what is x ? It is tempting to argue the following: since 30 is halfway between 20 and 40, the x -value should be halfway between 2 and 3: $x = 2.5$. This method of solution is called **linear interpolation**, since the point $x = 2.5$, $y = 30$ lies on the line through the points $(2, 20)$ and $(3, 40)$. While this provides a primitive estimate of the needed dosage, we should be able to do better. After all, this estimate does not take into account all of the information we have. (Think about what's missing here.) The points in Figure 0.47a suggest a curve that is curving upward. If this is the case, $x = 2.6$ or $x = 2.7$ may be a better estimate of the required dosage. In Figure 0.47b, we have sketched a smooth curve through the data points and indicated a graphical solution of the problem. There are numerous more advanced techniques (e.g., polynomial interpolation) developed by mathematicians to make the estimate of such quantities as accurate as possible. ■

EXERCISES 0.3

WRITING EXERCISES

1. Explain in words (and a picture) why the following is true: if $f(x)$ is increasing for all x , then f has an inverse.
2. Suppose the graph of a function passes the horizontal line test. Explain why you know that the function has an inverse (defined on the range of the function).
3. Radar works by bouncing a high-frequency electromagnetic pulse off of a moving object, then measuring the disturbance in the pulse as it is bounced back. Explain why this is an inverse problem by identifying the input and output.
4. Each human disease has a set of symptoms associated with it. Physicians attempt to solve an inverse problem: given the symptoms, they try to identify the disease causing the symptoms. Explain why this is not a well-defined inverse problem (i.e., logically it is not always possible to correctly identify diseases from symptoms alone).

In exercises 1–4, show that $f(g(x)) = x$ and $g(f(x)) = x$ for all x :

1. $f(x) = x^5$ and $g(x) = x^{1/5}$
2. $f(x) = 4x^3$ and $g(x) = \frac{1}{4}x^{1/3}$
3. $f(x) = 2x^3 + 1$ and $g(x) = \sqrt[3]{\frac{x-1}{2}}$
4. $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{1-2x}{x}$ ($x \neq 0, x \neq -2$)

T In exercises 5–12, determine whether or not the function is one-to-one. If it is, find the inverse and graph both the function and its inverse.

- | | |
|-----------------------------|-----------------------------|
| 5. $f(x) = x^3 - 2$ | 6. $f(x) = x^3 + 4$ |
| 7. $f(x) = x^5 - 1$ | 8. $f(x) = x^5 + 4$ |
| 9. $f(x) = x^4 + 2$ | 10. $f(x) = x^4 - 2x - 1$ |
| 11. $f(x) = \sqrt{x^3 + 1}$ | 12. $f(x) = \sqrt{x^2 + 1}$ |

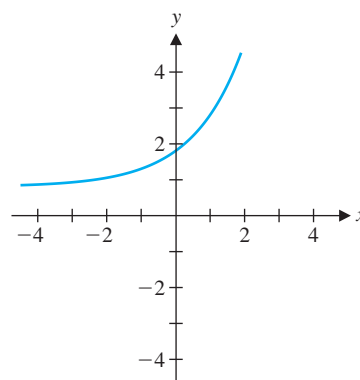
In exercises 13–18, assume that the function has an inverse. Without solving for the inverse, find the indicated function values.

13. $f(x) = x^3 + 4x - 1$, (a) $f^{-1}(-1)$, (b) $f^{-1}(4)$
14. $f(x) = x^3 + 2x + 1$, (a) $f^{-1}(1)$, (b) $f^{-1}(13)$
15. $f(x) = x^5 + 3x^3 + x$, (a) $f^{-1}(-5)$, (b) $f^{-1}(5)$

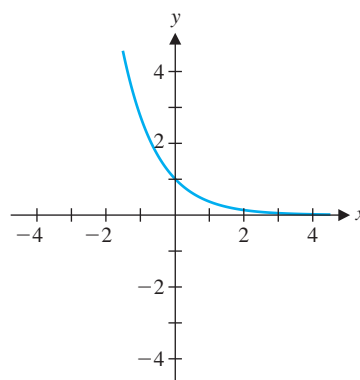
16. $f(x) = x^5 + 4x - 2$, (a) $f^{-1}(38)$, (b) $f^{-1}(3)$
17. $f(x) = \sqrt{x^3 + 2x + 4}$, (a) $f^{-1}(4)$, (b) $f^{-1}(2)$
18. $f(x) = \sqrt{x^5 + 4x^3 + 3x + 1}$, (a) $f^{-1}(3)$, (b) $f^{-1}(1)$

In exercises 19–22, use the given graph to graph the inverse function.

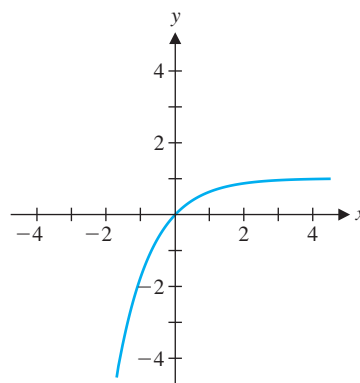
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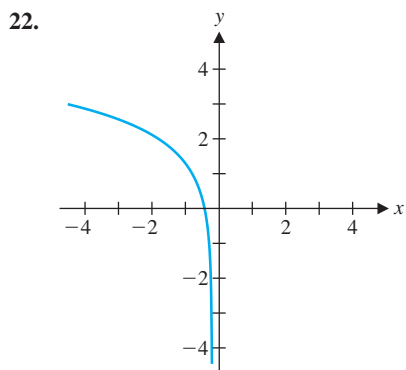


20.



21.





In exercises 23–26, use linear interpolation as in example 3.8 to estimate $f^{-1}(b)$. Use the apparent curving of the graph to conjecture whether the estimate is too high or too low.

23. $(1, 12), (2, 20), (3, 26), (4, 30), b = 23$
24. $(1, 12), (2, 10), (3, 6), (4, 0), b = 8$
25. $(1, 12), (2, 6), (3, 2), (4, 0), b = 5$
26. $(1, 12), (2, 20), (3, 36), (4, 50), b = 32$

T In exercises 27–36, use a graph to determine if the function is one-to-one. If it is, graph the inverse function.

27. $f(x) = x^3 - 5$
28. $f(x) = x^2 - 3$
29. $f(x) = x^3 + 2x - 1$
30. $f(x) = x^3 - 2x - 1$
31. $f(x) = x^5 - 3x^3 - 1$
32. $f(x) = x^5 + 4x^3 - 2$
33. $f(x) = \frac{1}{x+1}$
34. $f(x) = \frac{4}{x^2+1}$
35. $f(x) = \frac{x}{x+4}$
36. $f(x) = \frac{x}{\sqrt{x^2+4}}$

Exercises 37–46 involve inverse functions on restricted domains.

37. Show that $f(x) = x^2 (x \geq 0)$ and $g(x) = \sqrt{x} (x \geq 0)$ are inverse functions. Graph both functions.
38. Show that $f(x) = x^2 - 1 (x \geq 0)$ and $g(x) = \sqrt{x-1} (x \geq -1)$ are inverse functions. Graph both functions.
39. Graph $f(x) = x^2$ for $x \leq 0$ and verify that it is one-to-one. Find its inverse. Graph both functions.

40. Graph $f(x) = x^2 + 2$ for $x \leq 0$ and verify that it is one-to-one. Find its inverse. Graph both functions.
41. Graph $f(x) = (x-2)^2$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.
42. Graph $f(x) = (x+1)^4$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.
43. Graph $f(x) = \sqrt{x^2 - 2x}$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.
44. Graph $f(x) = \frac{x}{x^2 - 4}$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.
45. Graph $f(x) = \sin x$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.
46. Graph $f(x) = \cos x$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.

In exercises 47–52, discuss whether or not the function described has an inverse.

47. The income of a company varies with time.
48. The height of a person varies with time.
49. For a dropped ball, its height varies with time.
50. For a ball thrown upward, its height varies with time.
51. The shadow made by an object depends on its three-dimensional shape.
52. The number of calories burned depends on how fast a person runs.
53. Suppose that your boss informs you that you have been awarded a 10% raise. The next week, your boss announces that due to circumstances beyond her control, all employees will have their salaries cut by 10%. Are you as well off now as you were two weeks ago? Show that increasing by 10% and decreasing by 10% are not inverse processes. Find the inverse for adding 10%. (Hint: To add 10% to a quantity you can multiply by 1.10.)

EXPLORATORY EXERCISES

1. Find all values of k such that $f(x) = x^3 + kx + 1$ is one-to-one.
2. Find all values of k such that $f(x) = x^3 + 2x^2 + kx - 1$ is one-to-one.

0.4 TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

Many phenomena encountered in your daily life are **waves**. For instance, music is transmitted from radio stations in the form of electromagnetic waves. Your radio receiver decodes these electromagnetic waves and causes a thin membrane inside the speakers to vibrate. This vibration, in turn, creates pressure waves in the air. When these waves reach your ears, you hear the music from your radio (see Figure 0.48). Each of these waves is **periodic**, meaning that the basic shape of the wave is repeated over and over again. The mathematical description of such phenomena involves periodic functions, the most familiar of which are the trigonometric functions. First, we remind you of a basic definition.

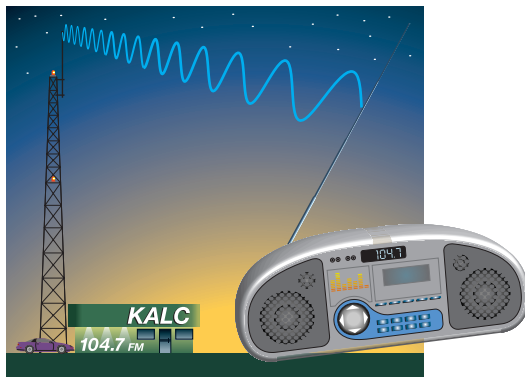


FIGURE 0.48
Radio and sound waves.

NOTES

When we discuss the period of a function, we most often focus on the fundamental period.

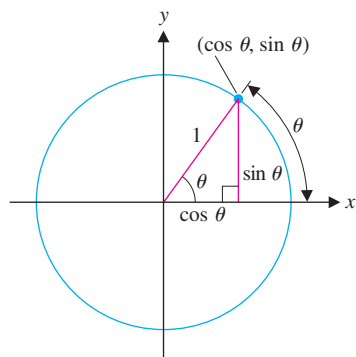


FIGURE 0.49
Definition of $\sin \theta$ and $\cos \theta$: $\cos \theta = x$ and $\sin \theta = y$.

DEFINITION 4.1

A function f is *periodic* of **period** T if

$$f(x + T) = f(x)$$

for all x such that x and $x + T$ are in the domain of f . The smallest such number $T > 0$ is called the **fundamental period**.

There are several equivalent ways of defining the sine and cosine functions. Regardless of how you were originally taught these functions, we want to emphasize a simple definition from which you can easily reproduce many of the basic properties of these functions. Referring to Figure 0.49, begin by drawing the unit circle $x^2 + y^2 = 1$. Let θ be the angle measured (counterclockwise) from the positive x -axis to the line segment joining the point (x, y) on the circle and the origin. Here, we measure θ in **radians**, where the radian measure of the angle θ is the length of the arc indicated in the figure. Again referring to Figure 0.49, we define $\sin \theta$ to be the y -coordinate of the point on the circle and $\cos \theta$ to be the x -coordinate of the point. Notice that from this definition, it follows that $\sin \theta$ and $\cos \theta$ are defined for all values of θ . That is, the domains of $f(\theta) = \sin \theta$ and of $g(\theta) = \cos \theta$ are both $-\infty < \theta < \infty$ (the entire real line). Observe that the range for each of these functions is the interval $[-1, 1]$.

Note that since the circumference of a circle ($C = 2\pi r$) of radius 1 is 2π , we have that 360° corresponds to 2π radians. Similarly, 180° corresponds to π radians, 90° corresponds to $\pi/2$ radians, and so on. In the table that follows, we list some common angles as measured in degrees together with the corresponding radian measure.

REMARK 4.1

Unless otherwise noted, we always measure angles in radians.

Angle in degrees	0°	30°	45°	60°	90°	135°	180°	270°	360°
Angle in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π

THEOREM 4.1

The functions $f(\theta) = \sin \theta$ and $g(\theta) = \cos \theta$ are periodic, of period 2π .

PROOF

Since a complete circle is 2π radians, adding 2π to any angle takes you all the way around the circle and back to the same point (x, y) . This says that

$$\sin(\theta + 2\pi) = \sin \theta$$

and

$$\cos(\theta + 2\pi) = \cos \theta,$$

for all values of θ . Furthermore, 2π is the smallest angle for which this is true. ■

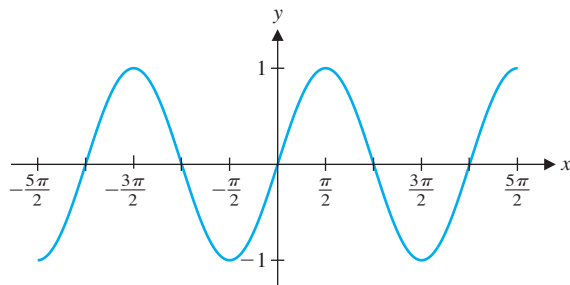


FIGURE 0.50a
 $y = \sin x$.

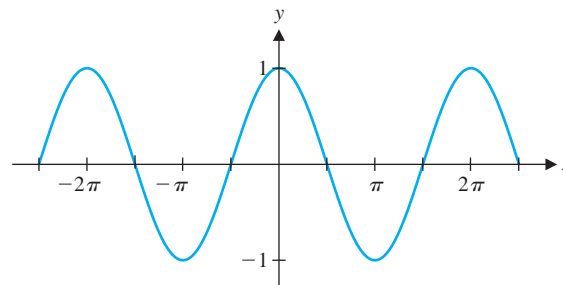


FIGURE 0.50b
 $y = \cos x$.

x	$\sin x$	$\cos x$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
π	0	-1
$\frac{3\pi}{2}$	-1	0
2π	0	1

You are likely already familiar with the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ shown in Figures 0.50a and 0.50b, respectively.

Notice that you could slide the graph of $y = \sin x$ slightly to the left or right and get an exact copy of the graph of $y = \cos x$. Specifically, you should observe that

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x.$$

REMARK 4.2

Instead of writing $(\sin \theta)^2$ or $(\cos \theta)^2$, we usually use the notation $\sin^2 \theta$ and $\cos^2 \theta$, respectively.

The accompanying table lists some common values of sine and cosine. Notice that many of these can be found by closely examining Figure 0.49.

EXAMPLE 4.1 Solving Equations Involving Sines and Cosines

Find all solutions of the equations (a) $2 \sin x - 1 = 0$ and (b) $\cos^2 x - 3 \cos x + 2 = 0$.

Solution For (a), notice that $2 \sin x - 1 = 0$ if $2 \sin x = 1$ or $\sin x = \frac{1}{2}$. From the unit circle, we find that $\sin x = \frac{1}{2}$ if $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$. Since $\sin x$ has period 2π , additional solutions are $\frac{\pi}{6} + 2\pi$, $\frac{5\pi}{6} + 2\pi$, $\frac{\pi}{6} + 4\pi$ and so on. A convenient way of indicating that any integer multiple of 2π can be added to either solution is to write: $x = \frac{\pi}{6} + 2n\pi$ or $x = \frac{5\pi}{6} + 2n\pi$, for any integer n . Part (b) may look rather difficult at first. However, notice that it looks like a quadratic equation using $\cos x$ instead of x . With this clue, you can factor the left-hand side to get

$$0 = \cos^2 x - 3 \cos x + 2 = (\cos x - 1)(\cos x - 2),$$

from which it follows that either $\cos x = 1$ or $\cos x = 2$. Since $-1 \leq \cos x \leq 1$ for all x , the equation $\cos x = 2$ has no solution. However, we get $\cos x = 1$ if $x = 0, 2\pi$ or any integer multiple of 2π . Notice that we can summarize all the solutions by writing $x = 2n\pi$, for any integer n . ■

We now give definitions of the remaining four trigonometric functions.

DEFINITION 4.2

The **tangent** function is defined by $\tan x = \frac{\sin x}{\cos x}$.

The **cotangent** function is defined by $\cot x = \frac{\cos x}{\sin x}$.

The **secant** function is defined by $\sec x = \frac{1}{\cos x}$.

The **cosecant** function is defined by $\csc x = \frac{1}{\sin x}$.

We give graphs of these functions in Figures 0.51a, 0.51b, 0.51c and 0.51d. Notice in each graph the locations of the vertical asymptotes. For the “co” functions $\cot x$ and $\csc x$, the division by $\sin x$ causes vertical asymptotes at $0, \pi, 2\pi$ and so on (where $\sin x = 0$). For $\tan x$ and $\sec x$, the division by $\cos x$ produces vertical asymptotes at $\pi/2, 3\pi/2, 5\pi/2$ and so on (where $\cos x = 0$). Once you have the vertical asymptotes in place, the graphs are relatively easy to draw.

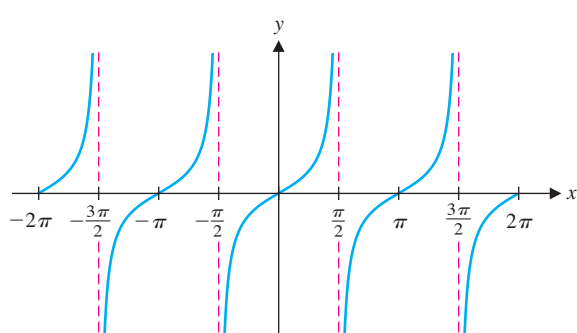


FIGURE 0.51a
 $y = \tan x$.

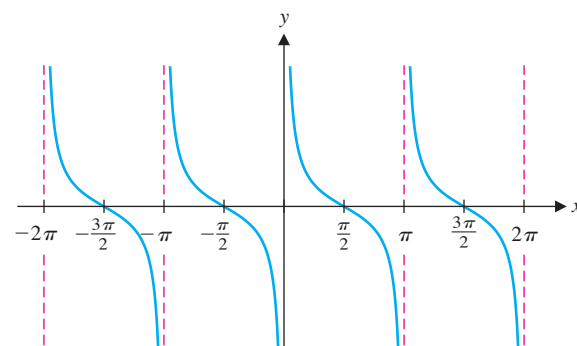


FIGURE 0.51b
 $y = \cot x$.

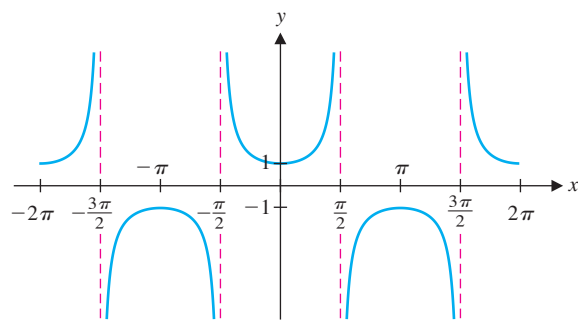


FIGURE 0.51c
 $y = \sec x$.

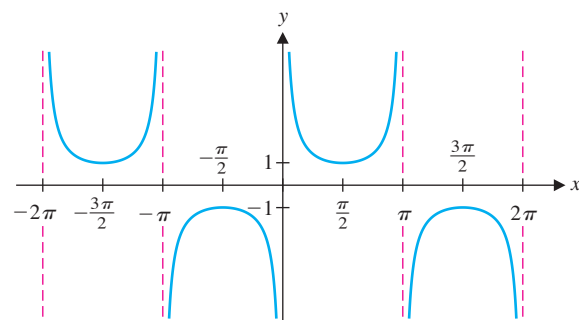


FIGURE 0.51d
 $y = \csc x$.

Notice that $\tan x$ and $\cot x$ are periodic, of period π , while $\sec x$ and $\csc x$ are periodic, of period 2π .

REMARK 4.3

Most calculators have keys for the functions $\sin x$, $\cos x$, and $\tan x$, but not for the other three trigonometric functions. This reflects the central role that $\sin x$, $\cos x$ and $\tan x$ play in applications. To calculate function values for the other three trigonometric functions, you can simply use the identities

$$\cot x = \frac{1}{\tan x}, \quad \sec x = \frac{1}{\cos x} \quad \text{and} \quad \csc x = \frac{1}{\sin x}.$$

It is important to learn the effect of slight modifications of these functions. We present a few ideas here and in the exercises.

EXAMPLE 4.2 Altering Amplitude and Period

Graph $y = 2 \sin x$ and $y = \sin 2x$ and describe how each differs from the graph of $y = \sin x$ (see Figure 0.52a).

Solution The graph of $y = 2 \sin x$ is given in Figure 0.52b. Notice that this graph is similar to the graph of $y = \sin x$, except that the y -values oscillate between -2 and 2

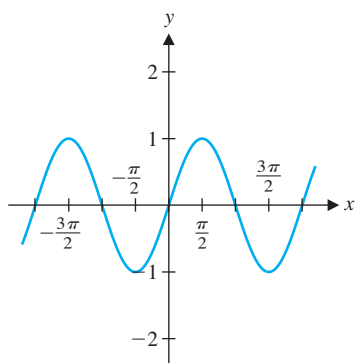


FIGURE 0.52a
 $y = \sin x$.

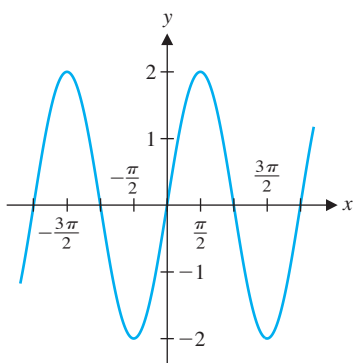


FIGURE 0.52b
 $y = 2 \sin x$.

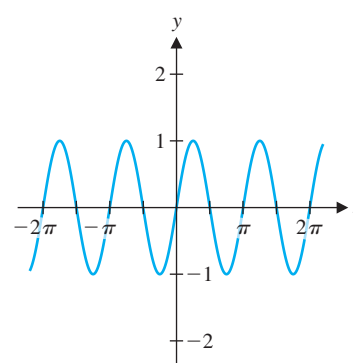


FIGURE 0.52c
 $y = \sin(2x)$.

instead of -1 and 1 . Next, the graph of $y = \sin 2x$ is given in Figure 0.52c. In this case, the graph is similar to the graph of $y = \sin x$, except that the period is π instead of 2π . (The oscillations occur twice as fast.)

The results in example 4.2 can be generalized. For $A > 0$, the graph of $y = A \sin x$ oscillates between $y = -A$ and $y = A$. In this case, we call A the **amplitude** of the sine curve. Notice that for any positive constant c , the period of $y = \sin cx$ is $2\pi/c$. Similarly, for the function $A \cos cx$, the amplitude is A and the period is $2\pi/c$.

The sine and cosine functions can be used to model sound waves. A pure tone (think of a single flute note) is a pressure wave described by the sinusoidal function $A \sin ct$. (Here, we are using the variable t , since the air pressure is a function of *time*.) The amplitude A determines how loud the tone is perceived to be, and the period determines the pitch of the note. In this setting, it is usually convenient to talk about the **frequency** $f = c/2\pi$ rather than the period. As you may know, the higher the frequency is, the higher the pitch of the note will be. (Frequency is measured in hertz, where 1 hertz equals 1 cycle per second.) Note that the frequency is simply the reciprocal of the period.

EXAMPLE 4.3 Finding Amplitude, Period, and Frequency

Find the amplitude, period, and frequency of (a) $f(x) = 4 \cos 3x$ and (b) $g(x) = 2 \sin(x/3)$.

Solution (a) For $f(x)$, the amplitude is 4, the period is $2\pi/3$ and the frequency is $3/(2\pi)$ (see Figure 0.53a). (b) For $g(x)$, the amplitude is 2, the period is $2\pi/(1/3) = 6\pi$ and the frequency is $1/(6\pi)$ (see Figure 0.53b).

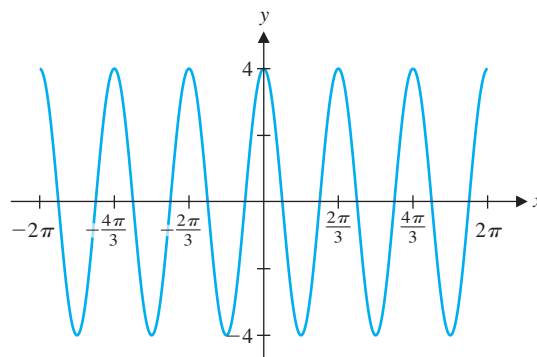


FIGURE 0.53a
 $y = 4 \cos 3x$.

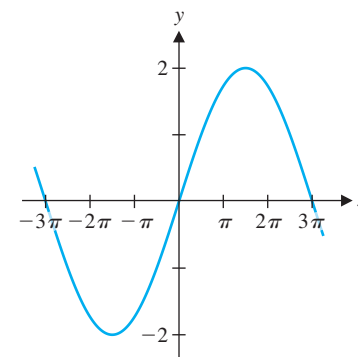


FIGURE 0.53b
 $y = 2 \sin(x/3)$.

As you will recall from your earlier study of trigonometry, there are numerous formulas, or **identities**, that are helpful in manipulating the trigonometric functions. You should observe from the definition of $\sin \theta$ and $\cos \theta$ (see Figure 0.49) that the Pythagorean Theorem gives us the familiar identity

$$\sin^2 \theta + \cos^2 \theta = 1,$$

since the hypotenuse of the indicated triangle is 1. This holds for any angle θ .

We list several important identities in Theorem 4.2.

THEOREM 4.2

For any real numbers α and β , the following identities hold:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha \quad (4.1)$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad (4.2)$$

$$\sin^2 \alpha = \frac{1}{2}(1 - \cos 2\alpha) \quad (4.3)$$

$$\cos^2 \alpha = \frac{1}{2}(1 + \cos 2\alpha). \quad (4.4)$$

From the basic identities summarized in Theorem 4.2, numerous other useful identities can be derived. We derive two of these in example 4.4.

EXAMPLE 4.4 Deriving New Trigonometric Identities

Derive the identities $\sin 2\theta = 2 \sin \theta \cos \theta$ and $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$.

Solution These can be obtained from formulas (4.1) and (4.2), respectively, by substituting $\alpha = \theta$ and $\beta = \theta$. Alternatively, the identity for $\cos 2\theta$ can be obtained by subtracting equation (4.3) from equation (4.4). ■

□ **The Inverse Trigonometric Functions**

We can expand the set of functions available to you by defining inverses to the trigonometric functions. To get started, let's again look at a graph of $y = \sin x$ (see Figure 0.50a). Notice that we cannot define an inverse function, since $\sin x$ is not one-to-one. Although the sine function as we have defined it does not have an inverse function, we can define one by modifying the domain of the sine. We do this by choosing a portion of the sine curve that passes the horizontal line test. In particular, we select an interval of x -values on which the sine increases from -1 to 1 , without repeating any of those values. If we restrict the domain to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then $y = \sin x$ is one-to-one there (see Figure 0.54) and hence, has an inverse. We thus define the **inverse sine** function by

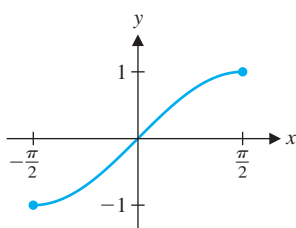


FIGURE 0.54
 $y = \sin x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$$y = \sin^{-1} x \quad \text{if and only if} \quad \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}. \quad (4.5)$$

It is convenient to think of this definition as follows. If $y = \sin^{-1} x$, then y is the angle (between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$) for which $\sin y = x$. You should note that we could have selected any interval on which $\sin x$ is one-to-one, but $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is the most convenient. To see that these are indeed inverse functions, you should observe that

$$\sin(\sin^{-1} x) = x, \quad \text{for all } x \in [-1, 1]$$

and

$$\sin^{-1}(\sin x) = x, \quad \text{for all } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \quad (4.6)$$

Read equation (4.6) very carefully. It *does not* say that $\sin^{-1}(\sin x) = x$ for *all* x , but rather, *only* for those in the restricted domain, $[-\frac{\pi}{2}, \frac{\pi}{2}]$. For instance, $\sin^{-1}(\sin \pi) \neq \pi$, since

$$\sin^{-1}(\sin \pi) = \sin^{-1}(0) = 0.$$

REMARK 4.4

Mathematicians often use the notation **arcsin** x in place of $\sin^{-1} x$. People will read $\sin^{-1} x$ interchangeably as “inverse sine of x ” or “arcsine of x .”

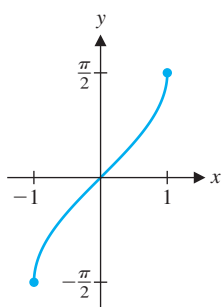


FIGURE 0.55
 $y = \sin^{-1} x$.

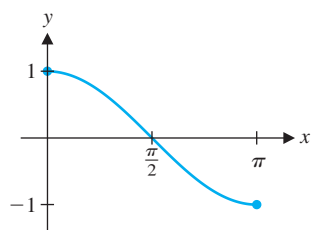


FIGURE 0.56
 $y = \cos x$ on $[0, \pi]$.

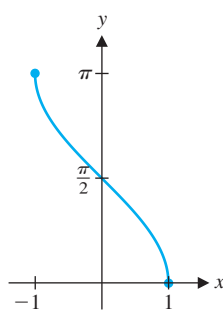


FIGURE 0.57
 $y = \cos^{-1} x$.

EXAMPLE 4.5 Evaluating the Inverse Sine Function

Evaluate (a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and (b) $\sin^{-1}\left(-\frac{1}{2}\right)$.

Solution For (a), we look for the angle θ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ for which $\sin \theta = \frac{\sqrt{3}}{2}$. Note that since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we have that $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$. For (b), note that $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ and $-\frac{\pi}{6} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Thus,

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}.$$

Judging by the preceding two examples, you might think that (4.5) is a roundabout way of defining a function. If so, you've got the idea exactly. In fact, we want to emphasize that what we know about the inverse sine function is principally through reference to the sine function. We will not have any other definition of arcsine, nor are there any algebraic formulas for this function. (These things are true of most inverse functions.) Further, you should recall from our discussion in section 0.3 that we can draw a graph of $y = \sin^{-1} x$ simply by reflecting the graph of $y = \sin x$ on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (from Figure 0.54) through the line $y = x$ (see Figure 0.55).

Turning to $y = \cos x$, can you think of how to restrict the domain to make the function one-to-one? Notice that restricting the domain to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, as we did for the inverse sine function will not work here. (Why not?) The simplest way to do this is to restrict its domain to the interval $[0, \pi]$ (see Figure 0.56). Consequently, we define the **inverse cosine** function by

$$y = \cos^{-1} x \quad \text{if and only if } \cos y = x \text{ and } 0 \leq y \leq \pi.$$

Note that here, we have

$$\cos(\cos^{-1} x) = x, \quad \text{for all } x \in [-1, 1]$$

and

$$\cos^{-1}(\cos x) = x, \quad \text{for all } x \in [0, \pi].$$

As with the definition of arcsine, it is helpful to think of $\cos^{-1} x$ as that angle θ in $[0, \pi]$ for which $\cos \theta = x$. As with $\sin^{-1} x$, it is common to use $\cos^{-1} x$ and $\arccos x$ interchangeably.

EXAMPLE 4.6 Evaluating the Inverse Cosine Function

Evaluate (a) $\cos^{-1}(0)$ and (b) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$.

Solution For (a), you will need to find that angle θ in $[0, \pi]$ for which $\cos \theta = 0$. It's not hard to see that $\cos^{-1}(0) = \frac{\pi}{2}$. If you calculate this on your calculator and get 90, your calculator is in degrees mode. In this event, you should immediately change it to radians mode. For (b), look for the angle $\theta \in [0, \pi]$ for which $\cos \theta = -\frac{\sqrt{2}}{2}$. Notice that $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\frac{3\pi}{4} \in [0, \pi]$. Consequently,

$$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}.$$

Once again, we obtain the graph of this inverse function by reflecting the graph of $y = \cos x$ on the interval $[0, \pi]$ (seen in Figure 0.56) through the line $y = x$ (see Figure 0.57).

We can define inverses for each of the four remaining trigonometric functions in similar ways. For $y = \tan x$, we restrict the domain to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Think about why the endpoints of this interval are not included (see Figure 0.58). Having done this, you should readily see that we define the **inverse tangent** function by

$$y = \tan^{-1} x \quad \text{if and only if } \tan y = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

The graph of $y = \tan^{-1} x$ is then as seen in Figure 0.59, found by reflecting the graph in Figure 0.58 through the line $y = x$.

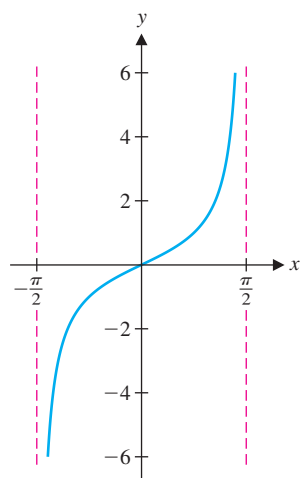


FIGURE 0.58
 $y = \tan x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

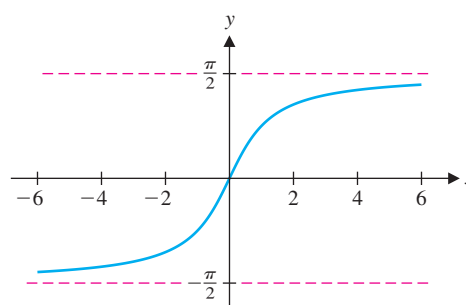


FIGURE 0.59
 $y = \tan^{-1} x$.

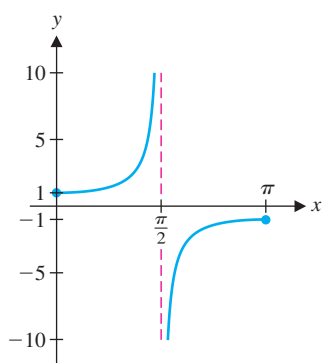


FIGURE 0.60
 $y = \sec x$ on $[0, \pi]$.

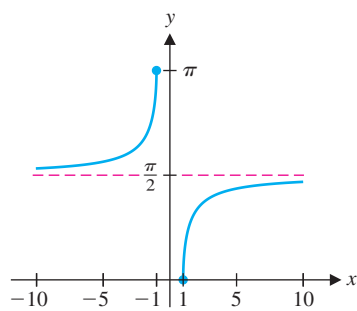


FIGURE 0.61
 $y = \sec^{-1} x$.

EXAMPLE 4.7 Evaluating an Inverse Tangent

Evaluate $\tan^{-1}(1)$.

Solution You must look for the angle, θ on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ for which $\tan \theta = 1$. This is easy enough. Since $\tan(\frac{\pi}{4}) = 1$ and $\frac{\pi}{4} \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we have that $\tan^{-1}(1) = \frac{\pi}{4}$. ■

We now turn to defining an inverse for $\sec x$. First, we must issue a disclaimer. As we have indicated, there are any number of ways to suitably restrict the domains of the trigonometric functions in order to make them one-to-one. With the first three we've seen, there has been an obvious choice of how to do this and there is general agreement among mathematicians on the choice of these intervals. In the case of $\sec x$, this is *not* true. There are several reasonable ways in which to suitably restrict the domain and different authors restrict these differently. We have (arbitrarily) chosen to restrict the domain to be $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$. You might initially think that this looks strange. Why not use all of $[0, \pi]$? You need only think about the definition of $\sec x$ to see why we needed to exclude the point $x = \frac{\pi}{2}$. See Figure 0.60 for a graph of $\sec x$ on this domain. (Note the vertical asymptote at $x = \frac{\pi}{2}$.) Consequently, we define the **inverse secant** function by

$$y = \sec^{-1} x \quad \text{if and only if } \sec y = x \text{ and } y \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi].$$

A graph of $\sec^{-1} x$ is shown in Figure 0.61.

REMARK 4.5

We can likewise define inverses to $\cot x$ and $\csc x$. As these functions are used only infrequently, we will omit them here and examine them in the exercises.

Function	Domain	Range
$\sin^{-1} x$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

EXAMPLE 4.8 Evaluating an Inverse Secant

Evaluate $\sec^{-1}(-\sqrt{2})$.

Solution You must look for the angle θ with $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, for which $\sec \theta = -\sqrt{2}$. Notice that if $\sec \theta = -\sqrt{2}$, then $\cos \theta = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$. Recall from example 4.6 that $\cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$. Further, the angle $\frac{3\pi}{4}$ is in the interval $(\frac{\pi}{2}, \pi]$ and so, $\sec^{-1}(-\sqrt{2}) = \frac{3\pi}{4}$.

Calculators do not usually have built-in functions for $\sec x$ or $\sec^{-1} x$. In this case, you must convert the desired secant value to a cosine value and use the inverse cosine function, as we did in example 4.8.

We summarize the three main inverse trigonometric functions in the margin.

In many applications, we need to calculate the length of one side of a right triangle using the length of another side and an **acute** angle (i.e., an angle between 0 and $\frac{\pi}{2}$ radians). We can do this rather easily, as in example 4.9.

EXAMPLE 4.9 Finding the Height of a Tower

A person 100 feet from the base of a water tower measures an angle of 60° from the ground to the top of the tower (see Figure 0.62). (a) Find the height of the tower. (b) What angle is measured if the person is 200 feet from the base?

Solution For (a), we first convert 60° to radians:

$$60^\circ = 60 \frac{\pi}{180} = \frac{\pi}{3} \text{ radians.}$$

We are given that the base of the triangle in Figure 0.62 is 100 feet. We must now compute the height of the tower h . Using the similar triangles indicated in Figure 0.62, we have that

$$\frac{\sin \theta}{\cos \theta} = \frac{h}{100},$$

so that the height of the tower is

$$h = 100 \frac{\sin \theta}{\cos \theta} = 100 \tan \theta = 100 \tan \frac{\pi}{3} = 100\sqrt{3} \approx 173 \text{ feet.}$$

For part (b), the similar triangles in Figure 0.62 give us

$$\tan \theta = \frac{h}{200} = \frac{100\sqrt{3}}{200} = \frac{\sqrt{3}}{2}.$$

Since $0 < \theta < \frac{\pi}{2}$, we have that

$$\theta = \tan^{-1} \left(\frac{\sqrt{3}}{2} \right) \approx 0.7137 \text{ radians (about 41 degrees).}$$

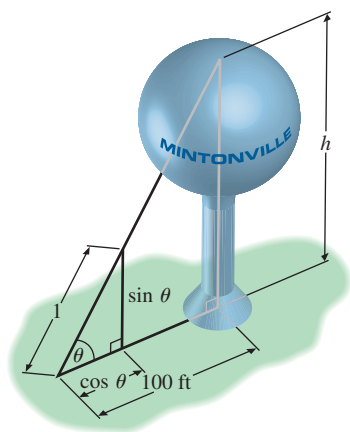


FIGURE 0.62
Height of a tower.

In example 4.10, we simplify expressions involving both trigonometric and inverse trigonometric functions.

EXAMPLE 4.10 Simplifying Expressions Involving Inverse Trigonometric Functions

Simplify (a) $\sin(\cos^{-1} x)$ and (b) $\tan(\cos^{-1} x)$.

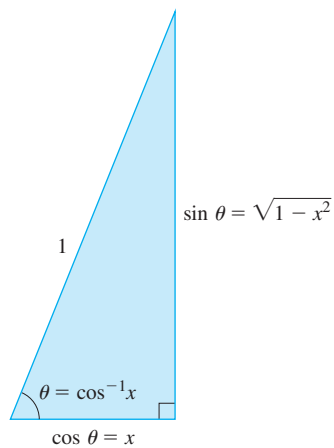


FIGURE 0.63
 $\theta = \cos^{-1}x$.

Solution Do not look for some arcane formula to help you out. *Think* first: $\cos^{-1}x$ is the angle (call it θ) for which $x = \cos \theta$. First, consider the case where $x > 0$. Looking at Figure 0.63, we have drawn a right triangle, with hypotenuse 1 and adjacent angle θ . From the definition of the sine and cosine, then, we have that the base of the triangle is $\cos \theta = x$ and the altitude is $\sin \theta$, which by the Pythagorean Theorem is

$$\sin(\cos^{-1}x) = \sin \theta = \sqrt{1-x^2}.$$

Wait! We have not yet finished part (a). Figure 0.63 shows $0 < \theta < \frac{\pi}{2}$, but by definition, $\theta = \cos^{-1}x$ could range from 0 to π . Does our answer change if $\frac{\pi}{2} < \theta < \pi$? To see that it doesn't change, note that if $0 \leq \theta \leq \pi$, then $\sin \theta \geq 0$. From the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$, we get

$$\sin \theta = \pm \sqrt{1 - \cos^2 \theta} = \pm \sqrt{1 - x^2}.$$

Since $\sin \theta \geq 0$, we must have

$$\sin \theta = \sqrt{1 - x^2},$$

for *all* values of x . For part (b), you can read from Figure 0.63 that

$$\tan(\cos^{-1}x) = \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\sqrt{1-x^2}}{x}.$$

Note that this last identity is valid, regardless of whether $x = \cos \theta$ is positive or negative. ■

EXERCISES 0.4

WRITING EXERCISES

- Many students are comfortable using degrees to measure angles and don't understand why they must learn radian measures. As discussed in the text, radians directly measure distance along the unit circle. Distance is an important aspect of many applications. In addition, we will see later that many calculus formulas are simpler in radians form than in degrees. Aside from familiarity, discuss any and all advantages of degrees over radians. On balance, which is better?
- A student graphs $f(x) = \cos x$ on a graphing calculator and gets what appears to be a straight line at height $y = 1$ instead of the usual cosine curve. Upon investigation, you discover that the calculator has graphing window $-10 \leq x \leq 10$, $-10 \leq y \leq 10$ and is in degrees mode. Explain what went wrong and how to correct it.
- Inverse functions are necessary for solving equations. The restricted range we had to use to define inverses of the trigonometric functions also restricts their usefulness in equation-solving. Explain how to use $\sin^{-1}x$ to find all solutions of the equation $\sin u = x$.
- Discuss how to compute $\sec^{-1}x$, $\csc^{-1}x$ and $\cot^{-1}x$ on a calculator that only has built-in functions for $\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$.
- In example 4.3, $f(x) = 4 \cos 3x$ has period $2\pi/3$ and $g(x) = 2 \sin(x/3)$ has period 6π . Explain why the sum $h(x) = 4 \cos 3x + 2 \sin(x/3)$ has period 6π .
- Give a different range for $\sec^{-1}x$ than that given in the text. For which x 's would the value of $\sec^{-1}x$ change? Using the calculator discussion in exercise 4, give one reason why we might have chosen the range that we did.

In exercises 1 and 2, convert the given radians measure to degrees.

- (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{4\pi}{3}$
- (a) $\frac{3\pi}{5}$ (b) $\frac{\pi}{7}$ (c) 2 (d) 3

In exercises 3 and 4, convert the given degrees measure to radians.

- (a) 180° (b) 270° (c) 120° (d) 30°
- (a) 40° (b) 80° (c) 450° (d) 390°

In exercises 5–14, find all solutions of the given equation.

- $2 \cos x - 1 = 0$
- $2 \sin x + 1 = 0$
- $\sqrt{2} \cos x - 1 = 0$
- $2 \sin x - \sqrt{3} = 0$

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9. $\sin^2 x - 4 \sin x + 3 = 0$ 10. $\sin^2 x - 2 \sin x - 3 = 0$
 11. $\sin^2 x + \cos x - 1 = 0$ 12. $\sin 2x - \cos x = 0$
 13. $\cos^2 x + \cos x = 0$ 14. $\sin^2 x - \sin x = 0$

T In exercises 15–24, sketch a graph of the function.

15. $f(x) = \sin 2x$ 16. $f(x) = \cos 3x$
 17. $f(x) = \tan 2x$ 18. $f(x) = \sec 3x$
 19. $f(x) = 3 \cos(x - \pi/2)$ 20. $f(x) = 4 \cos(x + \pi)$
 21. $f(x) = \sin 2x - 2 \cos 2x$ 22. $f(x) = \cos 3x - \sin 3x$
 23. $f(x) = \sin x \sin 12x$ 24. $f(x) = \sin x \cos 12x$

In exercises 25–32, identify the amplitude, period, and frequency.

25. $f(x) = 3 \sin 2x$ 26. $f(x) = 2 \cos 3x$
 27. $f(x) = 5 \cos 3x$ 28. $f(x) = 3 \sin 5x$
 29. $f(x) = 3 \cos(2x - \pi/2)$ 30. $f(x) = 4 \sin(3x + \pi)$
 31. $f(x) = -4 \sin x$ 32. $f(x) = -2 \cos 3x$

In exercises 33–36, prove that the given trigonometric identity is true.

33. $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \sin \beta \cos \alpha$
 34. $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$
 35. (a) $\cos(2\theta) = 2 \cos^2 \theta - 1$
 (b) $\cos(2\theta) = 1 - 2 \sin^2 \theta$
 36. (a) $\sec^2 \theta = \tan^2 \theta + 1$
 (b) $\csc^2 \theta = \cot^2 \theta + 1$

In exercises 37–46, evaluate the inverse function by sketching a unit circle and locating the correct angle on the circle.

37. $\cos^{-1} 0$ 38. $\tan^{-1} 0$
 39. $\sin^{-1}(-1)$ 40. $\cos^{-1}(1)$
 41. $\sec^{-1} 1$ 42. $\tan^{-1}(-1)$
 43. $\sec^{-1} 2$ 44. $\csc^{-1} 2$
 45. $\cot^{-1} 1$ 46. $\tan^{-1} \sqrt{3}$

47. Prove that, for some constant β ,

$$4 \cos x - 3 \sin x = 5 \cos(x + \beta).$$

Then, estimate the value of β .

48. Prove that, for some constant β ,

$$2 \sin x + \cos x = \sqrt{5} \sin(x + \beta).$$

Then, estimate the value of β .

In exercises 49–52, determine whether or not the function is periodic. If it is periodic, find the smallest (fundamental) period.

49. $f(x) = \cos 2x + 3 \sin \pi x$
 50. $f(x) = \sin x - \cos \sqrt{2}x$
 51. $f(x) = \sin 2x - \cos 5x$
 52. $f(x) = \cos 3x - \sin 7x$

In exercises 53–56, use the range for θ to determine the indicated function value.

53. $\sin \theta = \frac{1}{3}$, $0 \leq \theta \leq \frac{\pi}{2}$, find $\cos \theta$.
 54. $\cos \theta = \frac{4}{5}$, $0 \leq \theta \leq \frac{\pi}{2}$, find $\sin \theta$.
 55. $\sin \theta = \frac{1}{2}$, $\frac{\pi}{2} \leq \theta \leq \pi$, find $\cos \theta$.
 56. $\sin \theta = \frac{1}{2}$, $\frac{\pi}{2} \leq \theta \leq \pi$, find $\tan \theta$.

In exercises 57–64, use a triangle to simplify each expression. Where applicable, state the range of x 's for which the simplification holds.

57. $\cos(\sin^{-1} x)$ 58. $\cos(\tan^{-1} x)$
 59. $\tan(\sec^{-1} x)$ 60. $\cot(\cos^{-1} x)$
 61. $\sin(\cos^{-1} \frac{1}{2})$ 62. $\cos(\sin^{-1} \frac{1}{2})$
 63. $\tan(\cos^{-1} \frac{3}{5})$ 64. $\csc(\sin^{-1} \frac{2}{3})$

T In exercises 65–68, use a graphing calculator or computer to determine the number of solutions of each equation and numerically estimate the solutions (x is in radians).

65. $2 \cos x = 2 - x$ 66. $3 \sin x = x$
 67. $\cos x = x^2 - 2$ 68. $\sin x = x^2$

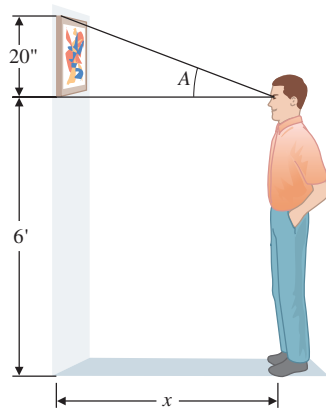
69. A person sitting 2 miles from a rocket launch site measures 20° up to the current location of the rocket. How high up is the rocket?

70. A person who is 6 feet tall stands 4 feet from the base of a light pole and casts a 2-foot-long shadow. How tall is the light pole?

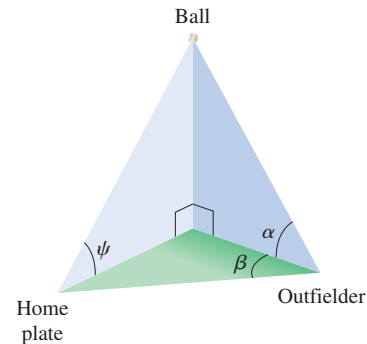
71. A surveyor stands 80 feet from the base of a building and measures an angle of 50° to the top of the steeple on top of the building. The surveyor figures that the center of the steeple lies 20 feet inside the front of the structure. Find the height of the steeple.

T 72. Suppose that the surveyor of exercise 71 estimates that the center of the steeple lies between 20' and 21' inside the front of the structure. Determine how much the extra foot would change the calculation of the height of the building.

T 73. A picture hanging in an art gallery has a frame 20 inches high and the bottom of the frame is 6 feet above the floor. A person who is 6 feet tall stands x feet from the wall. Let A be the angle formed by the ray from the person's eye to the bottom of the frame and the ray from the person's eye to the top of the frame. Write A as a function of x and graph $y = A(x)$.

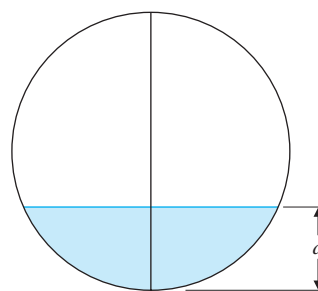


74. In golf, the goal is to hit a ball into a hole of diameter 4.5 inches. Suppose a golfer stands x feet from the hole trying to putt the ball into the hole. A first approximation of the margin of error in a putt is to measure the angle A formed by the ray from the ball to the right edge of the hole and the ray from the ball to the left edge of the hole. Find A as a function of x .
75. In an AC circuit, the voltage is given by $v(t) = v_p \sin 2\pi f t$, where v_p is the peak voltage and f is the frequency in Hz. A voltmeter actually measures an average (called the **root-mean-square**) voltage, equal to $v_p/\sqrt{2}$. If the voltage has amplitude 170 and period $\pi/30$, find the frequency and meter voltage.
76. An old-fashioned LP record player rotated records at $33\frac{1}{3}$ rpm (revolutions per minute). What is the period (in minutes) of the rotation? What is the period for a 45 rpm single?
77. Suppose that the ticket sales of an airline (in thousands of dollars) is given by $s(t) = 110 + 2t + 15 \sin(\frac{1}{6}\pi t)$, where t is measured in months. What real-world phenomenon might cause the fluctuation in ticket sales modeled by the sine term? Based on your answer, what month corresponds to $t = 0$? Disregarding seasonal fluctuations, by what amount is the airline's sales increasing annually?
- T** 78. Piano tuners sometimes start by striking a tuning fork and then the corresponding piano key. If the tuning fork and piano note each have frequency 8, then the resulting sound is $\sin 8t + \sin 8t$. Graph this. If the piano is slightly out-of-tune at frequency 8.1, the resulting sound is $\sin 8t + \sin 8.1t$. Graph this and explain how the piano tuner can hear the small difference in frequency.
79. Give precise definitions of $\csc^{-1} x$ and $\cot^{-1} x$.
80. In baseball, outfielders are able to easily track down and catch fly balls that have very long and high trajectories. Physicists have argued for years about how this is done. A recent explanation involves the following geometry.
 The player can catch the ball by running to keep the angle ψ constant (this makes it appear that the ball is moving in a straight line). Assuming that all triangles shown are right triangles, show that $\tan \psi = \frac{\tan \alpha}{\tan \beta}$ and then solve for ψ .

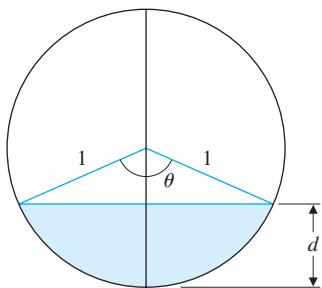


EXPLORATORY EXERCISES

- In his book and video series *The Ring of Truth*, physicist Philip Morrison performed an experiment to estimate the circumference of the earth. In Nebraska, he measured the angle to a bright star in the sky, drove 370 miles due south into Kansas, and measured the new angle to the star. Some geometry shows that the difference in angles, about 5.02° , equals the angle from the center of the earth to the two locations in Nebraska and Kansas. If the earth is perfectly spherical (it's not) and the circumference of the portion of the circle measured out by 5.02° is 370 miles, estimate the circumference of the earth. This experiment was based on a similar experiment by the ancient Greek scientist Eratosthenes. The ancient Greeks and the Spaniards of Columbus' day knew that the earth was round; they just disagreed about the circumference. Columbus argued for a figure about half of the actual value, since a ship couldn't survive on the water long enough to navigate the true distance.
- An oil tank with circular cross sections lies on its side. A stick is inserted in a hole at the top and used to measure the depth d of oil in the tank. Based on this measurement, the goal is to compute the percentage of oil left in the tank.



To simplify calculations, suppose the circle is a unit circle with center at $(0, 0)$. Sketch radii extending from the origin to the top of the oil. The area of oil at the bottom equals the area of the portion of the circle bounded by the radii minus the area of the triangle formed above.



Start with the triangle, which has area one-half base times height. Explain why the height is $1 - d$. Find a right triangle in the figure (there are two of them) with hypotenuse 1 (the radius of the circle) and one vertical side of length $1 - d$. The horizontal side has length equal to one-half the base of the larger triangle. Show that this equals $\sqrt{1 - (1 - d)^2}$. The area of the portion of the circle equals $\pi\theta/2\pi = \theta/2$, where θ is the angle at the top of the triangle. Find this angle as a function of d . (Hint: Go back to the right triangle used above with upper angle $\theta/2$.) Then find the area filled with oil and divide by π to get the portion of the tank filled with oil.

- T** 3. Computer graphics can be misleading. This exercise works best using a “disconnected” graph (individual dots, not connected). Graph $y = \sin x^2$ using a graphing window for which each pixel represents a step of 0.1 in the x - or y -direction. You should get the impression of a sine wave that oscillates more and more rapidly as you move to the left and right. Next, change the graphing window so that the middle of the original screen (probably $x = 0$) is at the far left of the new screen. You will probably see what appears to be a random jumble of dots. Continue to change the graphing window by increasing the x -values. Describe the patterns or lack of patterns that you see. You should find one pattern that looks like two rows of dots across the top and bottom of the screen; another pattern looks like the original sine wave. For each pattern that you find, pick adjacent points with x -coordinates a and b . Then change the graphing window so that $a \leq x \leq b$ and find the portion of the graph that is missing. Remember that, whether the points are connected or not, computer graphs always leave out part of the graph; it is part of your job to know whether the missing part is important or not.

0.5 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Some bacteria reproduce very quickly, as you may have discovered if you have ever had an infected cut or strep throat. Under the right circumstances, the number of bacteria in certain cultures will double in as little as an hour (or even less). In this section, we discuss some functions that can be used to model such rapid growth.

For instance, suppose that initially there are 100 bacteria at a given site, and the population doubles every hour. Call the population function $P(t)$, where t represents time (in hours) and start the clock running at time $t = 0$. The initial population of 100 means that $P(0) = 100$. After 1 hour, the population will double to 200, so that $P(1) = 200$. After another hour, the population will have doubled again to 400, making $P(2) = 400$. After a third hour, the population will have doubled again to 800, making $P(3) = 800$ and so on.

Let's compute the population after 10 hours. (Try guessing this now. Most people guess poorly on this type of problem.) You could calculate the population at 4 hours, 5 hours, and so on, or you could use the following shortcut. To find $P(1)$, you double the initial population, so that $P(1) = 2 \cdot 100$. To find $P(2)$, you double the population at time $t = 1$, so that $P(2) = 2 \cdot 2 \cdot 100 = 2^2 \cdot 100$. Similarly, $P(3) = 2^3 \cdot 100$. This pattern leads us to

$$P(10) = 2^{10} \cdot 100 = 102,400.$$

Depending on the organism, this is now a population that could cause some trouble!

The pattern just discovered suggests that the population can be modeled by the function

$$P(t) = 2^t \cdot 100.$$

We call $P(t)$ an **exponential** function because the variable t is in the exponent. There is a subtle question here: what is the domain of this function? We have so far used only integer values of t , but for what other values of t does $P(t)$ make sense? Certainly, rational powers

make sense, as in $P(1/2) = 2^{1/2} \cdot 100$, where you recognize that $2^{1/2} = \sqrt{2}$. This says that the number of bacteria in the culture after a half hour is

$$P(1/2) = 2^{1/2} \cdot 100 = \sqrt{2} \cdot 100 \approx 141.$$

In this context, the function is not intended to give the exact number of bacteria at a given time, but rather, to predict an approximate number.

In any case, it's a simple matter to interpret fractional powers as roots. For instance, $x^{1/2} = \sqrt{x}$, $x^{1/3} = \sqrt[3]{x}$, $x^{1/4} = \sqrt[4]{x}$, and so on. Further, we can write

$$x^{2/3} = \sqrt[3]{x^2} = (\sqrt[3]{x})^2,$$

$$x^{7/4} = \sqrt[4]{x^7} = (\sqrt[4]{x})^7,$$

$$x^{3.1} = x^{31/10} = \sqrt[10]{x^{31}}$$

and so on. But, what about irrational powers? These are somewhat harder to define, but they work exactly the way you would want them to. For instance, since π is between 3.14 and 3.15, 2^π is between $2^{3.14}$ and $2^{3.15}$. In this way, we define 2^x for x irrational to fill in the gaps in the graph of $y = 2^x$ for x rational. That is, if x is irrational and $a < x < b$, for rational numbers a and b , then $2^a < 2^x < 2^b$. This is the logic behind the definition of irrational powers.

If for some reason, you want to find the population after π hours, you can use your calculator or computer to obtain the approximate population:

$$P(\pi) = 2^\pi \cdot 100 \approx 882.$$

For your convenience, we summarize the usual rules of exponents next.

RULES OF EXPONENTS

- For any integers m and n ,

$$x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m.$$

- For any real number p ,

$$x^{-p} = \frac{1}{x^p}.$$

- For any real numbers p and q ,

$$(x^p)^q = x^{p \cdot q}.$$

- For any real numbers p and q ,

$$x^p \cdot x^q = x^{p+q}.$$

Throughout your calculus course, you will need to be able to quickly convert back and forth between exponential form and fractional or root form.

EXAMPLE 5.1 Converting Expressions to Exponential Form

Convert each to exponential form: (a) $3\sqrt{x^5}$, (b) $\frac{5}{\sqrt[3]{x}}$, (c) $\frac{3x^2}{2\sqrt{x}}$, and (d) $(2^x \cdot 2^{3+x})^2$.

Solution For (a), simply leave the 3 alone and convert the power:

$$3\sqrt{x^5} = 3x^{5/2}.$$

For (b), use a negative exponent to write x in the numerator:

$$\frac{5}{\sqrt[3]{x}} = 5x^{-1/3}.$$

For (c), first separate the constants from the variables and then simplify:

$$\frac{3x^2}{2\sqrt{x}} = \frac{3}{2} \frac{x^2}{x^{1/2}} = \frac{3}{2} x^{2-1/2} = \frac{3}{2} x^{3/2}.$$

For (d), first work inside the parentheses and then square:

$$(2^x \cdot 2^{3+x})^2 = (2^{x+3+x})^2 = (2^{2x+3})^2 = 2^{4x+6}. \blacksquare$$

The function in part (d) of example 5.1 is called an exponential function with a *base* of 2.

DEFINITION 5.1

For any positive constant $b \neq 1$, the function $f(x) = b^x$ is called an **exponential function**. Here, b is called the **base** and x is the **exponent**.

Be careful to distinguish between algebraic functions like $f(x) = x^3$ and $g(x) = x^{2/3}$ and exponential functions. Confusing these types of functions is a very common error. Notice that for exponential functions like $h(x) = 2^x$, the variable is in the exponent (hence the name), instead of in the base. Also, notice that the domain of an exponential function is the entire real line, $(-\infty, \infty)$, while the range is the open interval $(0, \infty)$.

While any positive real number can be used as a base for an exponential function, three bases are the most commonly used in practice. Base 2 arises naturally when analyzing processes that double at regular intervals (such as the bacteria at the beginning of this section). Also, most computers perform their calculations using base 2 arithmetic. Our standard counting system is base 10, so this base is commonly used. However, far and away the most useful base is the irrational number e . Like π , the number e has a surprising tendency to occur in important calculations. We define e by the following:

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n. \quad (5.1)$$

Note that equation (5.1) has at least two serious shortcomings. First, we have not yet said what the notation $\lim_{n \rightarrow \infty}$ means. (In fact, we won't define this until Chapter 1.) Second, it's unclear why anyone would ever define a number in such a strange way. We will not be in a position to answer the second question until Chapter 6 (but the answer is worth the wait).

It suffices for the moment to say that equation (5.1) means that e can be approximated by calculating values of $(1 + 1/n)^n$ for large values of n and that the larger the value of n , the closer the approximation will be to the actual value of e . In particular, if you look at the sequence of numbers $(1 + 1/2)^2$, $(1 + 1/3)^3$, $(1 + 1/4)^4$ and so on, they will get progressively closer and closer to (i.e., home-in on) the irrational number e (named by the famous mathematician Leonhard Euler).

To get an idea of the value of e , we compute several of these numbers:

$$\left(1 + \frac{1}{10}\right)^{10} = 2.5937 \dots,$$

$$\left(1 + \frac{1}{1000}\right)^{1000} = 2.7169\dots,$$

$$\left(1 + \frac{1}{10,000}\right)^{10,000} = 2.7181\dots$$

and so on. You can compute enough of these values to convince yourself that the first few digits of the decimal representation of e ($e \approx 2.718281828459\dots$) are correct.

EXAMPLE 5.2 Computing Values of Exponentials

Approximate e^4 , $e^{-1/5}$ and e^0 .

Solution From a calculator, we find that

$$e^4 = e \cdot e \cdot e \cdot e \approx 54.598.$$

From the usual rules of exponents,

$$e^{-1/5} = \frac{1}{e^{1/5}} = \frac{1}{\sqrt[5]{e}} \approx 0.81873.$$

(On a calculator, it is convenient to replace $-1/5$ with -0.2 .) Finally, $e^0 = 1$. ■

The graphs of the exponential functions summarize many of their important properties.

EXAMPLE 5.3 Sketching Graphs of Exponentials

Sketch the graphs of the exponential functions $y = 2^x$, $y = e^x$, $y = e^{2x}$, $y = e^{x/2}$, $y = (1/2)^x$ and $y = e^{-x}$.

Solution Using a calculator or computer, you should get graphs similar to those that follow.

Notice that each of the graphs in Figures 0.64a, 0.64b, 0.65a and 0.65b starts very near the x -axis (reading left to right), passes through the point $(0, 1)$ and then rises steeply. This is true for all exponentials with base greater than 1 and with a positive coefficient in the exponent. Note that the larger the base ($e > 2$) or the larger the coefficient in the exponent ($2 > 1 > 1/2$), the more quickly the graph rises to the right (and drops to the left). Note that the graphs in Figures 0.66a and 0.66b (on the following page) are the mirror images in the y -axis of Figures 0.64a and 0.64b, respectively. The graphs rise as you move to the left and drop toward the x -axis as you move to the right. It's worth noting that by the rules of exponents, $(1/2)^x = 2^{-x}$ and $(1/e)^x = e^{-x}$.

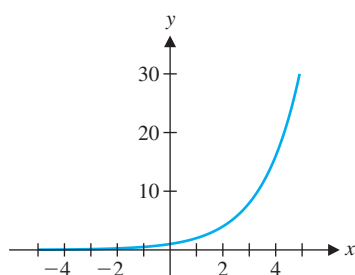


FIGURE 0.64a
 $y = 2^x$.

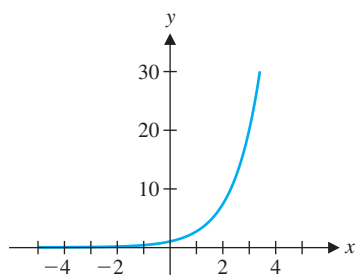


FIGURE 0.64b
 $y = e^x$.

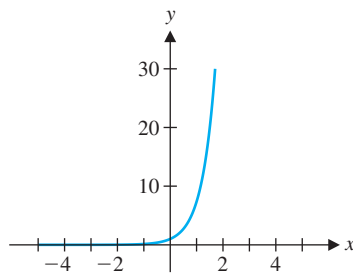


FIGURE 0.65a
 $y = e^{2x}$.

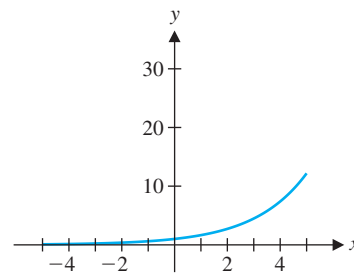


FIGURE 0.65b
 $y = e^{x/2}$.

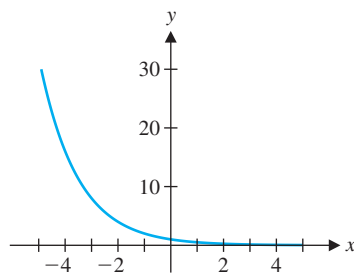


FIGURE 0.66a
 $y = (1/2)^x$.

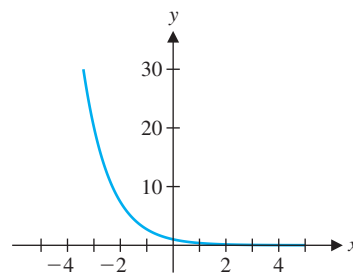


FIGURE 0.66b
 $y = e^{-x}$.

Notice in Figures 0.64–0.66 that each exponential function is one-to-one and hence, has an inverse function. We define the logarithmic functions to be inverses of the exponential functions.

DEFINITION 5.2

For any positive number $b \neq 1$, the **logarithm** function with base b , written $\log_b x$, is defined by

$$y = \log_b x \text{ if and only if } x = b^y.$$

That is, the logarithm $\log_b x$ gives the exponent to which you must raise the base b to get the given number x . For example,

$$\begin{aligned} \log_{10} 10 &= 1 \text{ (since } 10^1 = 10\text{),} \\ \log_{10} 100 &= 2 \text{ (since } 10^2 = 100\text{),} \\ \log_{10} 1000 &= 3 \text{ (since } 10^3 = 1000\text{)} \end{aligned}$$

and so on. The value of $\log_{10} 45$ is less clear than the preceding three values, but the idea is the same: you need to find the number y such that $10^y = 45$. The answer is between 1 and 2 (why?), but to be more precise, you will need to employ trial and error. (Of course, you can always use your calculator or computer to compute an approximate value, but that won't help you to understand what logarithms are all about.) You should get $\log_{10} 45 \approx 1.6532$. Later in the course, we introduce you to a powerful method for accurately approximating the values of such functions.

Observe from Definition 5.2 that for any base $b > 0$ ($b \neq 1$), if $y = \log_b x$, then $x = b^y > 0$. That is, the domain of $f(x) = \log_b x$ is the interval $(0, \infty)$. Likewise, the range of f is the entire real line, $(-\infty, \infty)$.

As with exponential functions, the most useful bases turn out to be 2, 10 and e . We usually abbreviate $\log_{10} x$ by $\log x$. Similarly, $\log_e x$ is usually abbreviated $\ln x$ (for **natural logarithm**).

EXAMPLE 5.4 Evaluating Logarithms

Without using your calculator, determine $\log(1/10)$, $\log(0.001)$, $\ln e$ and $\ln e^3$.

Solution Since $1/10 = 10^{-1}$, $\log(1/10) = -1$. Similarly, since $0.001 = 10^{-3}$, we have that $\log(0.001) = -3$. Since $\ln e = \log_e e^1$, $\ln e = 1$. Similarly, $\ln e^3 = 3$. ■

We want to emphasize the inverse relationship defined by Definition 5.2. That is, b^x and $\log_b x$ are inverse functions for any $b > 0$.

In particular, for the base e , we have

$$e^{\ln x} = x \quad \text{for any } x > 0 \quad \text{and} \quad \ln(e^x) = x \quad \text{for any } x. \quad (5.2)$$

We demonstrate this as follows. Let

$$y = \ln x = \log_e x.$$

By Definition 5.2, we have that

$$x = e^y = e^{\ln x}.$$

We can use this relationship between natural logarithms and exponentials to solve equations involving logarithms and exponentials, as in examples 5.5 and 5.6.

EXAMPLE 5.5 Solving a Logarithmic Equation

Solve the equation $\ln(x + 5) = 3$ for x .

Solution Taking the exponential of both sides of the equation and writing things backward (for convenience), we have

$$e^3 = e^{\ln(x+5)} = x + 5,$$

from (5.2). Subtracting 5 from both sides gives us

$$e^3 - 5 = x. \quad \blacksquare$$

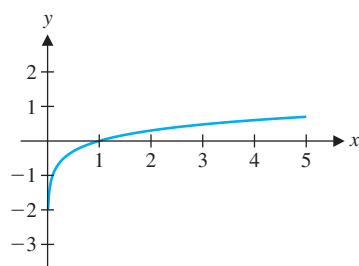


FIGURE 0.67a
 $y = \log x.$

EXAMPLE 5.6 Solving an Exponential Equation

Solve the equation $e^{x+4} = 7$ for x .

Solution Taking the natural logarithm of both sides and writing things backward (for simplicity), we have from (5.2) that

$$\ln 7 = \ln(e^{x+4}) = x + 4.$$

Subtracting 4 from both sides yields

$$\ln 7 - 4 = x. \quad \blacksquare$$

As always, graphs provide excellent visual summaries of the important properties of a function.

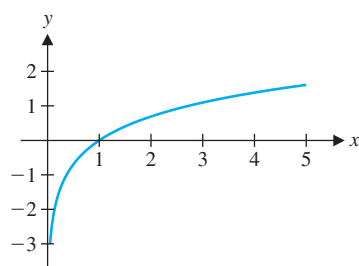


FIGURE 0.67b
 $y = \ln x.$

EXAMPLE 5.7 Sketching Graphs of Logarithms

Sketch graphs of $y = \log x$ and $y = \ln x$ and briefly discuss the properties of each.

Solution From a calculator or computer, you should obtain graphs resembling those in Figures 0.67a and 0.67b. Notice that both graphs appear to have a vertical asymptote at $x = 0$ (why would that be?), cross the x -axis at $x = 1$ and very gradually increase as x increases. Neither graph has any points to the left of the y -axis, since $\log x$ and $\ln x$ are defined only for $x > 0$. The two graphs are very similar, although not identical. \blacksquare

The properties just described graphically are summarized in Theorem 5.1.

THEOREM 5.1

For any positive base $b \neq 1$,

- (i) $\log_b x$ is defined only for $x > 0$,
- (ii) $\log_b 1 = 0$ and
- (iii) if $b > 1$, then $\log_b x < 0$ for $0 < x < 1$ and $\log_b x > 0$ for $x > 1$.

PROOF

- (i) Note that since $b > 0$, $b^y > 0$ for any y . So, if $\log_b x = y$, then $x = b^y > 0$.
- (ii) Since $b^0 = 1$ for any number $b \neq 0$, $\log_b 1 = 0$ (i.e., the exponent to which you raise the base b to get the number 1 is 0).
- (iii) We leave this as an exercise. ■

All logarithms share a set of defining properties, as stated in Theorem 5.2.

THEOREM 5.2

For any positive base $b \neq 1$ and positive numbers x and y , we have

- (i) $\log_b(xy) = \log_b x + \log_b y$,
- (ii) $\log_b(x/y) = \log_b x - \log_b y$
- (iii) $\log_b(x^y) = y \log_b x$.

As with most algebraic rules, each of these properties can dramatically simplify calculations when they apply.

EXAMPLE 5.8 Simplifying Logarithmic Expressions

Write each as a single logarithm: (a) $\log_2 27^x - \log_2 3^x$ and (b) $\ln 8 - 3 \ln(1/2)$.

Solution First, note that there is more than one order in which to work each problem. For part (a), we have $27 = 3^3$ and so, $27^x = (3^3)^x = 3^{3x}$. This gives us

$$\begin{aligned} \log_2 27^x - \log_2 3^x &= \log_2 3^{3x} - \log_2 3^x \\ &= 3x \log_2 3 - x \log_2 3 = 2x \log_2 3 = \log_2 3^{2x}. \end{aligned}$$

For part (b), note that $8 = 2^3$ and $1/2 = 2^{-1}$. Then,

$$\begin{aligned} \ln 8 - 3 \ln(1/2) &= 3 \ln 2 - 3(-\ln 2) \\ &= 3 \ln 2 + 3 \ln 2 = 6 \ln 2 = \ln 2^6 = \ln 64. \quad \blacksquare \end{aligned}$$

In some circumstances, it is beneficial to use the rules of logarithms to expand a given expression, as in the following example.

EXAMPLE 5.9 Expanding a Logarithmic Expression

Use the rules of logarithms to expand the expression $\ln\left(\frac{x^3 y^4}{z^5}\right)$.

Solution From Theorem 5.2, we have that

$$\begin{aligned}\ln\left(\frac{x^3y^4}{z^5}\right) &= \ln(x^3y^4) - \ln(z^5) = \ln(x^3) + \ln(y^4) - \ln(z^5) \\ &= 3\ln x + 4\ln y - 5\ln z. \quad \blacksquare\end{aligned}$$

Using the rules of exponents and logarithms, notice that we can rewrite any exponential as an exponential with base e , as follows. For any base $a > 0$, we have

$$a^x = e^{\ln(a^x)} = e^{x \ln a}. \quad (5.3)$$

This follows from Theorem 5.2 (iii) and the fact that $e^{\ln y} = y$, for all $y > 0$.

EXAMPLE 5.10 Rewriting Exponentials as Exponentials with Base e

Rewrite the exponentials 2^x , 5^x and $(2/5)^x$ as exponentials with base e .

Solution From (5.3), we have

$$\begin{aligned}2^x &= e^{\ln(2^x)} = e^{x \ln 2}, \\ 5^x &= e^{\ln(5^x)} = e^{x \ln 5}\end{aligned}$$

and

$$\left(\frac{2}{5}\right)^x = e^{\ln[(2/5)^x]} = e^{x \ln(2/5)}. \quad \blacksquare$$

Just as we were able to use the relationship between the natural logarithm and exponentials to rewrite an exponential with any positive base in terms of an exponential with base e , we can use these same properties to rewrite any logarithm in terms of natural logarithms, as follows. For any positive base b ($b \neq 1$), we will show that

$$\log_b x = \frac{\ln x}{\ln b}. \quad (5.4)$$

Let $y = \log_b x$. Then by Definition 5.2, we have that $x = b^y$. Taking the natural logarithm of both sides of this equation, we get by Theorem 5.2 (iii) that

$$\ln x = \ln(b^y) = y \ln b.$$

Dividing both sides by $\ln b$ (since $b \neq 1$ and $\ln b \neq 0$) gives us

$$y = \frac{\ln x}{\ln b},$$

establishing (5.4).

One use you will find for equation (5.4) is for computing logarithms with bases other than e or 10. More than likely, your calculator has keys only for $\ln x$ and $\log x$. We illustrate this idea in example 5.11.

EXAMPLE 5.11 Approximating the Value of Logarithms

Approximate the value of $\log_7 12$.

Solution From (5.4), we have

$$\log_7 12 = \frac{\ln 12}{\ln 7} \approx 1.2769894. \quad \blacksquare$$



Saint Louis Gateway Arch.

Hyperbolic Functions

There are two special combinations of exponential functions, called the **hyperbolic sine** and **hyperbolic cosine** functions, that have important applications. For instance, the Gateway Arch in Saint Louis was built in the shape of a hyperbolic cosine graph. (See the photograph in the margin.) The hyperbolic sine function is denoted by $\sinh(x)$ and the hyperbolic cosine function is denoted by $\cosh(x)$. These are defined to be

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Graphs of these functions are shown in Figures 0.68a and 0.68b. The hyperbolic functions (including the hyperbolic tangent, $\tanh x$, defined in the obvious way) are often convenient to use when solving equations. For now, we verify several basic properties that the hyperbolic functions satisfy in parallel with their trigonometric counterparts.

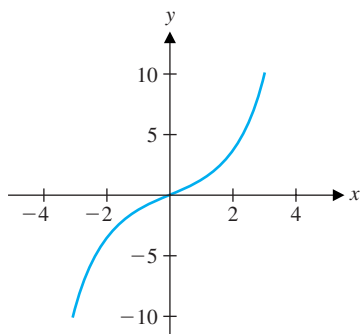


FIGURE 0.68a
 $y = \sinh x$.

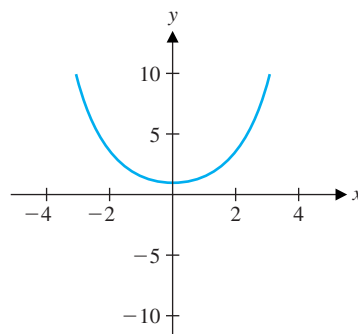


FIGURE 0.68b
 $y = \cosh x$.

EXAMPLE 5.12 Computing Values of Hyperbolic Functions

Compute $f(0)$, $f(1)$ and $f(-1)$ and determine how $f(x)$ and $f(-x)$ compare for each function: (a) $f(x) = \sinh x$ and (b) $f(x) = \cosh x$.

Solution For part (a), we have $\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1 - 1}{2} = 0$. Note that this means that $\sinh 0 = \sin 0 = 0$. Also, we have $\sinh 1 = \frac{e^1 - e^{-1}}{2} \approx 1.18$, while $\sinh(-1) = \frac{e^{-1} - e^1}{2} \approx -1.18$. Notice that $\sinh(-1) = -\sinh 1$. In fact, for any x ,

$$\sinh(-x) = \frac{e^{-x} - e^x}{2} = \frac{-(e^x - e^{-x})}{2} = -\sinh x.$$

[Recall that the same rule holds for the corresponding trigonometric function: $\sin(-x) = -\sin x$.] For part (b), we have $\cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1 + 1}{2} = 1$. Note that this means that $\cosh 0 = \cos 0 = 1$. Also, we have $\cosh 1 = \frac{e^1 + e^{-1}}{2} \approx 1.54$, while $\cosh(-1) = \frac{e^{-1} + e^1}{2} \approx 1.54$. Notice that $\cosh(-1) = \cosh 1$. In fact, for any x ,

$$\cosh(-x) = \frac{e^{-x} + e^x}{2} = \frac{e^x + e^{-x}}{2} = \cosh x.$$

[Again, the same rule holds for the corresponding trigonometric function: $\cos(-x) = \cos x$.]

□ Fitting a Curve to Data

You are familiar with the idea that two points determine a straight line. As we see in example 5.13, two points will also determine an exponential function.

EXAMPLE 5.13 Matching Data to an Exponential Curve

Find the exponential function of the form $f(x) = ae^{bx}$ that passes through the points $(0, 5)$ and $(3, 9)$.

Solution Notice that you can't simply solve this problem by inspection. That is, you can't just read off appropriate values for a and b . Instead, we solve for a and b , using the properties of logarithms and exponentials. First, for the graph to pass through the point $(0, 5)$, this means that

$$5 = f(0) = ae^{b \cdot 0} = a,$$

so that $a = 5$. Next, for the graph to pass through the point $(3, 9)$, we must have

$$9 = f(3) = ae^{3b} = 5e^{3b}.$$

To solve for the b in the exponent, we divide both sides of the last equation by 5 and take the natural logarithm of both sides, which yields

$$\ln\left(\frac{9}{5}\right) = \ln e^{3b} = 3b \ln e = 3b,$$

since $\ln e = 1$. Finally, dividing by 3 gives us the value for b :

$$b = \frac{1}{3} \ln\left(\frac{9}{5}\right).$$

Thus, $f(x) = 5e^{(1/3)\ln(9/5)x}$.

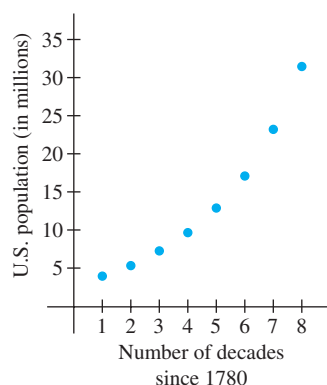


FIGURE 0.69a
U.S. Population 1790–1860.

Year	U.S. Population
1790	3,929,214
1800	5,308,483
1810	7,239,881
1820	9,638,453
1830	12,866,020
1840	17,069,453
1850	23,191,876
1860	31,443,321

Consider the population of the United States from 1790 to 1860, found in the accompanying table. A plot of these data points can be seen in Figure 0.69a (where the vertical scale represents the population in millions). This shows that the population was increasing, with larger and larger increases each decade. If you sketch an imaginary curve through these points, you will probably get the impression of a parabola or perhaps the right half of a cubic or exponential. And that's the question: is this data best modeled by a quadratic function, a cubic function, a fourth-order polynomial, an exponential function or what?

We can use the properties of logarithms from Theorem 5.2 to help determine whether a given set of data is modeled better by a polynomial or an exponential function, as follows. Suppose that the data actually comes from an exponential, say $y = ae^{bx}$ (i.e., the data points lie on the graph of this exponential). Then,

$$\ln y = \ln(ae^{bx}) = \ln a + \ln e^{bx} = \ln a + bx.$$

If you draw a new graph, where the horizontal axis shows values of x and the vertical axis corresponds to values of $\ln y$, then the graph will be the line $\ln y = bx + c$ (where the

constant $c = \ln a$). On the other hand, suppose the data actually came from a polynomial. If $y = bx^n$ (for any n), then observe that

$$\ln y = \ln(bx^n) = \ln b + \ln x^n = \ln b + n \ln x.$$

In this case, a graph with horizontal and vertical axes corresponding to x and $\ln y$, respectively, will look like the graph of a logarithm, $\ln y = n \ln x + c$. Such a **semi-log graph** (i.e., graphing $\ln y$ versus x) lets us distinguish the graph of an exponential from that of a polynomial: graphs of exponentials become straight lines, while graphs of polynomials (of degree ≥ 1) become logarithmic curves. Scientists and engineers frequently use semi-log graphs to help them gain an understanding of physical phenomena represented by some collection of data.

EXAMPLE 5.14 Using a Semi-Log Graph to Identify a Type of Function

Determine if the population of the United States from 1790 to 1860 was increasing exponentially or as a polynomial.

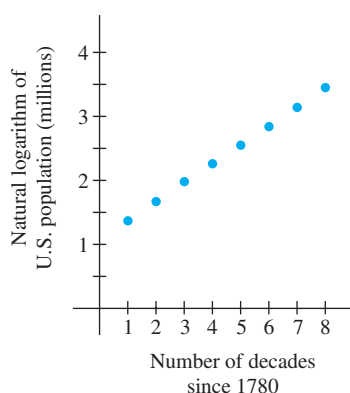


FIGURE 0.69b
Semi-log plot of U.S. population.

Solution As already indicated, the trick is to draw a semi-log graph. That is, instead of plotting $(1, 3.9)$ as the first data point, plot $(1, \ln 3.9)$ and so on. A semi-log plot of this data set is seen in Figure 0.69b. Although the points are not exactly colinear (how would you prove this?), the plot is very close to a straight line with $\ln y$ -intercept of 1 and slope 0.35. You should conclude that the population is well-modeled by an exponential function. (Keep in mind that here, as with most real problems, the data is somewhat imprecise and so, the points in the semi-log plot need *not* be perfectly colinear for you to conclude that the data is modeled quite well by an exponential.) The exponential model would be $y = P(t) = ae^{bt}$, where t represents the number of decades since 1780. Here, b is the slope and $\ln a$ is the $\ln y$ -intercept of the line in the semi-log graph. That is, $b = 0.35$ and $\ln a = 1$ (why?), so that $a = e$. The population is then modeled by

$$P(t) = e \cdot e^{0.35t} \text{ million.}$$

EXERCISES 0.5

WRITING EXERCISES

- Starting from a single cell, a human being is formed by 50 generations of cell division. Explain why after n divisions there are 2^n cells. Guess how many cells will be present after 50 divisions, then compute 2^{50} . Briefly discuss how rapidly exponential functions increase.
- Explain why the graphs of $f(x) = 2^{-x}$ and $g(x) = (\frac{1}{2})^x$ are the same.
- Compare $f(x) = x^2$ and $g(x) = 2^x$ for $x = \frac{1}{2}, x = 1, x = 2, x = 3$ and $x = 4$. In general, which function is bigger for large values of x ? For small values of x ?
- Compare $f(x) = 2^x$ and $g(x) = 3^x$ for $x = -2, x = -\frac{1}{2}$,

$x = \frac{1}{2}$ and $x = 2$. In general, which function is bigger for negative values of x ? For positive values of x ?

In exercises 1–6, convert each exponential expression into fractional or root form.

- | | | |
|--------------|--------------|---------------|
| 1. 2^{-3} | 2. 4^{-2} | 3. $3^{1/2}$ |
| 4. $6^{2/5}$ | 5. $5^{2/3}$ | 6. $4^{-2/3}$ |

In exercises 7–12, convert each expression into exponential form.

- | | | |
|---------------------|---------------------------|-----------------------------|
| 7. $\frac{1}{x^2}$ | 8. $\sqrt[3]{x^2}$ | 9. $\frac{2}{x^3}$ |
| 10. $\frac{4}{x^2}$ | 11. $\frac{1}{2\sqrt{x}}$ | 12. $\frac{3}{2\sqrt{x^3}}$ |

In exercises 13–16, find the integer value of the given expression without using a calculator.

13. $4^{3/2}$ 14. $8^{2/3}$ 15. $\frac{\sqrt{8}}{2^{1/2}}$ 16. $\frac{2}{(1/3)^2}$

T In exercises 17–20, use a calculator or computer to estimate each value.

17. $2e^{-1/2}$ 18. $4e^{-2/3}$ 19. $\frac{12}{e}$ 20. $\frac{14}{\sqrt{e}}$

T In exercises 21–30, sketch a graph of the given function.

21. $f(x) = e^{2x}$ 22. $f(x) = e^{3x}$
 23. $f(x) = 2e^{x/4}$ 24. $f(x) = e^{-x^2}$
 25. $f(x) = 3e^{-2x}$ 26. $f(x) = 10e^{-x/3}$
 27. $f(x) = \ln 2x$ 28. $f(x) = \ln x^2$
 29. $f(x) = e^{2 \ln x}$ 30. $f(x) = e^{-x/4} \sin x$

In exercises 31–40, solve the given equation for x .

31. $e^{2x} = 2$ 32. $e^{4x} = 3$
 33. $e^x(x^2 - 1) = 0$ 34. $xe^{-2x} + 2e^{-2x} = 0$
 35. $\ln 2x = 4$ 36. $2 \ln 3x = 1$
 37. $4 \ln x = -8$ 38. $x^2 \ln x - 9 \ln x = 0$
 39. $e^{2 \ln x} = 4$ 40. $\ln(e^{2x}) = 6$

In exercises 41 and 42, use the definition of logarithm to determine the value.

41. (a) $\log_3 9$ (b) $\log_4 64$ (c) $\log_3 \frac{1}{27}$
 42. (a) $\log_4 \frac{1}{16}$ (b) $\log_4 2$ (c) $\log_9 3$

T In exercises 43 and 44, use equation (5.4) to approximate the value.

43. (a) $\log_3 7$ (b) $\log_4 60$ (c) $\log_3 \frac{1}{24}$
 44. (a) $\log_4 \frac{1}{10}$ (b) $\log_4 3$ (c) $\log_9 8$

In exercises 45–50, rewrite the expression as a single logarithm.

45. $\ln 3 - \ln 4$ 46. $2 \ln 4 - \ln 3$
 47. $\frac{1}{2} \ln 4 - \ln 2$ 48. $3 \ln 2 - \ln \frac{1}{2}$
 49. $\ln \frac{3}{4} + 4 \ln 2$ 50. $\ln 9 - 2 \ln 3$

In exercises 51–54, find a function of the form $f(x) = ae^{bx}$ with the given function values.

51. $f(0) = 2, f(2) = 6$ 52. $f(0) = 3, f(3) = 4$
 53. $f(0) = 4, f(2) = 2$ 54. $f(0) = 5, f(1) = 2$

T 55. A fast-food restaurant gives every customer a game ticket. With each ticket, the customer has a 1-in-10 chance of winning a free meal. If you go 10 times, estimate your chances of winning at least one free meal. The exact probability is $1 - \left(\frac{9}{10}\right)^{10}$. Compute this number and compare it to your guess.

T 56. In exercise 55, if you had 20 tickets with a 1-in-20 chance of winning, would you expect your probability of winning to increase or decrease? Compute the probability $1 - \left(\frac{19}{20}\right)^{20}$ to find out.

57. In general, if you have n chances of winning with a 1-in- n chance on each try, the probability of winning at least once is $1 - \left(1 - \frac{1}{n}\right)^n$. As n gets larger, what number does this probability approach? (Hint: There is a very good reason that this question is in this section!)

58. If $y = a \cdot x^m$, show that $\ln y = \ln a + m \ln x$. If $v = \ln y$, $u = \ln x$ and $b = \ln a$, show that $v = mu + b$. Explain why the graph of v as a function of u would be a straight line. This graph is called the **log-log plot** of y and x .

T 59. For the given data, compute $v = \ln y$ and $u = \ln x$ and plot points (u, v) . Find constants m and b such that $v = mu + b$ and use the results of exercise 58 to find a constant a such that $y = a \cdot x^m$.

x	2.2	2.4	2.6	2.8	3.0	3.2
y	14.52	17.28	20.28	23.52	27.0	30.72

T 60. Repeat exercise 59 for the given data.

x	2.8	3.0	3.2	3.4	3.6	3.8
y	9.37	10.39	11.45	12.54	13.66	14.81

T 61. Construct a log-log plot (see exercise 58) of the U.S. population data in example 5.14. Compared to the semi-log plot of the data in Figure 0.69b, does the log-log plot look linear? Based on this, is the population data modeled better by an exponential function or a polynomial (power) function?

T 62. Construct a semi-log plot of the data in exercise 59. Compared to the log-log plot already constructed, does this plot look linear? Based on this, is this data better modeled by an exponential or power function?

63. The concentration $[H^+]$ of free hydrogen ions in a chemical solution determines the solution's pH, as defined by $\text{pH} = -\log [H^+]$. Find $[H^+]$ if the pH equals (a) 7, (b) 8, and (c) 9. For each increase in pH of 1, by what factor does $[H^+]$ change?

64. Gastric juice is considered an acid, with a pH of about 2.5. Blood is considered alkaline, with a pH of about 7.5. Compare the concentrations of hydrogen ions in the two substances (see exercise 63).

65. The Richter magnitude M of an earthquake is defined in terms of the energy E in joules released by the earthquake, with $\log_{10} E = 4.4 + 1.5M$. Find the energy for earthquakes with magnitudes (a) 4, (b) 5 and (c) 6. For each increase in M of one, by what factor does E change?

66. It puzzles some people who have not grown up around earthquakes that a magnitude 6 quake is considered much more severe than a magnitude 3 quake. Compare the amount of energy released in the two quakes.

58 CHAPTER 0 Preliminaries

67. The decibel level of a noise is defined in terms of the intensity I of the noise, with $\text{dB} = 10 \log_{10}(I/I_0)$. Here, $I_0 = 10^{-12} \text{ W/m}^2$ is the intensity of a barely audible sound. Compute the intensity levels of sounds with (a) $\text{dB} = 80$, (b) $\text{dB} = 90$ and (c) $\text{dB} = 100$. For each increase in decibels of 10, by what factor does I change?
68. At a basketball game, a courtside decibel meter shows crowd noises ranging from 60 dB to 110 dB. How much louder is the 110-dB noise compared to the 60-dB noise (see exercise 67)?
- T** 69. Use a graphing calculator to graph $y = xe^{-x}$, $y = xe^{-2x}$, $y = xe^{-3x}$ and so on. Estimate the locations of the maximum for each. In general, state a rule for the location of the maximum of $y = xe^{-kx}$.
- T** 70. In golf, the task is to hit a golf ball into a small hole. If the ground near the hole is not flat, the golfer must judge how much the ball will curve. Suppose the golfer is at the point $(-13, 0)$, the hole is at the point $(0, 0)$ and the path of the ball is, for $-13 \leq x \leq 0$, $y = -1.672x + 72 \ln(1 + 0.02x)$. Show that the ball goes in the hole and estimate the point on the y -axis at which the golfer aimed.

Exercises 71–76 refer to the hyperbolic functions.

71. Show that the range of the hyperbolic cosine is $\cosh x \geq 1$ and the range of the hyperbolic sine is the entire real line.
72. Show that $\cosh^2 x - \sinh^2 x = 1$ for all x .
- T** 73. The Saint Louis Gateway Arch is both 630 feet wide and 630 feet tall. (Most people think that it looks taller than it is wide.) One model for the outline of the arch is $y = 757.7 - 127.7 \cosh\left(\frac{x}{127.7}\right)$ for $y \geq 0$. Use a graphing calculator to approximate the x - and y -intercepts and determine if the model has the correct horizontal and vertical measurements.
- T** 74. To model the outline of the arch with a parabola, you can start with $y = -c(x + 315)(x - 315)$ for some constant c . Explain why this gives the correct x -intercepts. Determine the constant c that gives a y -intercept of 630. Graph this parabola and the hyperbolic cosine in exercise 73 on the same axes. Are the graphs nearly identical or very different?
75. Find all solutions of $\sinh(x^2 - 1) = 0$.

76. Find all solutions of $\cosh(3x + 2) = 0$.
77. On a standard piano, the A below middle C produces a sound wave with frequency 220 Hz (cycles per second). The frequency of the A one octave higher is 440 Hz. In general, doubling the frequency produces the same note an octave higher. Find an exponential formula for the frequency f as a function of the number of octaves x above the A below middle C.
78. There are 12 notes in an octave on a standard piano. Middle C is 3 notes above A (see exercise 77). If the notes are tuned equally, this means that middle C is a quarter-octave above A. Use $x = \frac{1}{4}$ in your formula from exercise 77 to estimate the frequency of middle C.

 EXPLORATORY EXERCISES

- T** 1. Graph $y = x^2$ and $y = 2^x$ and approximate the two positive solutions of the equation $x^2 = 2^x$. Graph $y = x^3$ and $y = 3^x$ and approximate the two positive solutions of the equation $x^3 = 3^x$. Explain why $x = a$ will always be a solution of $x^a = a^x$, $a > 0$. What is different about the role of $x = 2$ as a solution of $x^2 = 2^x$ compared to the role of $x = 3$ as a solution of $x^3 = 3^x$? To determine the a -value at which the change occurs, graphically solve $x^a = a^x$ for $a = 2.1, 2.2, \dots, 2.9$ and note that $a = 2.7$ and $a = 2.8$ behave differently. Continue to narrow down the interval of change by testing $a = 2.71, 2.72, \dots, 2.79$. Then guess the exact value of a .
- T** 2. Graph $y = \ln x$ and describe the behavior near $x = 0$. Then graph $y = x \ln x$ and describe the behavior near $x = 0$. Repeat this for $y = x^2 \ln x$, $y = x^{1/2} \ln x$, and $y = x^a \ln x$ for a variety of positive constants a . Because the function “blows up” at $x = 0$, we say that $y = \ln x$ has a **singularity** at $x = 0$. The **order** of the singularity at $x = 0$ of a function $f(x)$ is the smallest value of a such that $y = x^a f(x)$ doesn’t have a singularity at $x = 0$. Determine the order of the singularity at $x = 0$ for (a) $f(x) = \frac{1}{x}$, (b) $f(x) = \frac{1}{x^2}$, and (c) $f(x) = \frac{1}{x^3}$. The higher the order of the singularity, the “worse” the singularity is. Based on your work, how bad is the singularity of $y = \ln x$ at $x = 0$?

 0.6 TRANSFORMATIONS OF FUNCTIONS

You are now familiar with a long list of functions including polynomials, rational functions, trigonometric functions, exponentials and logarithms. Many students at this stage expect that calculus and higher-level mathematics courses will consist of defining even more functions and solving more equations. Actually, one important goal for this course is to more fully understand the properties of the functions we have already defined. To a large extent, you

will build your understanding by examining a few key properties of functions. To briefly get an idea of where we are headed, recall what the function $f(x) = x^2$ represents. To understand this function, you first had to understand how to multiply two numbers together. From practice multiplying “3 times 3” and “6 times 6” and so on, the idea of multiplying any number by itself became natural. The function x^2 is, in this sense, simply an abstraction. We have put multiplication into the broader context of functions. Now that we have functions, we need to make the manipulations of functions as natural as possible.

The first few function manipulations we define are straightforward.

DEFINITION 6.1

Suppose that $f(x)$ and $g(x)$ are functions with domains D_1 and D_2 , respectively. The functions $f + g$, $f - g$ and $f \cdot g$ are defined by

$$(f + g)(x) = f(x) + g(x),$$

$$(f - g)(x) = f(x) - g(x),$$

and

$$(f \cdot g)(x) = f(x) \cdot g(x),$$

for all x in $D_1 \cap D_2$ (i.e., $x \in D_1$, and $x \in D_2$). The function $\frac{f}{g}$ is defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$

for all x in $D_1 \cap D_2$ such that $g(x) \neq 0$.

In example 6.1, we examine various combinations of several simple functions.

EXAMPLE 6.1 Combinations of Functions

If $f(x) = x - 3$ and $g(x) = \sqrt{x - 1}$, determine the functions $f + g$, $3f - g$ and $\frac{f}{g}$, stating the domains of each.

Solution First, note that the domain of f is the entire real line and the domain of g is the set of all $x \geq 1$. Now,

$$(f + g)(x) = x - 3 + \sqrt{x - 1}$$

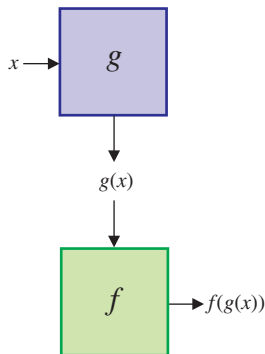
and

$$(3f - g)(x) = 3(x - 3) - \sqrt{x - 1} = 3x - 9 - \sqrt{x - 1}.$$

Notice that the domain of both $(f + g)$ and $(3f - g)$ is $\{x | x \geq 1\}$. For

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x - 3}{\sqrt{x - 1}},$$

the domain is $\{x | x > 1\}$, where we have added the restriction $x \neq 1$ to avoid dividing by 0. ■



$$(f \circ g)(x) = f(g(x)).$$

Definition 6.1 and example 6.1 show us how to do arithmetic with functions. An operation on functions that does not directly correspond to arithmetic is the *composition* of two functions.

DEFINITION 6.2

The **composition** of functions $f(x)$ and $g(x)$, written $f \circ g$, is defined by

$$(f \circ g)(x) = f(g(x)),$$

for all x such that x is in the domain of g and $g(x)$ is in the domain of f .

The composition of two functions is a two-step process, as indicated in the margin schematic. Be careful to notice what this definition is saying. In particular, for $f(g(x))$ to be defined, you first need $g(x)$ to be defined, so x must be in the domain of g . Next, f must be defined at the point $g(x)$, so that the number $g(x)$ will need to be in the domain of f .

EXAMPLE 6.2 Finding the Composition of Two Functions

For $f(x) = x^2 + 1$ and $g(x) = \sqrt{x - 2}$, find the compositions $f \circ g$ and $g \circ f$ and identify the domain of each.

Solution First, we have

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) = f(\sqrt{x - 2}) \\ &= (\sqrt{x - 2})^2 + 1 = x - 2 + 1 = x - 1. \end{aligned}$$

It's tempting to write that the domain of $f \circ g$ is the entire real line, but look more carefully. Note that for x to be in the domain of g , we must have $x \geq 2$. The domain of f is the whole real line, so this places no further restrictions on the domain of $f \circ g$. Even though the final expression $x - 1$ is defined for all x , the domain of $(f \circ g)$ is $\{x \mid x \geq 2\}$. For the second composition,

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) = g(x^2 + 1) \\ &= \sqrt{(x^2 + 1) - 2} = \sqrt{x^2 - 1}. \end{aligned}$$

The resulting square root requires $x^2 - 1 \geq 0$ or $|x| \geq 1$. Since the “inside” function f is defined for all x , the domain of $g \circ f$ is $\{x \in \mathbb{R} \mid |x| \geq 1\}$, which we write in interval notation as $(-\infty, -1] \cup [1, \infty)$. ■

As you progress through the calculus, you will often find yourself needing to recognize that a given function is a composition of simpler functions. For now, it is an important skill to practice.

EXAMPLE 6.3 Identifying Compositions of Functions

Identify functions f and g such that the given function can be written as $(f \circ g)(x)$ for each of (a) $\sqrt{x^2 + 1}$, (b) $(\sqrt{x} + 1)^2$, (c) $\sin x^2$ and (d) $\cos^2 x$. Note that more than one answer is possible for each function.

Solution (a) Notice that $x^2 + 1$ is *inside* the square root. So, one choice is to have $g(x) = x^2 + 1$ and $f(x) = \sqrt{x}$.

(b) Here, $\sqrt{x} + 1$ is *inside* the square. So, one choice is $g(x) = \sqrt{x} + 1$ and $f(x) = x^2$.

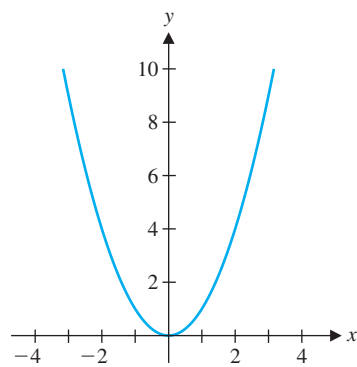


FIGURE 0.70a
 $y = x^2$.

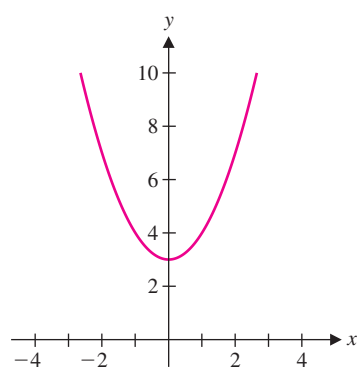


FIGURE 0.70b
 $y = x^2 + 3$.

(c) The function can be rewritten as $\sin(x^2)$, with x^2 clearly *inside* the sine function. Then, $g(x) = x^2$ and $f(x) = \sin x$ is one choice.

(d) The function as written is shorthand for $(\cos x)^2$. So, one choice is $g(x) = \cos x$ and $f(x) = x^2$.

In general, it is quite difficult to take the graphs of $f(x)$ and $g(x)$ and produce the graph of $f(g(x))$. If one of the functions f and g is linear, however, there is a simple graphical procedure for graphing the composition. Such **linear transformations** are explored in the remainder of this section.

The first case is to take the graph of $f(x)$ and produce the graph of $f(x) + c$ for some constant c . You should be able to deduce the general result from the following example.

EXAMPLE 6.4 Vertical Translation of a Graph

Graph $y = x^2$ and $y = x^2 + 3$; compare and contrast the graphs.

Solution You can probably sketch these by hand. You should get something like those in Figures 0.70a and 0.70b. Both figures show parabolas opening up. The main obvious difference is that x^2 has a y -intercept of 0 and $x^2 + 3$ has a y -intercept of 3. Some thought should convince you that this is an important clue. For *any* given value of x , the point on the graph of $y = x^2 + 3$ will be plotted exactly 3 units higher than the corresponding point on the graph of $y = x^2$. This is shown in Figure 0.71a.

In Figure 0.71b, the two graphs are shown on the same set of axes. To many people, it does not look like the top graph is the same as the bottom graph moved up 3 units. This is an unfortunate optical illusion. Humans usually mentally judge distance between curves as the shortest distance between the curves. For these parabolas, the shortest distance is vertical at $x = 0$ but becomes increasingly horizontal as you move away from the y -axis. The distance of 3 between the parabolas is measured vertically.

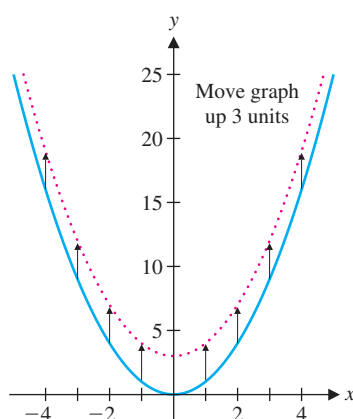


FIGURE 0.71a
 Translate graph up.

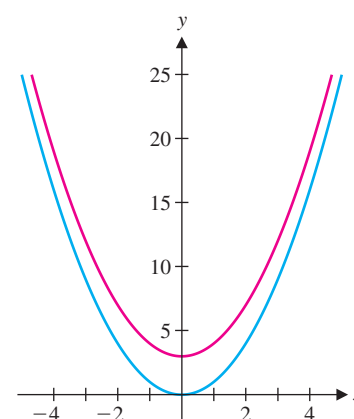


FIGURE 0.71b
 $y = x^2$ and $y = x^2 + 3$.

In general, the graph of $y = f(x) + c$ is exactly the same as the graph of $y = f(x)$ shifted up (if $c > 0$) or down (if $c < 0$) by $|c|$ units. We usually refer to $f(x) + c$ as a **vertical translation** (up or down, by $|c|$ units).

In example 6.5, we explore what happens if a constant is added to x .

EXAMPLE 6.5 A Horizontal Translation

Compare and contrast the graphs of $y = x^2$ and $y = (x - 1)^2$.

Solution The graphs are shown in Figures 0.72a and 0.72b, respectively.

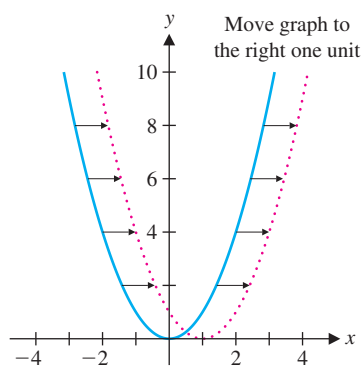


FIGURE 0.73
Translation to the right.

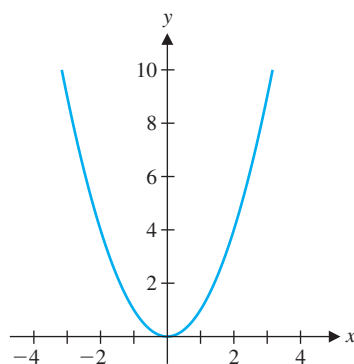


FIGURE 0.72a
 $y = x^2$.

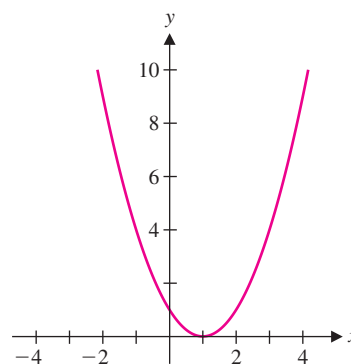


FIGURE 0.72b
 $y = (x - 1)^2$.

Notice that the graph of $y = (x - 1)^2$ appears to be the same as the graph of $y = x^2$, except that it is shifted 1 unit to the right. This should make sense for the following reason. Pick a value of x , say $x = 13$. The value of $(x - 1)^2$ at $x = 13$ is 12^2 , the same as the value of x^2 at $x = 12$, 1 unit to the left. Observe that this same pattern would hold for any x you choose. A simultaneous plot of the two functions (see Figure 0.73) shows this. ■

In general, for $c > 0$, the graph of $y = f(x - c)$ is the same as the graph of $y = f(x)$ shifted c units to the right. Likewise (again, for $c > 0$), you get the graph of $f(x + c)$ by moving the graph of $y = f(x)$ to the left c units. We usually refer to $f(x - c)$ and $f(x + c)$ as **horizontal translations** (to the right and left, respectively, by c units).

To avoid confusion on which way to translate the graph of $y = f(x)$, focus on what makes the argument (the quantity inside the parentheses) zero. For $f(x)$, this is $x = 0$, but for $f(x - c)$ you must have $x = c$ to get $f(0)$ [i.e., the same y -value as $f(x)$ when $x = 0$]. This says that the point on the graph of $y = f(x)$ at $x = 0$ corresponds to the point on the graph of $y = f(x - c)$ at $x = c$.

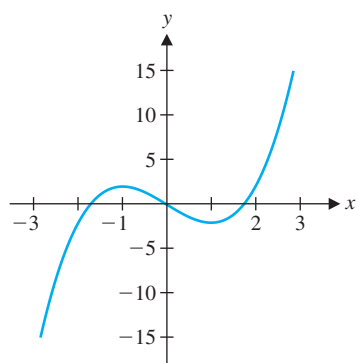


FIGURE 0.74a
 $y = f(x)$.

EXAMPLE 6.6 Comparing Vertical and Horizontal Translations

Given the graph of $y = f(x)$ shown in Figure 0.74a, sketch the graphs of $y = f(x) - 2$ and $y = f(x - 2)$.

Solution To graph $y = f(x) - 2$, simply translate the original graph down 2 units, as shown in Figure 0.74b. To graph $y = f(x - 2)$, simply translate the original graph to the right 2 units (so that the x -intercept at $x = 0$ in the original graph corresponds to an x -intercept at $x = 2$ in the translated graph), as seen in Figure 0.74c.

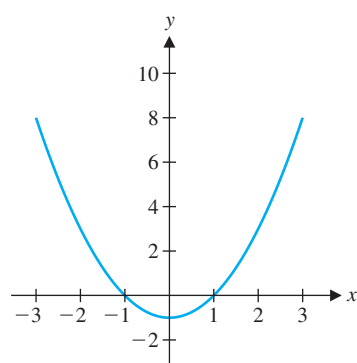


FIGURE 0.75a
 $y = x^2 - 1$.

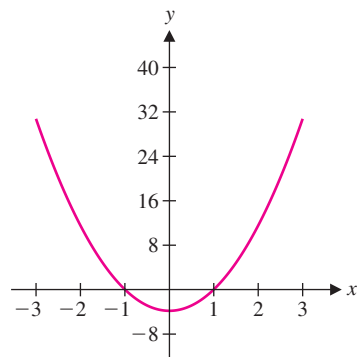


FIGURE 0.75b
 $y = 4(x^2 - 1)$.

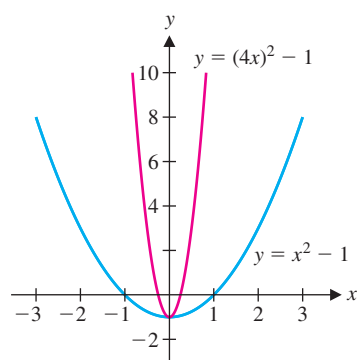


FIGURE 0.75e
 $y = x^2 - 1$ and $y = (4x)^2 - 1$.

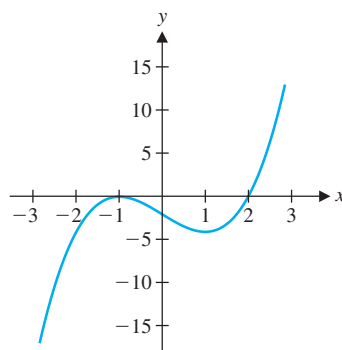


FIGURE 0.74b
 $y = f(x) - 2$.

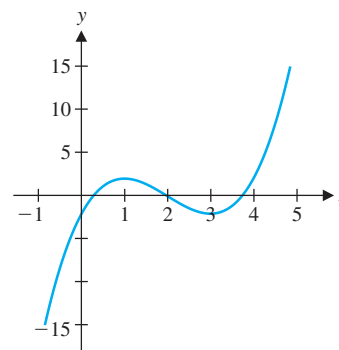


FIGURE 0.74c
 $y = f(x - 2)$.

Example 6.7 explores the effect of multiplying or dividing x or y by a constant.

EXAMPLE 6.7 Comparing Some Related Graphs

Compare and contrast the graphs of $y = x^2 - 1$, $y = 4(x^2 - 1)$ and $y = (4x)^2 - 1$.

Solution The first two graphs are shown in Figures 0.75a and 0.75b, respectively.

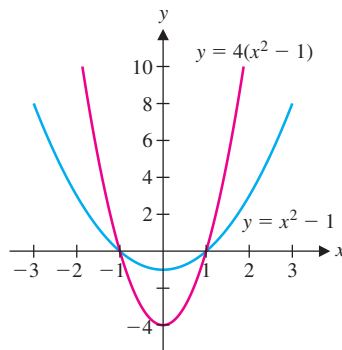


FIGURE 0.75c
 $y = x^2 - 1$ and $y = 4(x^2 - 1)$.

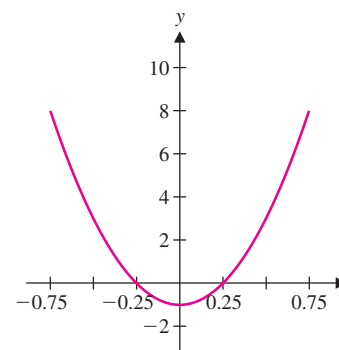


FIGURE 0.75d
 $y = (4x)^2 - 1$.

These graphs look identical until you compare the scales on the y -axes. The values in Figure 0.75b are four times as large, resulting from the multiplication of the original function by 4. The effect looks different when the functions are plotted on the same scale, as in Figure 0.75c. Here, the parabola $y = 4(x^2 - 1)$ looks thinner. Note that the x -intercepts remain the same but the y -intercepts are different. (Why would that be?) The graph of $y = (4x)^2 - 1$ is shown in Figure 0.75d.

Can you spot the difference here? In this case, the x -scale has now changed, by the same factor of 4 as in the function. To see this, note that substituting $x = 1/4$ into $(4x)^2 - 1$ produces $(1)^2 - 1$, exactly the same as substituting $x = 1$ into the original function. When plotted on the same set of axes (as in Figure 0.75e), the parabola $y = (4x)^2 - 1$ looks thinner. Note that here, the x -intercepts are different, but the y -intercepts are the same. (Why would that be?)

We can generalize the observations made in example 6.7. Before reading our explanation, try to state a general rule for yourself. How are the graphs of the functions $cf(x)$ and $f(cx)$ related to the graph of $y = f(x)$?

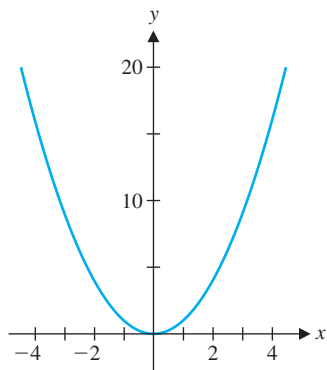


FIGURE 0.76a
 $y = x^2$.

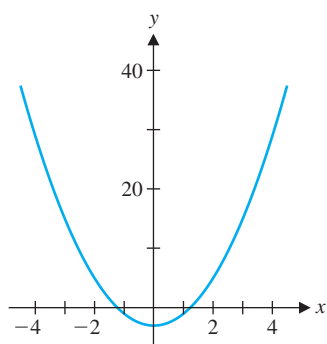


FIGURE 0.76b
 $y = 2x^2 - 3$.

Based on example 6.7, notice that to obtain a graph of $y = cf(x)$ for some constant $c > 0$, you can take the graph of $y = f(x)$ and multiply the scale on the y -axis by c . To obtain a graph of $y = f(cx)$ for some constant $c > 0$, you can take the graph of $y = f(x)$ and multiply the scale on the x -axis by $1/c$.

These basic rules can be combined to understand more complicated graphs.

EXAMPLE 6.8 A Translation and a Stretching

Describe how to get the graph of $y = 2x^2 - 3$ from the graph of $y = x^2$.

Solution You can get from x^2 to $2x^2 - 3$ by multiplying by 2 and then subtracting 3. In terms of the graph, this has the effect of multiplying the y -scale by 2 and then shifting the graph down by 3 units (see the graphs in Figures 0.76a and 0.76b). ■

EXAMPLE 6.9 A Translation Involving Both x and y Scales

Describe how to get the graph of $y = x^2 + 4x + 3$ from the graph of $y = x^2$.

Solution We can again relate this (and the graph of every quadratic) to the graph of $y = x^2$. We must first **complete the square**. Recall that in this process, you take the coefficient of x (4), divide by $2(4/2 = 2)$ and square the result ($2^2 = 4$). Add and subtract this number and then, rewrite the x terms as a perfect square. We have

$$y = x^2 + 4x + 3 = (x^2 + 4x + 4) - 4 + 3 = (x + 2)^2 - 1.$$

To graph this function, take the parabola $y = x^2$ (see Figure 0.77a) and translate the graph 2 units to the left and 1 unit down (see Figure 0.77b).

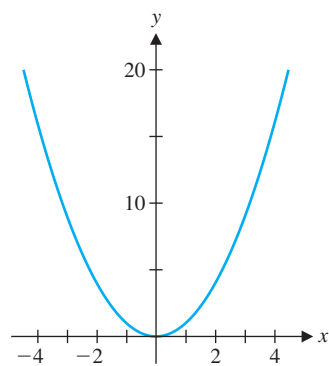


FIGURE 0.77a
 $y = x^2$.

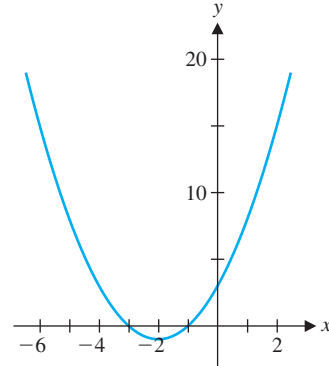


FIGURE 0.77b
 $y = (x + 2)^2 - 1$.

The following table summarizes our discoveries in this section.

Transformations of $f(x)$

Transformation	Form	Effect on Graph
Vertical translation	$f(x) + c$	$ c $ units up ($c > 0$) or down ($c < 0$)
Horizontal translation	$f(x + c)$	$ c $ units left ($c > 0$) or right ($c < 0$)
Vertical scale	$cf(x)$ ($c > 0$)	multiply vertical scale by c
Horizontal scale	$f(cx)$ ($c > 0$)	divide horizontal scale by c

You will explore additional transformations in the exercises.

EXERCISES 0.6

WRITING EXERCISES

- The restricted domain of example 6.2 may be puzzling. Consider the following analogy. Suppose you have an airplane flight from New York to Los Angeles with a stop for refueling in Minneapolis. If bad weather has closed the airport in Minneapolis, explain why your flight will be canceled (or at least rerouted) even if the weather is great in New York and Los Angeles.
- Explain why the graphs of $y = 4(x^2 - 1)$ and $y = (4x)^2 - 1$ in Figures 0.75c and 0.75e appear “thinner” than the graph of $y = x^2 - 1$.
- As illustrated in example 6.9, completing the square can be used to rewrite any quadratic function in the form $a(x - d)^2 + e$. Using the transformation rules in this section, explain why this means that all parabolas (with $a > 0$) will look essentially the same.
- Explain why the graph of $y = f(x + 4)$ is obtained by moving the graph of $y = f(x)$ four units to the left, instead of to the right.

In exercises 1–6, find the compositions $f \circ g$ and $g \circ f$ and identify their respective domains.

- $f(x) = x + 1$, $g(x) = \sqrt{x - 3}$
- $f(x) = x - 2$, $g(x) = \sqrt{x + 1}$
- $f(x) = e^x$, $g(x) = \ln x$
- $f(x) = \sqrt{1 - x}$, $g(x) = \ln x$
- $f(x) = x^2 + 1$, $g(x) = \sin x$
- $f(x) = \frac{1}{x^2 - 1}$, $g(x) = x^2 - 2$

In exercises 7–16, identify functions $f(x)$ and $g(x)$ such that the given function equals $(f \circ g)(x)$.

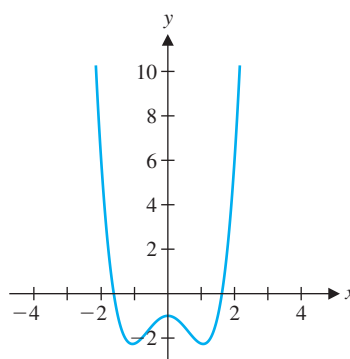
- | | |
|------------------------|-------------------------|
| 7. $\sqrt{x^4 + 1}$ | 8. $\sqrt[3]{x + 3}$ |
| 9. $\frac{1}{x^2 + 1}$ | 10. $\frac{1}{x^2} + 1$ |
| 11. $(4x + 1)^2 + 3$ | 12. $4(x + 1)^2 + 3$ |
| 13. $\sin^3 x$ | 14. $\sin x^3$ |
| 15. $e^{x^2 + 1}$ | 16. $e^{4x - 2}$ |

In exercises 17–22, identify functions $f(x)$, $g(x)$ and $h(x)$ such that the given function equals $[f \circ (g \circ h)](x)$.

- | | |
|-----------------------------------|-----------------------------|
| 17. $\frac{3}{\sqrt{\sin x + 2}}$ | 18. $\sqrt{e^{4x} + 1}$ |
| 19. $\cos^3(4x - 2)$ | 20. $\ln \sqrt{x^2 + 1}$ |
| 21. $4e^{x^2} - 5$ | 22. $[\tan^{-1}(3x + 1)]^2$ |

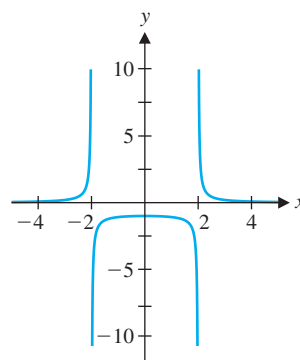
In exercises 23–30, use the graph of $y = f(x)$ given in the figure to graph the indicated function.

- | | | |
|-----------------|-----------------|----------------|
| 23. $f(x) - 3$ | 24. $f(x + 2)$ | 25. $f(x - 3)$ |
| 26. $f(x) + 2$ | 27. $f(2x)$ | 28. $3f(x)$ |
| 29. $4f(x) - 1$ | 30. $3f(x + 2)$ | |



In exercises 31–38, use the graph of $y = f(x)$ given in the figure to graph the indicated function.

- | | | |
|-----------------|-----------------|-------------|
| 31. $f(x - 4)$ | 32. $f(x + 3)$ | 33. $f(2x)$ |
| 34. $f(2x - 4)$ | 35. $f(3x + 3)$ | 36. $3f(x)$ |
| 37. $2f(x) - 4$ | 38. $3f(x) + 3$ | |



In exercises 39–44, complete the square and explain how to transform the graph of $y = x^2$ into the graph of the given function.

- | | |
|----------------------------|----------------------------|
| 39. $f(x) = x^2 + 2x + 1$ | 40. $f(x) = x^2 - 4x + 4$ |
| 41. $f(x) = x^2 + 2x + 4$ | 42. $f(x) = x^2 - 4x + 2$ |
| 43. $f(x) = 2x^2 + 4x + 4$ | 44. $f(x) = 3x^2 - 6x + 2$ |

T In exercises 45–48, graph the given function and compare to the graph of $y = x^2 - 1$.

45. $f(x) = -2(x^2 - 1)$ 46. $f(x) = -3(x^2 - 1)$

47. $f(x) = -3(x^2 - 1) + 2$ 48. $f(x) = -2(x^2 - 1) - 1$

T In exercises 49–52, graph the given function and compare to the graph of $y = (x - 1)^2 - 1 = x^2 - 2x$.

49. $f(x) = (-x)^2 - 2(-x)$

50. $f(x) = (-2x)^2 - 2(-2x)$

51. $f(x) = (-x)^2 - 2(-x) + 1$

52. $f(x) = (-3x)^2 - 2(-3x) - 3$

53. Based on exercises 45–48, state a rule for transforming the graph of $y = f(x)$ into the graph of $y = cf(x)$ for $c < 0$.

54. Based on exercises 49–52, state a rule for transforming the graph of $y = f(x)$ into the graph of $y = f(cx)$ for $c < 0$.

55. Sketch the graph of $y = |x|^3$. Explain why the graph of $y = |x|^3$ is identical to that of $y = x^3$ to the right of the y -axis. For $y = |x|^3$, describe how the graph to the left of the y -axis compares to the graph to the right of the y -axis. In general, describe how to draw the graph of $y = f(|x|)$ given the graph of $y = f(x)$.

56. For $y = x^3$, describe how the graph to the left of the y -axis compares to the graph to the right of the y -axis. Show that for $f(x) = x^3$, we have $f(-x) = -f(x)$. In general, if you have the graph of $y = f(x)$ to the right of the y -axis and $f(-x) = -f(x)$ for all x , describe how to graph $y = f(x)$ to the left of the y -axis.

T 57. **Iterations** of functions are important in a variety of applications. To iterate $f(x)$, start with an initial value x_0 and compute $x_1 = f(x_0)$, $x_2 = f(x_1)$, $x_3 = f(x_2)$ and so on. For example, with $f(x) = \cos x$ and $x_0 = 1$, the **iterates** are $x_1 = \cos 1 \approx 0.54$, $x_2 = \cos x_1 \approx \cos 0.54 \approx 0.86$, $x_3 \approx \cos 0.86 \approx 0.65$ and so on. Keep computing iterates and show that they get closer and closer to 0.739085. Then pick your own x_0 (any number you like) and show that the iterates with this new x_0 also converge to 0.739085.

T 58. Referring to exercise 57, show that the iterates of a function can be written as $x_1 = f(x_0)$, $x_2 = f(f(x_0))$, $x_3 = f(f(f(x_0)))$ and so on. Graph $y = \cos(\cos x)$, $y = \cos(\cos(\cos x))$ and $y = \cos(\cos(\cos(\cos x)))$. The graphs should look more and more like a horizontal line. Use the result of exercise 57 to identify the limiting line.

T 59. Compute several iterates of $f(x) = \sin x$ (see exercise 57) with a variety of starting values. What happens to the iterates in the long run?

T 60. Repeat exercise 59 for $f(x) = x^2$.

T 61. In cases where the iterates of a function (see exercise 57) repeat a single number, that number is called a **fixed point**. Explain why any fixed point must be a solution of the equation $f(x) = x$. Find all fixed points of $f(x) = \cos x$ by solving the equation $\cos x = x$. Compare your results to that of exercise 57.

T 62. Find all fixed points of $f(x) = \sin x$ (see exercise 61). Compare your results to those of exercise 59.

EXPLORATORY EXERCISES

1. You have explored how completing the square can transform any quadratic function into the form $y = a(x - d)^2 + e$. We concluded that all parabolas with $a > 0$ look alike. To see that the same statement is not true of cubic polynomials, graph $y = x^3$ and $y = x^3 - 3x$. In this exercise, you will use completing the cube to determine how many different cubic graphs there are. To see what “completing the cube” would look like, first show that $(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$. Use this result to transform the graph of $y = x^3$ into the graphs of (a) $y = x^3 - 3x^2 + 3x - 1$ and (b) $y = x^3 - 3x^2 + 3x + 2$. Show that you can’t get a simple transformation to $y = x^3 - 3x^2 + 4x - 2$. However, show that $y = x^3 - 3x^2 + 4x - 2$ can be obtained from $y = x^3 + x$ by basic transformations. Show that the following statement is true: any cubic ($y = ax^3 + bx^2 + cx + d$) can be obtained with basic transformations from $y = ax^3 + kx$ for some constant k .

2. In many applications, it is important to take a section of a graph (e.g., some data) and extend it for predictions or other analysis. For example, suppose you have an electronic signal equal to $f(x) = 2x$ for $0 \leq x \leq 2$. To predict the value of the signal at $x = -1$, you would want to know whether or not the signal was periodic. If the signal is periodic, explain why $f(-1) = 2$ would be a good prediction. In some applications, you would assume that the function is *even*. That is, $f(x) = f(-x)$ for all x . In this case, you want $f(x) = 2(-x) = -2x$ for $-2 \leq x \leq 0$.

Graph the *even extension* $f(x) = \begin{cases} -2x & \text{if } -2 \leq x \leq 0 \\ 2x & \text{if } 0 \leq x \leq 2 \end{cases}$.

Find the even extension for (a) $f(x) = x^2 + 2x + 1$, $0 \leq x \leq 2$ and (b) $f(x) = e^{-x}$, $0 \leq x \leq 2$.

3. Similar to the even extension discussed in exploratory exercise 2, applications sometimes require a function to be *odd*; that is, $f(-x) = -f(x)$. For $f(x) = x^2$, $0 \leq x \leq 2$, the odd extension requires that for $-2 \leq x \leq 0$, $f(x) = -f(-x) = -(-x)^2 = -x^2$ so that $f(x) = \begin{cases} -x^2 & \text{if } -2 \leq x \leq 0 \\ x^2 & \text{if } 0 \leq x \leq 2 \end{cases}$. Graph $y = f(x)$ and discuss how to graphically rotate the right half of the graph to get the left half of the graph. Find the odd extension for (a) $f(x) = x^2 + 2x$, $0 \leq x \leq 2$ and (b) $f(x) = e^{-x} - 1$, $0 \leq x \leq 2$.

0.7 PARAMETRIC EQUATIONS AND POLAR COORDINATES



RoboCup is an international competition of robot soccer teams. These are not the remote-controlled attack robots you may have seen on television. Instead, the robots are engineered and programmed to automatically respond to the positions of the ball, goal and other players. As programmers of such robots, we would be frustrated by the limitations of the x - y system (called **rectangular coordinates**) we have used in the first six sections of this chapter. First of all, the path followed by a robot is unlikely to pass the vertical line test for a function. Even if it did, the path of the robot is irrelevant unless we know *when* the robot was at each point. Further, what a robot needs to know about a competitor are its distance and direction, not its x - and y -coordinates.

In this section, we introduce two alternatives to functions in rectangular coordinates. Parametric equations will enable us to track the position of an object as a function of time. Polar coordinates enable us to keep track of points in terms of distance and direction. These alternative descriptions give us a great deal of needed flexibility for attacking a variety of problems. Many complicated looking graphs have a very simple description in parametric equations or polar coordinates.

□ Parametric Equations

To describe the position of an object (such as a robot) in terms of time, we need functions $g(t)$ and $h(t)$ that give the object's x - and y -coordinates, respectively, at any given time t . So, from this description, we not only know the path the object follows, but we also know *when* it passes through each point. This is the motivation for the following general definition of parametric equations.

Given any pair of functions $g(t)$ and $h(t)$ defined on the same domain D , the equations

$$x = g(t), \quad y = h(t)$$

are called **parametric equations**. Notice that for each choice of t , the parametric equations specify a point $(x, y) = (g(t), h(t))$ in the xy -plane. The collection of all such points is called the **graph** of the parametric equations. The graph is a curve in the xy -plane, referred to as a **parametric curve**.

The choice of the letter t to denote the independent variable (called the **parameter**) should make you think of *time*. In many situations, this is exactly what the parameter represents. In some applications, the parameter has an interpretation other than time; in others, it has no physical meaning at all. In general, the parameter can be *any* quantity that is convenient for describing the relationship between x and y . In our first example, we can simplify our discussion by eliminating the parameter.

EXAMPLE 7.1 Graphing a Parametric Curve

Sketch the curve defined by the parametric equations $x = 6 - t^2$, $y = t/2$, for $-2 \leq t \leq 4$.

Solution In the table that follows, we list a number of values of the parameter t and the corresponding values of x and y .

t	-2	-1	0	1	2	3	4
x	2	5	6	5	2	-3	-10
y	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2

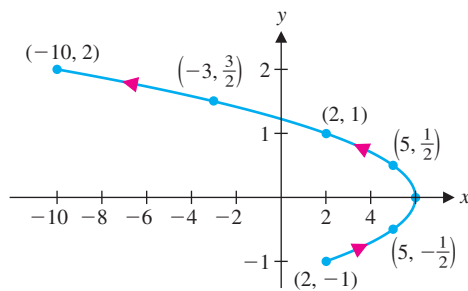


FIGURE 0.78

$$x = 6 - t^2, y = \frac{t}{2}, -2 \leq t \leq 4.$$

We have plotted these points and connected them with a smooth curve in Figure 0.78. You might also notice that we can easily eliminate the parameter here, by solving for t in terms of y . We have $t = 2y$, so that $x = 6 - 4y^2$. The graph of this last equation is a parabola opening to the left. The plane curve we're looking for is the portion of this parabola corresponding to $-2 \leq t \leq 4$. From the table, notice that this corresponds to $-1 \leq y \leq 2$, so that the plane curve is the portion of the parabola indicated in Figure 0.78, where we have also indicated a number of points on the curve. ■

You probably noticed the small arrows drawn on top of the plane curve in Figure 0.78. These indicate the **orientation** of the curve (i.e., the direction of increasing t). In the case where t represents time and the curve represents the path of an object, notice that the orientation indicates the direction followed by the object as it traverses the path, as in the following example.

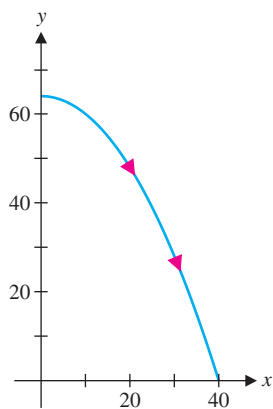


FIGURE 0.79
Path of projectile.

EXAMPLE 7.2 The Path of a Projectile

The path of a projectile thrown horizontally with initial speed of 20 ft/s from a height of 64 feet, has parametric equations $x = 20t$, $y = 64 - 16t^2$, for $0 \leq t \leq 2$. Sketch the path.

Solution As in example 7.1, we can solve for t , to get $t = x/20$, so that $y = 64 - \frac{x^2}{25}$.

With $0 \leq t \leq 2$, we get $0 \leq x \leq 40$. The portion of the parabola $y = 64 - \frac{x^2}{25}$ with $0 \leq x \leq 40$ is shown in Figure 0.79. Note that in this case, the orientation indicated in the graph gives the direction of motion. Although we used the x - y equation for graphing, observe that the parametric equations also tell us *when* the object is located at a given point and indicates the direction of motion. ■

Graphing calculators and computer algebra systems sketch a parametric graph by plotting points corresponding to a large number of values of the parameter t and then connecting the plotted points with a smooth curve. The appearance of the resulting graph depends greatly on the graphing window used *and also* on the particular choice of t -values. This can be seen in the following example.



EXAMPLE 7.3 Parametric Equations Involving Sines and Cosines

Sketch the curve defined by the parametric equations

$$x = 2 \cos t, y = 2 \sin t, \quad \text{for (a) } 0 \leq t \leq 2\pi \quad \text{and} \quad \text{(b) } 0 \leq t \leq \pi \quad (7.1)$$

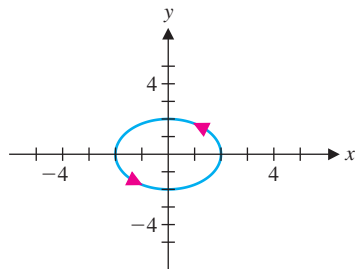


FIGURE 0.80a
 $x = 2 \cos t, y = 2 \sin t$.

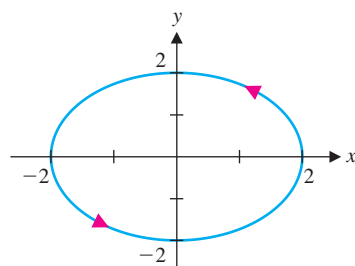


FIGURE 0.80b
 $x = 2 \cos t, y = 2 \sin t$.

REMARK 7.1

To sketch a parametric graph on a CAS, you may need to write the equations in **vector format**. For instance, in the case of example 7.3, instead of entering $x = 2 \cos t$ and $y = 2 \sin t$, you would enter the ordered pair of functions $(2 \cos t, 2 \sin t)$.

Solution The default graph produced by most graphing calculators looks something like the curve shown in Figure 0.80a (where we have added arrows indicating the orientation). With some thought, we can improve this sketch. First, notice that since $x = 2 \cos t$, x ranges between -2 and 2 . Similarly, y ranges between -2 and 2 . Changing the graphing window to $-2.1 \leq x \leq 2.1$ and $-2.1 \leq y \leq 2.1$ produces the curve shown in Figure 0.80b. The curve still looks like an ellipse, but with some more thought we can identify it as a circle. Rather than eliminate the parameter by solving for t in terms of either x or y , instead notice from (7.1) that

$$x^2 + y^2 = 4 \cos^2 t + 4 \sin^2 t = 4(\cos^2 t + \sin^2 t) = 4.$$

So, the plane curve lies on the circle of radius 2 centered at the origin. In fact, it's the whole circle, as we can see by recognizing what the parameter represents in this case. Recall from the definition of sine and cosine that if (x, y) is a point on the unit circle and θ is the angle from the positive x -axis to the line segment joining (x, y) and the origin, then we define $\cos \theta = x$ and $\sin \theta = y$. Since we have $x = 2 \cos t$ and $y = 2 \sin t$, the parameter t corresponds to the angle θ . Further, the curve is the entire circle of radius 2, traced out as the angle t ranges from 0 to 2π . A “square” graphing window is one with the same scale on the x - and y -axes (not necessarily the same x and y ranges, but the same scales!). Some programs generate such a window when you specify an **aspect ratio** of 1. Such a square window gives us the circle seen in Figure 0.80b. Finally, what would change if the domain were limited to $0 \leq t \leq \pi$? Since we've identified t as the angle as measured from the positive x -axis, it should be clear that you will now get the top half of the circle of radius 2, as shown in Figure 0.80d.

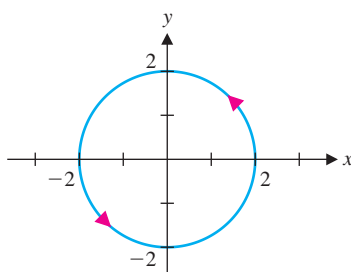


FIGURE 0.80c
 A circle.

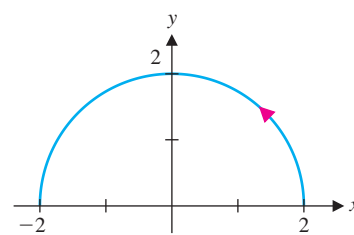


FIGURE 0.80d
 Top semicircle.

Simple modifications to the parametric equations in example 7.3 will produce a variety of circles and ellipses. We explore this in example 7.4 and the exercises.

EXAMPLE 7.4 More Circles and Ellipses Defined by Parametric Equations

Identify the plane curves (a) $x = 2 \cos t, y = 3 \sin t$, (b) $x = 2 + 4 \cos t, y = 3 + 4 \sin t$ and (c) $x = 3 \cos 2t, y = 3 \sin 2t$, all for $0 \leq t \leq 2\pi$.

Solution A computer-generated sketch of (a) is shown in Figure 0.81a. It's difficult to determine from the sketch whether the curve is an ellipse or simply a distorted graph of a circle. You should rule out a circle, since the parametric equations produce x -values between -2 and 2 and y -values between -3 and 3 . To verify that this is an ellipse, notice that

$$\frac{x^2}{4} + \frac{y^2}{9} = \frac{4 \cos^2 t}{4} + \frac{9 \sin^2 t}{9} = \cos^2 t + \sin^2 t = 1.$$

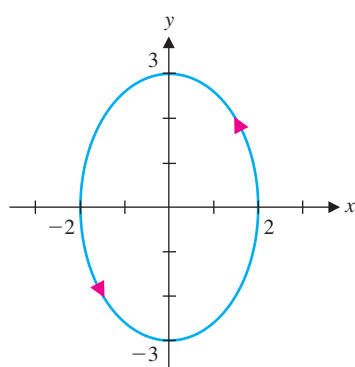


FIGURE 0.81a
 $x = 2 \cos t, y = 3 \sin t$.

A computer-generated sketch of (b) is shown in Figure 0.81b. You should verify that this is the circle $(x - 2)^2 + (y - 3)^2 = 16$. Finally, a computer sketch of (c) is shown in Figure 0.81c. You should verify that this is the circle $x^2 + y^2 = 9$. So, what is the role of the 2 in the argument of cosine and sine? If you sketched this on a calculator, you may have noticed that the circle was completed long before the calculator finished graphing. Because of the 2, a complete circle corresponds to $0 \leq 2t \leq 2\pi$ or $0 \leq t \leq \pi$. With the domain $0 \leq t \leq 2\pi$, the circle is traced out twice. You might say that the factor of 2 in the argument doubles the speed with which the curve is traced.

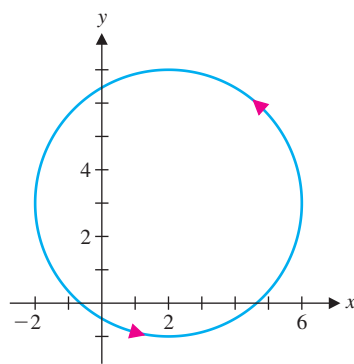


FIGURE 0.81b
 $x = 2 + 4 \cos t, y = 3 + 4 \sin t.$

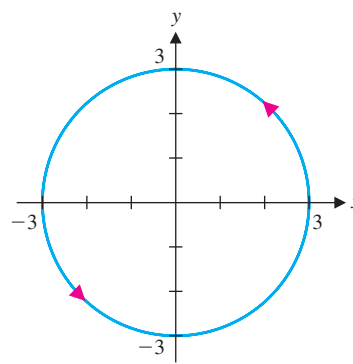


FIGURE 0.81c
 $x = 3 \cos 2t, y = 3 \sin 2t.$

REMARK 7.2

Look carefully at the plane curves in examples 7.3 and 7.4 until you can identify the roles of each of the constants in the equations $x = a + b \cos ct, y(t) = d + e \sin ct$. These interpretations are important in applications. In particular, (a, d) is the center of an ellipse with axes of length $2b$ and $2e$.

In example 7.5, we see how to find parametric equations for a line segment.

EXAMPLE 7.5 Parametric Equations for a Line Segment

Find parametric equations for the line segment joining the points $(1, 2)$ and $(4, 7)$.

Solution For a line segment, notice that the parametric equations can be linear functions. That is,

$$x = a + bt, \quad y = c + dt,$$

for some constants a, b, c and d . (Eliminate the parameter t to see why this generates a line.) The simplest way to choose these constants is to have $t = 0$ correspond to the starting point $(1, 2)$. Note that if $t = 0$, the equations reduce to $x = a$ and $y = c$. To start our segment at $x = 1$ and $y = 2$, we set $a = 1$ and $c = 2$. Now note that with $t = 1$, the equations are $x = a + b$ and $y = c + d$. To produce the endpoint $(4, 7)$, we must have $a + b = 4$ and $c + d = 7$. With $a = 1$ and $c = 2$, solve to get $b = 3$ and $d = 5$. We now have that

$$x = 1 + 3t, \quad y = 2 + 5t, \quad \text{for } 0 \leq t \leq 1$$

is a pair of parametric equations describing the line segment. ■

In general, for parametric equations of the form $x = a + bt, y = c + dt$, notice that you can always choose a and c to be the x - and y -coordinates, respectively, of the starting point (since $x = a, y = c$ corresponds to $t = 0$). Then b is the difference in x -coordinates (endpoint minus starting point) and d is the difference in y -coordinates. With these choices, the line segment is always sketched out for $0 \leq t \leq 1$.

REMARK 7.3

Note that in example 7.5, as for all parametric equations, there are an infinite number of choices of parameters that produce a given curve. For instance, you can easily verify that the parametric equations

$$x = -2 + 3t, \quad y = -3 + 5t, \quad \text{for } 1 \leq t \leq 2$$

and

$$x = t, \quad y = \frac{1 + 5t}{3}, \quad \text{for } 1 \leq t \leq 4$$

also produce the line segment of example 7.5. We say that each of these pairs of parametric equations is a different **parameterization** of the curve.

As we illustrate with example 7.6, every equation of the form $y = f(x)$ can be simply written in parametric equations.

EXAMPLE 7.6 Parametric Equations from an x - y Equation

Find parametric equations for the portion of the parabola $y = x^2$ from $(-1, 1)$ to $(3, 9)$.

Solution Any equation of the form $y = f(x)$ can be converted to parametric form by simply letting x equal t . Here, this gives us $y = x^2 = t^2$. Then

$$x = t, \quad y = t^2, \quad \text{for } -1 \leq t \leq 3,$$

is a parametric representation of the curve. (Of course, you can use the letter x as the parameter instead of the letter t if you prefer.) ■

We can describe many plane curves parametrically that are unlike anything you've seen so far in your studies. Many of these are difficult to draw by hand, but can be easily plotted with a graphing calculator or CAS.

EXAMPLE 7.7 Some Unusual Plane Curves

Sketch the plane curves (a) $x = t^2 - 2, y = t^3 - t$ and (b) $x = t^3 - t, y = t^4 - 5t^2 + 4$.

Solution A sketch of (a) is shown in Figure 0.82a. This is clearly not the graph of any function (consider the vertical line test to see why), so converting to an x - y equation would not be particularly helpful. You should, however, examine the parametric equations to see if important portions of the graph have been left out (e.g., is there supposed to be anything to the left of $x = -2$?). In this case, $x = t^2 - 2 \geq -2$ for all t and $y = t^3 - t$ has no maximum or minimum (think about why). It seems that most of the graph is indeed shown in Figure 0.82a. A computer sketch of (b) is shown in Figure 0.82b. Again, this is not the graph of a function. To get an idea of the scope of the graph, note that $x = t^3 - t$ has no maximum or minimum. To find the minimum of $y = t^4 - 5t^2 + 4$, note that

$$t^4 - 5t^2 + 4 = (t^2 - 5/2)^2 - \frac{9}{4}.$$

You should conclude that $y \geq -\frac{9}{4}$, as indicated in Figure 0.82b. ■

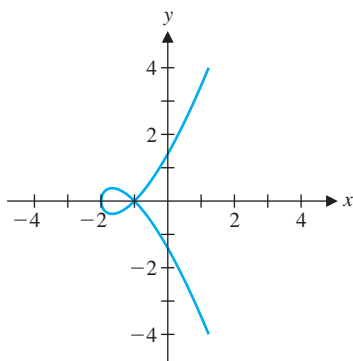


FIGURE 0.82a
 $x = t^2 - 2, y = t^3 - t.$

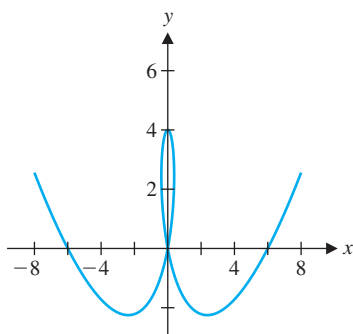


FIGURE 0.82b
 $x = t^3 - t, y = t^4 - 5t^2 + 4.$

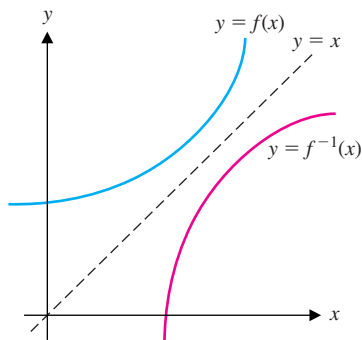


FIGURE 0.83a
 $y = f(x)$ and $y = f^{-1}(x)$.

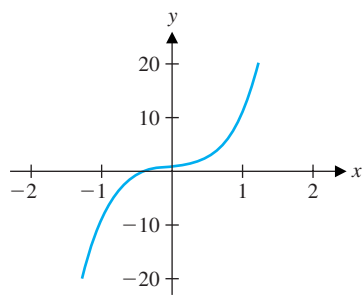


FIGURE 0.83b
 $y = x^5 + 8x^3 + x + 1$.

In section 0.3, we found the relationship between graphs of one-to-one functions and their inverses. Recall that you can reflect the graph of $y = f(x)$ across the line $y = x$ to get the graph of $y = f^{-1}(x)$. (See Figure 0.83a.) Using parametric equations, this can be easily implemented on a graphing calculator.

EXAMPLE 7.8 Drawing the Graph of an Unknown Inverse Function

Draw a graph of the inverse function of $f(x) = x^5 + 8x^3 + x + 1$.

Solution Given the graph of $f(x)$ (see Figure 0.83b), we could certainly draw by hand part of the graph of the inverse. However, recall that if (x, y) is a point on the graph of $f(x)$ [i.e., $y = f(x)$], then (y, x) is a point on the graph of $f^{-1}(x)$ [i.e., $x = f^{-1}(y)$]. The simplest choice of parametric equations for $y = f(x)$ is

$$\begin{aligned} x &= t \\ y &= f(t). \end{aligned}$$

Swapping x and y now gives parametric equations for $f^{-1}(x)$, namely,

$$\begin{aligned} x &= f(t) \\ y &= t. \end{aligned}$$

In this case, we want $x = t^5 + 8t^3 + t + 1$ and $y = t$. The graph of $f^{-1}(x)$ along with the graphs of $f(x)$ and $y = x$ are shown in Figures 0.83c and 0.83d.

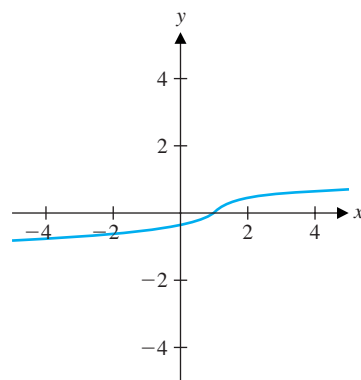


FIGURE 0.83c
 $y = f^{-1}(x)$.

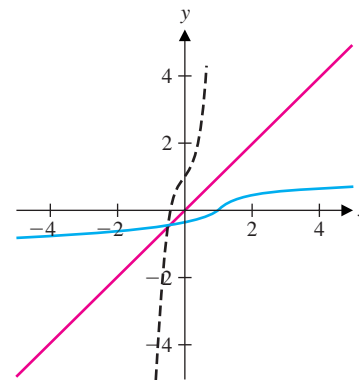


FIGURE 0.83d
 $y = f(x)$ and $y = f^{-1}(x)$.

You should now have some idea of the flexibility of parametric equations. This flexibility alone increases our ability to solve problems and makes the study of parametric equations worthwhile.

Polar Coordinates

The search for a more convenient description of a graph leads us to some important ideas. Recall that in our discussion of robot soccer, we observed that in the heat of the action, information about the distance and direction to the goal is more important than the x - and y -coordinates of the goal. This kind of description is the idea behind polar coordinates.

You are familiar with the rectangular coordinates (x, y) of a point, where we identify a point by its horizontal displacement x and vertical displacement y from the origin.

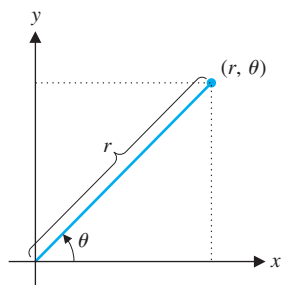


FIGURE 0.84
Polar coordinates.

An alternative description of a point in the xy -plane consists of specifying the distance r from the point to the origin and the angle θ (in radians) measured from the positive x -axis counterclockwise to the ray connecting the point and the origin (see Figure 0.84). We describe the point by the ordered pair (r, θ) and refer to r and θ as **polar coordinates** for the point.

EXAMPLE 7.9 Converting from Polar to Rectangular Coordinates

Plot the points with the indicated polar coordinates and determine the corresponding rectangular coordinates (x, y) , for: (a) $(2, 0)$, (b) $(3, \frac{\pi}{2})$, (c) $(-3, \frac{\pi}{2})$ and (d) $(2, \pi)$.

Solution (a) Notice that the angle $\theta = 0$ locates the point on the positive x -axis. At a distance of $r = 2$ units from the origin, this corresponds to the point $(2, 0)$ in rectangular coordinates (see Figure 0.85a).

(b) The angle $\theta = \frac{\pi}{2}$ locates points on the positive y -axis. At a distance of $r = 3$ units from the origin, this corresponds to the point $(0, 3)$ in rectangular coordinates (see Figure 0.85b).

(c) The angle is the same as in (b), but a negative value of r indicates that the point is located 3 units in the opposite direction, at the point $(0, -3)$ in rectangular coordinates (see Figure 0.85b).

(d) The angle $\theta = \pi$ corresponds to the negative x -axis. The distance of $r = 2$ units from the origin gives us the point $(-2, 0)$ in rectangular coordinates (see Figure 0.85c).

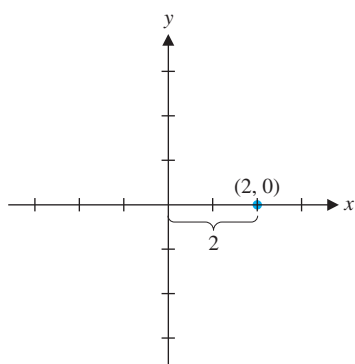


FIGURE 0.85a
The point $(2, 0)$ in polar coordinates.

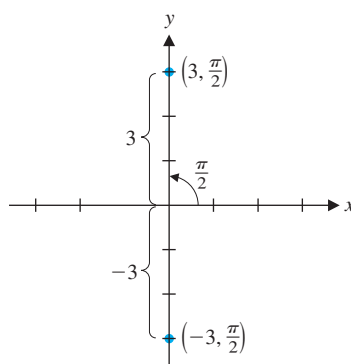


FIGURE 0.85b
The points $(3, \frac{\pi}{2})$ and $(-3, \frac{\pi}{2})$ in polar coordinates.

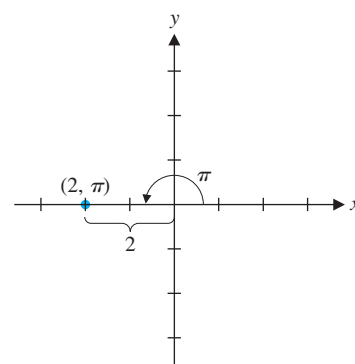


FIGURE 0.85c
The point $(2, \pi)$ in polar coordinates.

EXAMPLE 7.10 Converting from Rectangular to Polar Coordinates

Find all polar coordinate representations of the rectangular point $(1, 1)$.

Solution From Figure 0.86a (on the following page), notice that the point lies on the line $y = x$, which makes an angle of $\frac{\pi}{4}$ with the positive x -axis. From the distance formula, we get that $r = \sqrt{1^2 + 1^2} = \sqrt{2}$. This says that we can write the point as $(\sqrt{2}, \frac{\pi}{4})$ in polar coordinates. Referring to Figure 0.86b, notice that we can specify the same point by using a negative value of r , $r = -\sqrt{2}$ with the angle $\frac{5\pi}{4}$. (Think about this some.) Notice further, that the angle $\frac{9\pi}{4} = \frac{\pi}{4} + 2\pi$ corresponds to the same ray shown in Figure 0.86a

(see Figure 0.86c). In fact, all of the polar points $(\sqrt{2}, \frac{\pi}{4} + 2n\pi)$ and $(-\sqrt{2}, \frac{5\pi}{4} + 2n\pi)$ for any integer n correspond to the same point in the xy -plane.

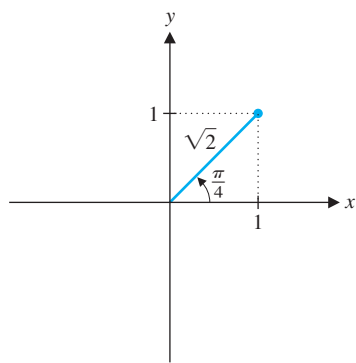


FIGURE 0.86a
Polar coordinates for the point $(1, 1)$.

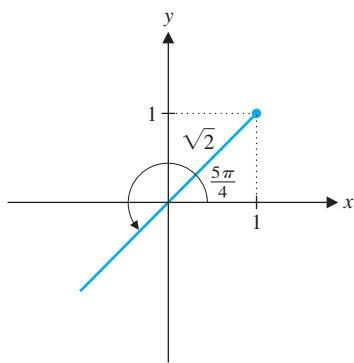


FIGURE 0.86b
An alternative polar representation of $(1, 1)$.

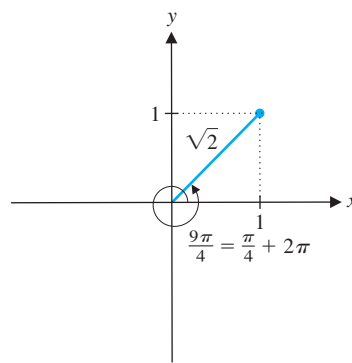


FIGURE 0.86c
Another polar representation of the point $(1, 1)$.

REMARK 7.4

Given a distance r and angle θ , there is exactly one point in the xy -plane with the polar coordinates (r, θ) . However, as we saw in example 7.10, for a given point (x, y) in the plane, there are an infinite number of possible polar coordinate representations. In particular, you can use both positive and negative values of r . Also, for a given angle θ , the angles $\theta \pm 2\pi$, $\theta \pm 4\pi$ and so on, all correspond to the same ray and can also be used. For convenience, we use the notation $\theta + 2n\pi$ (for any integer n) to represent all of these possible angles.

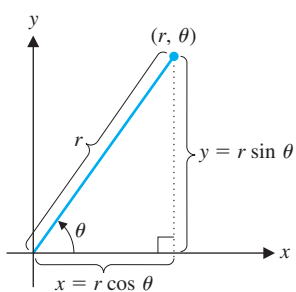


FIGURE 0.87
Converting from polar to rectangular coordinates.

Referring to Figure 0.87, notice that it is a simple matter to find the rectangular coordinates (x, y) of a point specified in polar coordinates as (r, θ) . From the usual definitions for $\sin \theta$ and $\cos \theta$, we get

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta. \quad (7.2)$$

As we've already observed, a given point (x, y) in the plane will have infinitely many polar coordinate representations. From equations (7.2), notice that

$$x^2 + y^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2(\cos^2 \theta + \sin^2 \theta) = r^2$$

and for $x \neq 0$,

$$\frac{y}{x} = \frac{r \sin \theta}{r \cos \theta} = \frac{\sin \theta}{\cos \theta} = \tan \theta.$$

That is, every polar coordinate representation (r, θ) of the point (x, y) , where $x \neq 0$ must satisfy

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan \theta = \frac{y}{x}. \quad (7.3)$$

Notice that since there's more than one choice of r and θ , we cannot actually solve equations (7.3) to produce formulas for r and θ . In particular, while you might be tempted to write $\theta = \tan^{-1}(\frac{y}{x})$, this is not the only possible choice. Remember that for (r, θ) to be a polar representation of the point (x, y) , θ can be *any* angle for which $\tan \theta = \frac{y}{x}$, while $\tan^{-1}(\frac{y}{x})$ gives you only one angle θ in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Finding polar coordinates for a given point is typically a process involving some graphing and some thought.

EXAMPLE 7.11 Converting from Rectangular to Polar Coordinates

Find all polar coordinate representations for the rectangular points (a) $(2, 3)$ and (b) $(-3, 1)$.

Solution (a) With $x = 2$ and $y = 3$, we have from (7.3) that

$$r^2 = x^2 + y^2 = 2^2 + 3^2 = 13,$$

so that $r = \pm\sqrt{13}$. Also,

$$\tan \theta = \frac{y}{x} = \frac{3}{2}.$$

One solution of this (the most obvious solution) is $\theta = \tan^{-1}(\frac{3}{2}) \approx 0.98$ radians. To determine which choice of r corresponds to this angle, note that $(2, 3)$ is located in the first quadrant (see Figure 0.88a). Since 0.98 radians also puts you in the first quadrant, this angle corresponds to the positive value of r , so that $(\sqrt{13}, \tan^{-1}(\frac{3}{2}))$ is one polar representation of the point. The negative choice of r corresponds to an angle one half-circle (i.e., π radians), away (see Figure 0.88b), so that another representation is $(-\sqrt{13}, \tan^{-1}(\frac{3}{2}) + \pi)$. Every other polar representation is found by adding multiples of 2π to the two angles used above. That is, every polar representation of the point $(2, 3)$ must have the form $(\sqrt{13}, \tan^{-1}(\frac{3}{2}) + 2n\pi)$ or $(-\sqrt{13}, \tan^{-1}(\frac{3}{2}) + 2n\pi)$, for some integer choice of n .

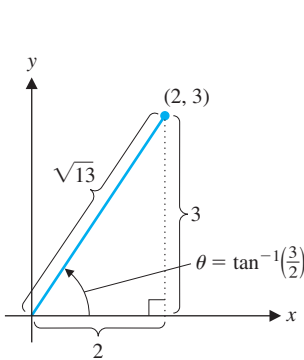


FIGURE 0.88a
The point $(2, 3)$.

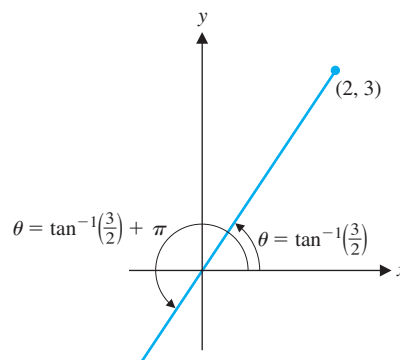


FIGURE 0.88b
Negative value of r .

(b) For the point $(-3, 1)$, we have $x = -3$ and $y = 1$. From (7.3), we have

$$r^2 = x^2 + y^2 = (-3)^2 + 1^2 = 10,$$

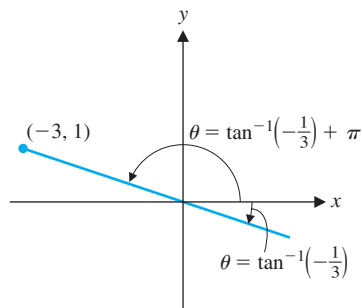


FIGURE 0.89
The point $(-3, 1)$.

so that $r = \pm\sqrt{10}$. Further,

$$\tan \theta = \frac{y}{x} = \frac{1}{-3},$$

so that the most obvious choice for the polar angle is $\theta = \tan^{-1}(-\frac{1}{3}) \approx -0.32$, which lies in the fourth quadrant. Since the point $(-3, 1)$ is in the second quadrant, this choice of the angle corresponds to the negative value of r (see Figure 0.89). The positive value of r then corresponds to the angle $\theta = \tan^{-1}(-\frac{1}{3}) + \pi$. Observe that all polar coordinate representations must then be of the form $(-\sqrt{10}, \tan^{-1}(-\frac{1}{3}) + 2n\pi)$ or $(\sqrt{10}, \tan^{-1}(-\frac{1}{3}) + \pi + 2n\pi)$, for some integer choice of n .

Observe that the conversion from polar coordinates to rectangular coordinates is completely straightforward, as in example 7.12.

REMARK 7.5

Notice that for any point (x, y) specified in rectangular coordinates ($x \neq 0$), we can always write the point in polar coordinates using either of the polar angles $\tan^{-1}(\frac{y}{x})$ or $\tan^{-1}(\frac{y}{x}) + \pi$. You can determine which angle corresponds to $r = \sqrt{x^2 + y^2}$ and which corresponds to $r = -\sqrt{x^2 + y^2}$ by looking at the quadrant in which the point lies.

EXAMPLE 7.12 Converting from Polar to Rectangular Coordinates

Find the rectangular coordinates for the polar points (a) $(3, \frac{\pi}{6})$ and (b) $(-2, 3)$.

Solution For (a), we have from (7.2) that

$$x = r \cos \theta = 3 \cos \frac{\pi}{6} = \frac{3\sqrt{3}}{2}$$

and

$$y = r \sin \theta = 3 \sin \frac{\pi}{6} = \frac{3}{2}.$$

The rectangular point is then $(\frac{3\sqrt{3}}{2}, \frac{3}{2})$. For (b), we have

$$x = r \cos \theta = -2 \cos 3 \approx 1.98$$

and

$$y = r \sin \theta = -2 \sin 3 \approx -0.28.$$

The rectangular point is $(-2 \cos 3, -2 \sin 3)$, which is located at approximately $(1.98, -0.28)$.

The **graph** of a polar equation $r = f(\theta)$ is the set of all points (x, y) for which $x = r \cos \theta$, $y = r \sin \theta$ and $r = f(\theta)$. In other words, the graph of a polar equation is a graph in the xy -plane of all those points whose polar coordinates satisfy the given equation. We begin by sketching two very simple (and familiar) graphs. The key to drawing the graph of a polar equation is to always keep in mind what the polar coordinates represent.

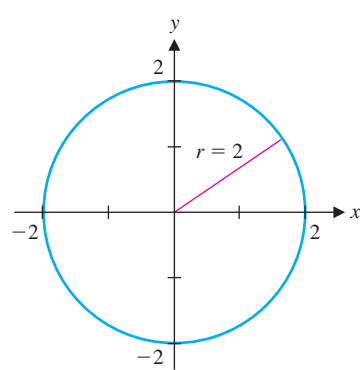


FIGURE 0.90a
The circle $r = 2$.

EXAMPLE 7.13 Some Simple Graphs in Polar Coordinates

Sketch the graphs of (a) $r = 2$ and (b) $\theta = \pi/3$.

Solution For (a), notice that $2 = r = \sqrt{x^2 + y^2}$, and so, we want all points whose distance from the origin is 2 (with any polar angle θ). Of course, this is the definition of a circle of radius 2 with center at the origin (see Figure 0.90a). For (b), notice that $\theta = \pi/3$ specifies all points with a polar angle of $\pi/3$ from the positive x -axis (at any

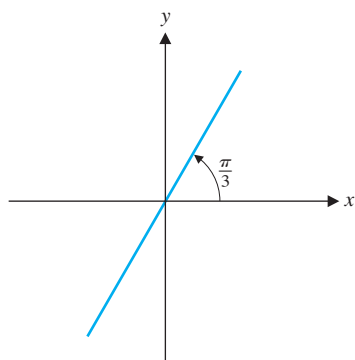


FIGURE 0.90b
The line $\theta = \frac{\pi}{3}$.

distance r from the origin). Including negative values for r , this defines a line with slope $\tan \pi/3 = \sqrt{3}$ (see Figure 0.90b).

REMARK 7.6

Calculators sketch polar graphs of $r = f(\theta)$ by computing the value of $f(\theta)$ for numerous values of θ at regularly spaced values of θ and then plotting the resulting points (x, y) . You should be aware that the appearance of a calculator plot depends on the x - y graphing window specified and also on the range of displayed values of θ . When drawing polar graphs, you should identify any values of θ corresponding to $r = 0$ or to where r reaches a maximum or minimum. In addition, you should identify the range of values of θ that produces one copy of the polar curve, when this is appropriate.

It turns out that many familiar curves have simple polar equations.

EXAMPLE 7.14 A Surprisingly Simple Polar Graph

Sketch the graph of the polar equation $r = \sin \theta$.

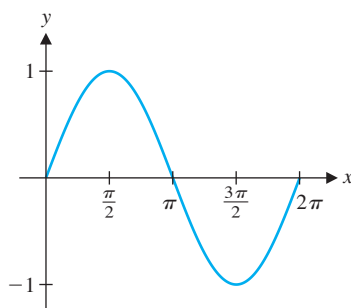


FIGURE 0.91
 $y = \sin x$ plotted in *rectangular* coordinates.

Solution For reference, we first sketch a graph of the sine function in **rectangular** coordinates on the interval $[0, 2\pi]$ (see Figure 0.91). Notice that on the interval $0 \leq \theta \leq \frac{\pi}{2}$, $\sin \theta$ increases from 0 to its maximum value of 1. Then, on the interval $\frac{\pi}{2} \leq \theta \leq \pi$, $\sin \theta$ decreases from 1 to 0. When plotting the polar graph, keep in mind that $r = 0$ corresponds to the origin. Next, on the interval $\pi \leq \theta \leq \frac{3\pi}{2}$, $\sin \theta$ decreases from 0 to its minimum value of -1 . Since the values of r are negative, remember that this means that the points plotted are in the *opposite* quadrant (i.e., the first quadrant). Notice that this traces out the same curve in the first quadrant as we've already drawn for $0 \leq \theta \leq \frac{\pi}{2}$. Likewise, taking θ in the interval $\frac{3\pi}{2} \leq \theta \leq 2\pi$ retraces the portion of the curve in the second quadrant. Since $\sin \theta$ is periodic of period 2π , taking further values of θ simply retraces portions of the curve that we have already traced. A sketch of the polar graph is shown in Figure 0.92. We now verify that this curve is actually a circle. Notice that if we multiply the equation $r = \sin \theta$ through by r , we get

$$r^2 = r \sin \theta.$$

You should immediately recognize from (7.2) and (7.3) that $y = r \sin \theta$ and $r^2 = x^2 + y^2$. This gives us the rectangular equation

$$x^2 + y^2 = y$$

or

$$0 = x^2 + y^2 - y.$$

Completing the square, we get

$$0 = x^2 + \left(y^2 - y + \frac{1}{4}\right) - \frac{1}{4}$$

and, adding $\frac{1}{4}$ to both sides,

$$\left(\frac{1}{2}\right)^2 = x^2 + \left(y - \frac{1}{2}\right)^2.$$

This is the rectangular equation for the circle of radius $\frac{1}{2}$ centered at the point $(0, \frac{1}{2})$, which is what we see in Figure 0.92.

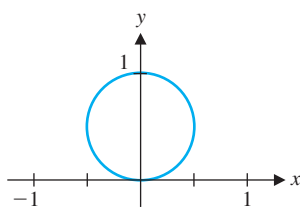


FIGURE 0.92
The circle $r = \sin \theta$.

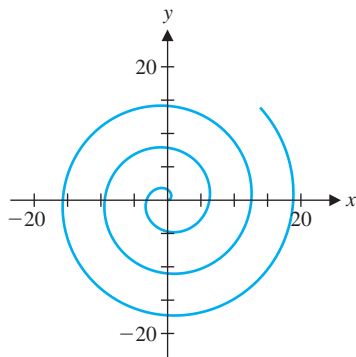


FIGURE 0.93
The spiral $r = \theta$, $\theta \geq 0$.

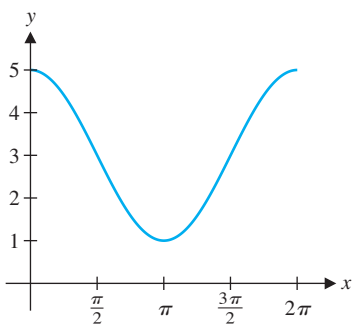


FIGURE 0.94
 $y = 3 + 2 \cos x$ in rectangular coordinates.

The graphs of many polar equations are not the graphs of *any* functions of the form $y = f(x)$, as in example 7.15.

EXAMPLE 7.15 An Archimedean Spiral

Sketch the graph of the polar equation $r = \theta$, for $\theta \geq 0$.

Solution Notice that here, as θ increases, so too does r . That is, as the polar angle increases, the distance from the origin also increases accordingly. This produces the spiral (an example of an **Archimedean spiral**) seen in Figure 0.93.

The graph in example 7.16 is in the general class known as **limaçons**. This class of graphs is defined by $r = a \pm b \sin \theta$ or $r = a \pm b \cos \theta$ for positive constants a and b . Additional graphs of limaçons are shown in Appendix A.

EXAMPLE 7.16 A Limaçon

Sketch the graph of the polar equation $r = 3 + 2 \cos \theta$.

Solution We begin by sketching the graph of $y = 3 + 2 \cos x$ in rectangular coordinates on the interval $[0, 2\pi]$ to use as a reference (see Figure 0.94). Notice that in this case, we have $r = 3 + 2 \cos \theta > 0$ for all values of θ . Further, the maximum value of r is 5 (corresponding to when $\cos \theta = 1$ at $\theta = 0, 2\pi$, etc.) and the minimum value of r is 1 (corresponding to when $\cos \theta = -1$ at $\theta = \pi, 3\pi$, etc.). In this case, the graph is traced out with $0 \leq \theta \leq 2\pi$. We summarize the intervals of increase and decrease for r in the following table.

Interval	$\cos \theta$	$r = 3 + 2 \cos \theta$
$[0, \frac{\pi}{2}]$	Decreases from 1 to 0	Decreases from 5 to 3
$[\frac{\pi}{2}, \pi]$	Decreases from 0 to -1	Decreases from 3 to 1
$[\pi, \frac{3\pi}{2}]$	Increases from -1 to 0	Increases from 1 to 3
$[\frac{3\pi}{2}, 2\pi]$	Increases from 0 to 1	Increases from 3 to 5

In Figures 0.95a–0.95d, we show how the sketch progresses through each interval indicated in the table, with the completed figure (called a *limaçon*) shown in Figure 0.95d.

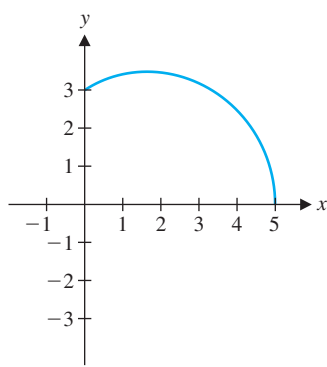


FIGURE 0.95a
 $0 \leq \theta \leq \frac{\pi}{2}$.

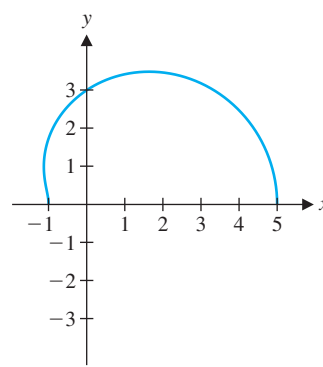


FIGURE 0.95b
 $0 \leq \theta \leq \pi$.

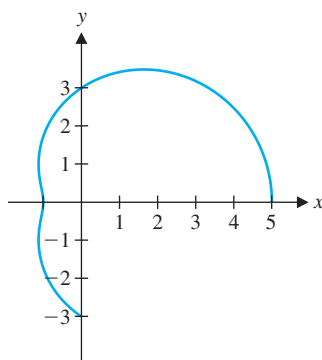


FIGURE 0.95c
 $0 \leq \theta \leq \frac{3\pi}{2}$.

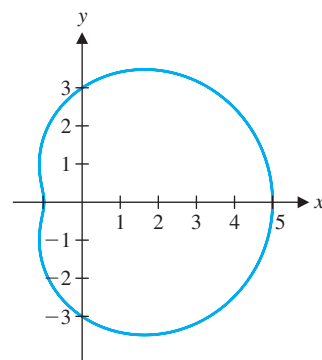


FIGURE 0.95d
 $0 \leq \theta \leq 2\pi$.

EXERCISES 0.7

WRITING EXERCISES

- Interpret in words the roles of each of the constants in the parametric equations $\begin{cases} x = a_1 + b_1 \cos(ct) \\ y = a_2 + b_2 \sin(ct) \end{cases}$.
- As indicated in remark 7.3, a given curve can be described by numerous sets of parametric equations. Explain why several different equations can all be correct. (Hint: Emphasize the fact that t is a dummy variable.)
- Suppose a point has polar representation (r, θ) . Explain why another polar representation of the same point is $(-r, \theta + \pi)$.
- After working with rectangular coordinates for so long, the idea of polar representations may seem slightly awkward. However, polar representations are entirely natural in many settings. For instance, if you were on a ship at sea and another ship was approaching you, explain whether you would use a polar representation (distance and bearing) or a rectangular representation (distance east-west and distance north-south).

In exercises 1–12, sketch the graph defined by the given parametric equations and find a corresponding x - y equation for the curve.

- $\begin{cases} x = 3 \cos t \\ y = 3 \sin t \end{cases}$
- $\begin{cases} x = 2 \cos t \\ y = 3 \sin t \end{cases}$
- $\begin{cases} x = 1 + 2 \cos t \\ y = -2 + 2 \sin t \end{cases}$
- $\begin{cases} x = 2 \sin t \\ y = 3 \cos t \end{cases}$
- $\begin{cases} x = -1 + 2t \\ y = 3t \end{cases}$
- $\begin{cases} x = 4 + 3t \\ y = 2 - 4t \end{cases}$

- $\begin{cases} x = 1 + t \\ y = t^2 + 2 \end{cases}$
- $\begin{cases} x = 2 - t \\ y = t^2 + 1 \end{cases}$
- $\begin{cases} x = t^2 - 1 \\ y = 2t \end{cases}$
- $\begin{cases} x = t^2 + 1 \\ y = t - 1 \end{cases}$
- $\begin{cases} x = t^2 - 1 \\ y = t^2 + 1 \end{cases}$
- $\begin{cases} x = \cos t \\ y = 3 \cos t - 1 \end{cases}$

T In exercises 13–20, use your CAS or graphing calculator to sketch the plane curves defined by the given parametric equations.

- $\begin{cases} x = t^3 - 2t \\ y = t^2 - 3 \end{cases}$
- $\begin{cases} x = t^3 - 2t \\ y = t^2 - 3t \end{cases}$
- $\begin{cases} x = t^2 - 1 \\ y = t^4 - 4t \end{cases}$
- $\begin{cases} x = t^2 - 1 \\ y = t^4 - 4t^2 \end{cases}$
- $\begin{cases} x = \cos t - \frac{1}{2} \cos 3t \\ y = \sin t - \frac{1}{2} \sin 3t \end{cases}$
- $\begin{cases} x = \cos t - \frac{1}{2} \cos 4t \\ y = \sin t - \frac{1}{2} \sin 4t \end{cases}$
- $\begin{cases} x = \cos 2t \\ y = \sin 7t \end{cases}$
- $\begin{cases} x = \cos 2t \\ y = \sin \pi t \end{cases}$

In exercises 21–26, match the parametric equations with the corresponding plane curve displayed below. Give reasons for your choices.

- $\begin{cases} x = t^2 - 1 \\ y = t^4 \end{cases}$
- $\begin{cases} x = t - 1 \\ y = t^3 \end{cases}$
- $\begin{cases} x = t^2 - 1 \\ y = \sin t \end{cases}$
- $\begin{cases} x = t^2 - 1 \\ y = \sin 2t \end{cases}$
- $\begin{cases} x = \cos 3t \\ y = \sin 2t \end{cases}$
- $\begin{cases} x = 3 \cos t \\ y = 2 \sin t \end{cases}$

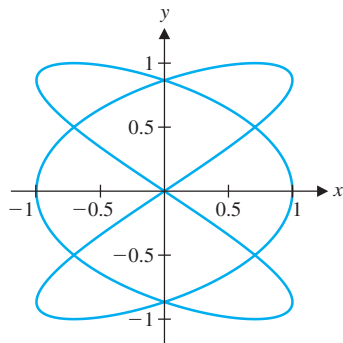


FIGURE A

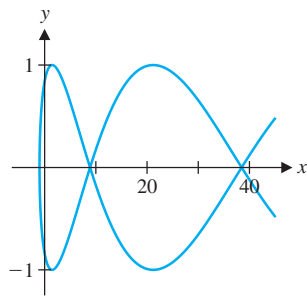


FIGURE B

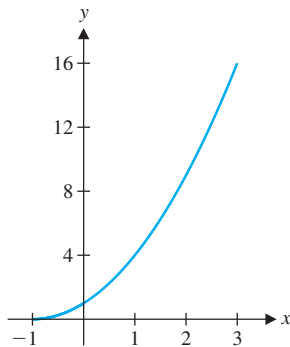


FIGURE C

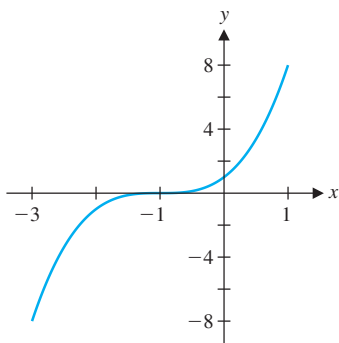


FIGURE D

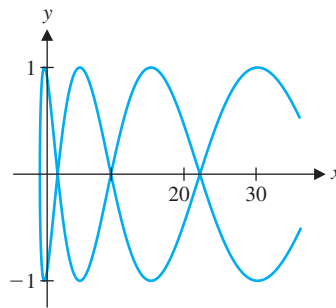


FIGURE E

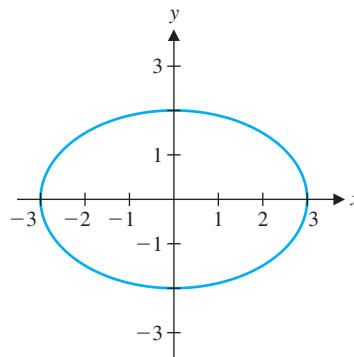


FIGURE F

In exercises 27–32, find parametric equations describing the given curve.

27. (a) The line segment from $(0, 1)$ to $(3, 4)$
 (b) The line segment from $(3, 1)$ to $(1, 3)$
28. (a) The portion of the parabola $y = x^2 + 1$ from $(1, 2)$ to $(2, 5)$
 (b) The portion of the parabola $y = 2x^2 - 1$ from $(0, -1)$ to $(2, 7)$
29. (a) The circle of radius 3 centered at $(2, 1)$, drawn counterclockwise.
 (b) The circle of radius 5 centered at $(-1, 3)$, drawn counterclockwise.
30. (a) The ellipse with vertices $(2, 0)$, $(0, 4)$, $(-2, 0)$ and $(0, -4)$
 (b) The ellipse with vertices $(0, 3)$, $(2, 4)$, $(4, 3)$ and $(2, 2)$
31. (a) The inverse of $f(x) = x^5 + 2x^3 + 4x - 2$
 (b) The inverse of $f(x) = 4x^3 + 2x$
32. (a) The inverse of $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
 (b) The inverse of $f(x) = \tan^{-1} x$

In exercises 33–38, plot the given polar points (r, θ) and find their rectangular representation.

- | | | |
|----------------------------|-----------------|---------------------------|
| 33. $(2, 0)$ | 34. $(2, \pi)$ | 35. $(-2, \pi)$ |
| 36. $(-3, \frac{3\pi}{2})$ | 37. $(3, -\pi)$ | 38. $(5, -\frac{\pi}{2})$ |

In exercises 39–44, find all polar coordinates representations of the given rectangular point.

39. $(2, -2)$ 40. $(-1, 1)$
 41. $(2, -1)$ 42. $(3, 4)$
 43. $(-\sqrt{3}, 1)$ 44. $(0, -4)$

In exercises 45–48, find rectangular coordinates for the given polar point.

45. $(2, -\frac{\pi}{3})$ 46. $(0, 3)$
 47. $(3, \frac{\pi}{8})$ 48. $(-3, 1)$

In exercises 49–56, sketch the graph of the polar equation and find a corresponding x - y equation.

49. $r = 4$ 50. $r = \sqrt{3}$
 51. $\theta = \pi/6$ 52. $\theta = 3\pi/4$
 53. $r = \cos \theta$ 54. $r = 2 \cos \theta$
 55. $r = 3 \sin \theta$ 56. $r = 2 \sin \theta$

In exercises 57–64, sketch the graph and identify all values of θ where $r = 0$ and a range of values of θ that produces one copy of the graph.

57. $r = \cos 2\theta$ 58. $r = \cos 3\theta$
 59. $r = 3 + 2 \sin \theta$ 60. $r = 2 - 2 \cos \theta$
 61. $r = 2 - 4 \sin \theta$ 62. $r = 2 + 2 \sin \theta$
 63. $r = \frac{1}{4}\theta$ 64. $r = e^{\theta/4}$

65. Based on your graphs in exercises 53 and 54, conjecture the graph of $r = a \cos \theta$ for any positive constant a .

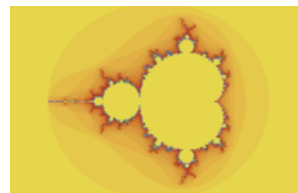
66. Based on the graphs in exercises 57 and 58 and others (try $r = \cos 4\theta$ and $r = \cos 5\theta$), conjecture the graph of $r = \cos n\theta$ for any positive integer n .

In exercises 67–72, find a polar equation corresponding to the given rectangular equation.

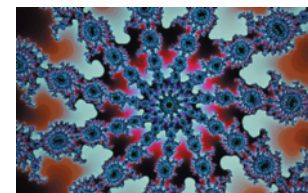
67. $y^2 - x^2 = 4$ 68. $x^2 + y^2 = 9$
 69. $x^2 + y^2 = 16$ 70. $x^2 + y^2 = x$
 71. $y = 3$ 72. $x = 2$

T 73. Compare the graphs of $\begin{cases} x = \cos 2t \\ y = \sin t \end{cases}$ and $\begin{cases} x = \cos t \\ y = \sin 2t \end{cases}$. Use the identities $\cos 2t = \cos^2 t - \sin^2 t$ and $\sin 2t = 2 \cos t \sin t$ to find x - y equations for each graph.

T 74. Sketch the graph of $\begin{cases} x = \frac{1}{2} \cos t - \frac{1}{4} \cos 2t \\ y = \frac{1}{2} \sin t - \frac{1}{4} \sin 2t \end{cases}$. This heart-shaped curve is the largest feature of the **Mandelbrot set**, one of the most famous mathematical sets. Portions of the Mandelbrot set have been turned into colorful T-shirts and posters that you may have seen.



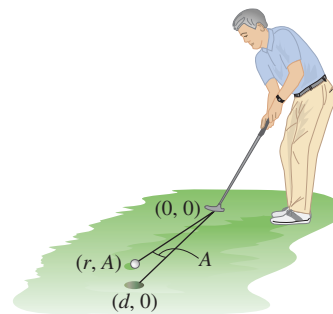
Mandelbrot set.



Mandelbrot zoom.

To progress further on a sketch of the Mandelbrot set, add the circle $\begin{cases} x = -1 + \frac{1}{4} \cos t \\ y = \frac{1}{4} \sin t \end{cases}$ to your initial sketch.

75. One situation where polar coordinates apply directly to sports is in making a golf putt. The two factors that the golfer tries to control are distance (determined by speed) and direction (usually called the “line”). Suppose a putter is d feet from the hole, which has radius $h = \frac{1}{6}$. Show that the path of the ball will intersect the hole if the angle A in the figure satisfies $-\sin^{-1}(h/d) < A < \sin^{-1}(h/d)$.



76. The distance r that the golf ball in exercise 75 travels also needs to be controlled. The ball must reach the front of the hole. In rectangular coordinates, the hole has equation $(x - d)^2 + y^2 = h^2$, so the left side of the hole is $x = d - \sqrt{h^2 - y^2}$. Show that this converts in polar coordinates to $r = d \cos \theta - \sqrt{d^2 \cos^2 \theta - (d^2 - h^2)}$. (Hint: Substitute for x and y , isolate the square root term, square both sides, combine r^2 terms and use the quadratic formula.)

77. The golf putt in exercises 75 and 76 will not go in the hole if it is hit too hard. Suppose that the putt would go $r = d + c$ feet if it did not go in the hole ($c > 0$). For a putt hit toward the center of the hole, define b to be the largest value of c such that the putt goes in (i.e., if the ball is hit more than b feet past the hole, it is hit too hard). Experimental evidence (see Dave Pelz’s *Putt Like the Pros*) shows that at other angles A , the distance r must be less than $d + b \left(1 - \left[\frac{A}{\sin^{-1}(h/d)} \right]^2 \right)$. The results of exercises 75 and 76 define limits for the angle A and distance r of a successful putt. Identify the functions $r_1(A)$ and $r_2(A)$ such that $r_1(A) < r < r_2(A)$ and constants A_1 and A_2 such that $A_1 < A < A_2$.

- T** 78. Take the general result of exercise 77 and apply it to a putt of $d = 15$ feet with a value of $b = 4$ feet. Visualize this by graphing the region

$$15 \cos \theta - \sqrt{225 \cos^2 \theta - (225 - 1/36)}$$

$$< r < 15 + 4 \left(1 - \left[\frac{\theta}{\sin^{-1}(1/90)} \right]^2 \right)$$

with $-\sin^{-1}(1/90) < \theta < \sin^{-1}(1/90)$. A good choice of graphing windows is $13.8 \leq x \leq 19$ and $-0.5 \leq y \leq 0.5$.

79. Find parametric equations for the path traced out by a specific point on a circle of radius r rolling left to right. (Hint: First, find parametric equations for the center of the circle, then add to these parametric equations for the point going around the center of the circle.) This curve is called a **cycloid**.
80. Find parametric equations for the path traced out by a specific point inside the circle as the circle rolls from left to right. (Hint: If r is the radius of the circle, let $d < r$ be the distance from the point to the center.) This curve is called a **trochoid**.
81. A microphone is set up at a specific place in a concert hall. Due to reflections from walls and other surfaces, sound bounces to the microphone from a variety of directions and is recorded by the microphone with a variety of amplitudes. If the amplitude r is given as a function of the direction θ , the resulting function is called the **polar pattern** of the microphone. If $r = 2.5 + 2.5 \cos \theta$, in which direction (which value of θ) do you think the microphone is pointed? If you were standing behind the microphone ($\theta = \pi$), would it be safe for you to talk?
82. Compare the properties of the microphone in exercise 81 to a microphone with polar pattern $r = 3.6 + 1.4 \cos \theta$.

EXPLORATORY EXERCISES

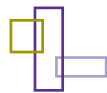
- T** 1. The Flying Zucchini Circus Troupe has a human cannonball act, shooting a performer from a cannon into a specially padded seat of a turning Ferris wheel. The Ferris wheel has a radius of 40 feet and rotates counterclockwise at one revolution per minute. The special seat starts at ground level. Carefully explain why parametric equations for the seat are $\begin{cases} x = 40 \cos(\frac{\pi}{30}t - \frac{\pi}{2}) \\ y = 40 + 40 \sin(\frac{\pi}{30}t - \frac{\pi}{2}) \end{cases}$. The cannon is located 200 feet left of the Ferris wheel with the muzzle 10 feet aboveground. The performer is launched 35 seconds after the wheel starts turning with an initial velocity of 100 ft/s at an angle of $\frac{\pi}{5}$ above the horizontal. Carefully explain why parametric equations for the human cannonball are $\begin{cases} x = (100 \cos \frac{\pi}{5})(t - 35) - 200 \\ y = -16(t - 35)^2 + (100 \sin \frac{\pi}{5})(t - 35) + 10 \end{cases}$ ($t \geq 35$). Determine whether the act is safe or the Flying Zucchini comes down squash.
- T** 2. Rework exploratory exercise 1 with initial velocity 135 ft/s, launch angle 30° and a 27-second delay. How close does the Flying Zucchini get to the special seat? Given that a Ferris wheel seat actually has height, width and depth, do you think that this is close enough? Repeat with (a) initial velocity 75 ft/s, launch angle 47° and 47.25-second delay; (b) initial velocity 118 ft/s, launch angle 35° and 28-second delay. Develop criteria for a safe and exciting human cannonball act. Consider each of the following: Should the launch velocity be large or small? Should the seat be high or low when the cannonball lands? Should the human have a positive or negative vertical velocity at landing? How close (vertically and horizontally) should the human need to get to the center of the seat? Based on your criteria, which of the launches in this exercise is the best? Find an initial velocity, launch angle and launch delay that is better.

REVIEW EXERCISES

CONCEPTS

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Slope of line	Parallel lines	Perpendicular lines
Domain	Intercepts	Zeros of function
Graphing window	Local maximum	Vertical asymptote
Inverse function	One-to-one function	Periodic function
Sine function	Cosine function	Arcsine function
e	Exponential function	Logarithm
Composition	Parametric equations	Polar coordinates



REVIEW EXERCISES

TRUE OR FALSE

State whether each statement is true or false and briefly explain why. If the statement is false, try to “fix it” by modifying the given statement to a new statement that is true.

- For a graph, you can compute the slope using any two points and get the same value.
- All graphs must pass the vertical line test.
- A cubic function has a graph with one local maximum and one local minimum.
- If a function has no local maximum or minimum, then it is one-to-one.
- The graph of the inverse of f can be obtained by reflecting the graph of f across the diagonal $y = x$.
- If f is a trigonometric function, then the solution of the equation $f(x) = 1$ is $f^{-1}(1)$.
- Exponential and logarithmic functions are inverses of each other.
- All quadratic functions have graphs that look like the parabola $y = x^2$.
- Polar coordinates are a specific example of parametric equations.
- Every curve has an infinite number of parametric representations.

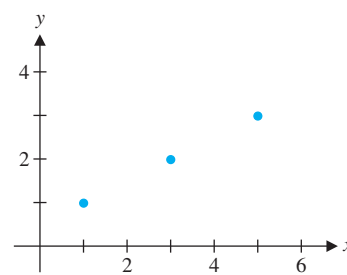
In exercises 1 and 2, find the slope of the line through the given points.

- $(2, 3), (0, 7)$
- $(1, 4), (3, 1)$

In exercises 3 and 4, determine if the lines are parallel, perpendicular or neither.

- $y = 3x + 1$ and $y = 3(x - 2) + 4$
- $y = -2(x + 1) - 1$ and $y = \frac{1}{2}x + 2$
- Determine if the points $(1, 2), (2, 4)$ and $(0, 6)$ form the vertices of a right triangle.
- The data represents populations at various times. Plot the points, discuss any patterns and predict the population at the next time: $(0, 2100), (1, 3050), (2, 4100)$ and $(3, 5050)$.

- Find an equation of the line through the given points and compute the y -coordinate corresponding to $x = 4$.



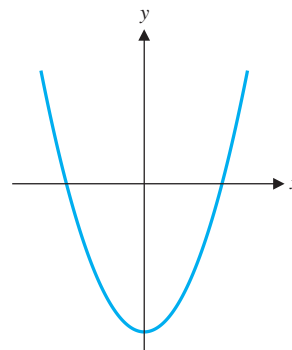
- For $f(x) = x^2 - 3x - 4$, compute $f(0)$, $f(2)$ and $f(4)$.

In exercises 9 and 10, find an equation of the line with given slope and point.

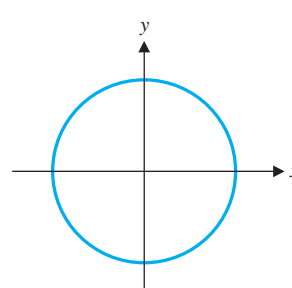
- $m = -\frac{1}{3}, (-1, -1)$
- $m = \frac{1}{4}, (0, 2)$

In exercises 11 and 12, use the vertical line test to determine if the curve is the graph of a function.

11.



12.



In exercises 13 and 14, find the domain of the given function.

- $f(x) = \sqrt{4 - x^2}$
- $f(x) = \frac{x - 2}{x^2 - 2}$

REVIEW EXERCISES



In exercises 15–28, sketch a graph of the function showing extrema, intercepts and asymptotes.

15. $f(x) = x^2 + 2x - 8$

16. $f(x) = x^3 - 6x + 1$

17. $f(x) = x^4 - 2x^2 + 1$

18. $f(x) = x^5 - 4x^3 + x - 1$

19. $f(x) = \frac{4x}{x+2}$

20. $f(x) = \frac{x-2}{x^2-x-2}$

21. $f(x) = \sin 3x$

22. $f(x) = \tan 4x$

23. $f(x) = \sin x + 2 \cos x$

24. $f(x) = \sec 2x$

25. $f(x) = 4e^{2x}$

26. $f(x) = 3e^{-4x}$

27. $f(x) = \ln 3x$

28. $f(x) = e^{\ln 2x}$

29. Determine all intercepts of $y = x^2 + 2x - 8$ (see exercise 15).

30. Determine all intercepts of $y = x^4 - 2x^2 + 1$ (see exercise 17).

31. Find all vertical asymptotes of $y = \frac{4x}{x+2}$.

32. Find all vertical asymptotes of $y = \frac{x-2}{x^2-x-2}$.

In exercises 33–36, find or estimate all zeros of the given function.

33. $f(x) = x^2 - 3x - 10$

34. $f(x) = x^3 + 4x^2 + 3x$

35. $f(x) = x^3 - 3x^2 + 2$

36. $f(x) = x^4 - 3x - 2$

In exercises 37 and 38, determine the number of solutions.

37. $\sin x = x^3$

38. $\sqrt{x^2+1} = x^2 - 1$

39. A surveyor stands 50 feet from a telephone pole and measures an angle of 34° to the top. How tall is the pole?

40. Find $\sin \theta$ given that $0 < \theta < \frac{\pi}{2}$ and $\cos \theta = \frac{1}{5}$.

41. Convert to fractional or root form: (a) $5^{-1/2}$ (b) 3^{-2} .

42. Convert to exponential form: (a) $\frac{2}{\sqrt{x}}$ (b) $\frac{3}{x^2}$.

43. Rewrite $\ln 8 - 2 \ln 2$ as a single logarithm.

44. Solve the equation for x : $e^{\ln 4x} = 8$.

In exercises 45 and 46, solve the equation for x .

45. $3e^{2x} = 8$

46. $2 \ln 3x = 5$.

In exercises 47 and 48, find $f \circ g$ and $g \circ f$ and identify their respective domains.

47. $f(x) = x^2, g(x) = \sqrt{x-1}$

48. $f(x) = x^2, g(x) = \frac{1}{x^2-1}$

In exercises 49 and 50, identify functions $f(x)$ and $g(x)$ such that $(f \circ g)(x)$ equals the given function.

49. e^{3x^2+2}

50. $\sqrt{\sin x + 2}$

In exercises 51 and 52, complete the square and explain how to transform the graph of $y = x^2$ into the graph of the given function.

51. $f(x) = x^2 - 4x + 1$

52. $f(x) = x^2 + 4x + 6$

In exercises 53–56, determine if the function is one-to-one. If so, find its inverse.

53. $x^3 - 1$

54. e^{-4x}

55. e^{2x^2}

56. $x^3 - 2x + 1$

In exercises 57–60, graph the inverse without solving for the inverse.

57. $x^5 + 2x^3 - 1$

58. $x^3 + 5x + 2$

59. $\sqrt{x^3+4x}$

60. e^{x^3+2x}

In exercises 61–64, evaluate the quantity using the unit circle.

61. $\sin^{-1} 1$

62. $\cos^{-1}(-\frac{1}{2})$

63. $\tan^{-1}(-1)$

64. $\csc^{-1}(-2)$

In exercises 65–68, simplify the expression using a triangle.

65. $\sin(\sec^{-1} 2)$

66. $\tan(\cos^{-1}(4/5))$

67. $\sin^{-1}(\sin(3\pi/4))$

68. $\cos^{-1}(\sin(-\pi/4))$

In exercises 69 and 70, find all solutions of the equation.

69. $\sin 2x = 1$

70. $\cos 3x = \frac{1}{2}$

In exercises 71–74, sketch the plane curve defined by the parametric equations and find a corresponding x - y equation for the curve.

71. $\begin{cases} x = -1 + 3 \cos t \\ y = 2 + 3 \sin t \end{cases}$

72. $\begin{cases} x = 2 - t \\ y = 1 + 3t \end{cases}$



REVIEW EXERCISES

73. $\begin{cases} x = t^2 + 1 \\ y = t^4 \end{cases}$ 74. $\begin{cases} x = \cos t \\ y = \cos^2 t - 1 \end{cases}$

In exercises 75–78, sketch the plane curves defined by the parametric equations.

75. $\begin{cases} x = \cos 2t \\ y = \sin 6t \end{cases}$ 76. $\begin{cases} x = \cos 6t \\ y = \sin 2t \end{cases}$
 77. $\begin{cases} x = \cos 2t \cos t \\ y = \cos 2t \sin t \end{cases}$ 78. $\begin{cases} x = \cos 2t \cos 3t \\ y = \cos 2t \sin 3t \end{cases}$

In exercises 79–82, match the parametric equations with the corresponding plane curve.

79. $\begin{cases} x = t^2 - 1 \\ y = t^3 \end{cases}$ 80. $\begin{cases} x = t^3 \\ y = t^2 - 1 \end{cases}$
 81. $\begin{cases} x = \cos 2t \cos t \\ y = \cos 2t \sin t \end{cases}$ 82. $\begin{cases} x = \cos(t + \cos t) \\ y = \cos(t + \sin t) \end{cases}$

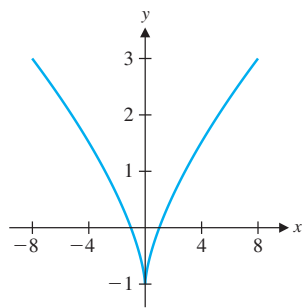


FIGURE A

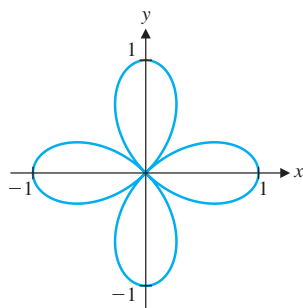


FIGURE B

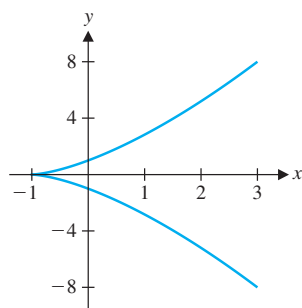


FIGURE C

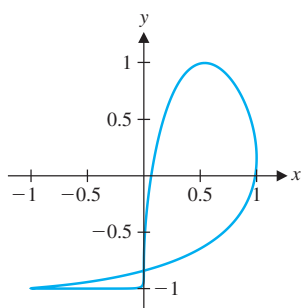


FIGURE D

In exercises 83 and 84, find parametric equations for the given curve.

83. The line segment from (2, 1) to (4, 7)
 84. The portion of the parabola $y = x^2 + 1$ from (1, 2) to (3, 10)

In exercises 85 and 86, sketch the graph of the polar equation and find a corresponding x - y equation.

85. $r = 3 \cos \theta$ 86. $r = 2 \sec \theta$

In exercises 87–94, sketch the graph and identify all values of θ where $r = 0$ and a range of values of θ that produces one copy of the graph.

87. $r = 2 \sin \theta$ 88. $r = 2 - 2 \sin \theta$
 89. $r = 2 - 3 \sin \theta$ 90. $r = \cos 3\theta + \sin 2\theta$
 91. $r^2 = 4 \sin 2\theta$ 92. $r = e^{\cos \theta} - 2 \cos 4\theta$
 93. $r = \frac{2}{1 + 2 \sin \theta}$ 94. $r = \frac{2}{1 + 2 \cos \theta}$

In exercises 95 and 96, find a polar equation corresponding to the rectangular equation.

95. $x^2 + y^2 = 9$ 96. $(x - 3)^2 + y^2 = 9$

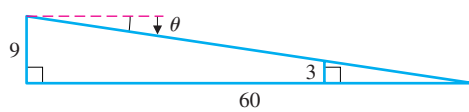
CONNECTIONS

- Sketch a graph of any function $y = f(x)$ that has an inverse. (Your choice.) Sketch the graph of the inverse function $y = f^{-1}(x)$. Then sketch the graph of $y = g(x) = f(x + 2)$. Sketch the graph of $y = g^{-1}(x)$ and use the graph to determine a formula for $g^{-1}(x)$ in terms of $f^{-1}(x)$. Repeat this for $h(x) = f(x) + 3$ and $k(x) = f(x - 4) + 5$.
- In tennis, a serve must clear the net and then land inside of a box drawn on the other side of the net. In this exercise, you will explore the margin of error for successfully serving. First, consider a straight serve (this essentially means a serve hit infinitely hard) struck 9 feet above the ground. Call the starting point (0, 9). The back of the service box is 60 feet away, at (60, 0). The top of the net is 3 feet above the ground and 39 feet from the server, at (39, 3). Find the service angle θ (i.e., the angle as measured from the horizontal) for the triangle formed by the points (0, 9), (0, 0) and (60, 0). Of course, most serves curve down due to gravity. Ignoring air resistance, the path of the ball struck at angle θ and initial speed v ft/s is $y = -\frac{16}{(v \cos \theta)^2} x^2 - (\tan \theta)x + 9$. To hit the back of the service line, you need $y = 0$ when $x = 60$. Substitute in these values along with $v = 120$. Multiply by $\cos^2 \theta$ and replace $\sin \theta$ with $\sqrt{1 - \cos^2 \theta}$. Replacing $\cos \theta$ with z gives you an algebraic equation in z . Numerically estimate z . Similarly, substitute $x = 39$ and $y = 3$ and find an equation for $w = \cos \theta$.

REVIEW EXERCISES



Numerically estimate w . The margin of error for the serve is given by $\cos^{-1} z < \theta < \cos^{-1} w$.



3. Baseball players often say that an unusually fast pitch rises or even hops up as it reaches the plate. One explanation of this illusion involves the players' inability to track the ball all the way to the plate. The player must compensate by predicting where

the ball will be when it reaches the plate. Suppose the height of a pitch when it reaches home plate is $h = -(240/v)^2 + 6$ feet for a pitch with velocity v ft/s. (This equation takes into consideration gravity but not air resistance.) Halfway to the plate, the height would be $h = -(120/v)^2 + 6$ feet. Compare the halfway heights for pitches with $v = 132$ and $v = 139$ (about 90 and 95 mph, respectively). Would a batter be able to tell much difference between them? Now compare the heights at the plate. Why might the batter think that the faster pitch hopped up right at the plate? How many inches did the faster pitch hop?