Part I: 單選, 填充, 是非題 (5 points for each problem)

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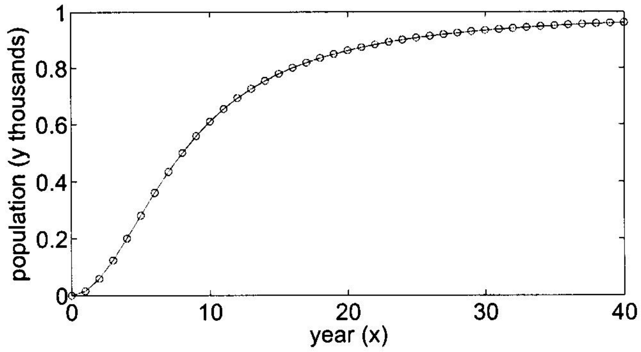
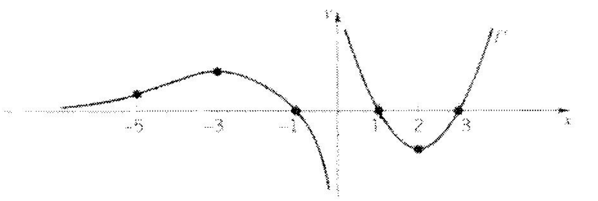
1. (5 pts) The plot below shows the growth of the population on an small island. Choose a better model from the options for this data? (Hint: find critical points, inflection points and asymptotes)

(A) $\frac{x}{x+8}$ (B) $\frac{x^2}{x^2-8^2}$ (C) $\frac{x}{x-8}$ (D) $\frac{x^2}{x^2+8^2}$



2. (5 pts) $f(x)$ is a **continuous** function on $(-\infty, \infty)$ and the graph of its **derivative**, $f'(x)$, is shown in the figure below. (Note: $\lim_{x\to-\infty} f'(x) = 0$; $\lim_{x\to\infty} f'(x) = \infty$)



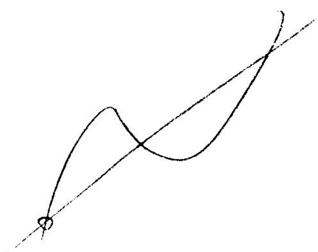
Answer the following True/False questions (True $\Rightarrow$ ◯ ; False $\Rightarrow$ $\times$).

- __O__ $f(x)$ has a horizontal asymptote.
- __X__ $f(x)$ has a vertical asymptote.
- __X__ $(1, f(1))$ is an inflection point.
- __O__ $(2, f(2))$ is an inflection point.
- __O__ f has a local minimum at $x = 0$.

C

3. (5 pts)A Region is bounded by two curves: $y = x^3$ and $y = x$. Set up a definite integral representing the area of the region

A) $\int_{-1}^{1} x - x^3\,dx$
B) $\int_{-1}^{0} x - x^3\,dx + \int_{0}^{1} x^3 - x\,dx$,
C) $\int_{-1}^{0} x^3 - x\,dx + \int_{0}^{1} x - x^3\,dx$
D) $\int_{-1}^{1} x^3 - x\,dx$



Part II: Problem-Solving Problems (計算與證明題 **Show all work**)

4. (10 pts) The sequence $\{x_n\}$ is recursively defined.

$$x_{n+1} = \frac{4x_n^2}{4+x_n^2}$$

(a) (2 pts) Find all equilibria (fixed points) of $\{x_n\}$.

(b) (8 pts) Determine the stability of the equilibria (fixed points).

a) $x = \frac{4x^2}{4+x^2} \Rightarrow x^3 + 4x = 4x^2 \Rightarrow x(x-2)^2 = 0$

$x = 0$ or 2

b) $f(x) = \frac{4x^2}{4+x^2} = 4 - \frac{16}{4+x^2}$, $f'(x) = \frac{16 \cdot 2x}{(4+x^2)^2}$

$f'(0) = 0$, $|f'(0)| < 1 \Rightarrow$ stable

$f'(2) = \frac{16 \cdot 2 \cdot 2}{(4+2^2)^2} = \frac{4 \cdot 16}{64} = 1$, $|f'(2)| = 1$ => Neutral

http://cellular.ci.ulsa.mx/comun/complex/node20.html

5. (10 pts) Given that $F(x) = \int_1^{x^2} \sqrt{1+t^2}\,dt$, for $x \geq 0$,

(a) $F'(x) =$?

(b) $(F^{-1})'(0) =$?

Note: since $\sqrt{1+t^2} > 0$, $F(x)$ is monotone and $F^{-1}(x)$ is well-defined.

(Let $G(x) = \int_1^x \sqrt{1+t^2}\,dt$, $G'(x) = \sqrt{1+x^4}$. $F(x) = G(x^2)$.)

a) $F'(x) = \sqrt{1+(x^2)^2} \cdot (x^2)' = 2x\sqrt{1+x^4}$

b) $\left(\begin{array}{l} F(F^{-1}(x)) = x \\ F'(F^{-1}(x)) \cdot [F^{-1}(x)]' = 1, \quad [F^{-1}(x)]' = \frac{1}{F'(F^{-1}(x))} \end{array} \right)$

$F^{-1\prime}(0) = \frac{1}{F'(F^{-1}(0))}$

Let $x = F^{-1}(0) \Rightarrow F(x) = 0 \Rightarrow \int_1^{x^2} \sqrt{1+t^2}\,dt = 0$

$\Rightarrow x^2 = 1 \Rightarrow x = 1 \quad (x \geq 0)$

$(F^{-1})'(0) = \frac{1}{F'(F^{-1}(0))} = \frac{1}{F'(1)} = \frac{1}{2\sqrt{1+1}} = \frac{1}{2\sqrt{2}}$

6. (5 pts) Use formulas for indefinite integrals to evaluate $\int \frac{1}{x^2-4x+11}\,dx$.

$a=1,\ b=-4,\ c=11$

$b^2-4ac = 16-44<0$

$\Rightarrow \int \frac{1}{x^2-4x+11}dx = \frac{2}{\sqrt{4\cdot 1\cdot 11-(-4)^2}} \tan^{-1}\frac{2\cdot x-4}{\sqrt{44-16}} + C = \frac{2}{\sqrt{28}}\tan^{-1}\frac{2x-4}{\sqrt{28}} + C$

$= \frac{1}{\sqrt{7}}\tan^{-1}\frac{x-2}{\sqrt{7}} + C$ #

7. (5 pts) $\frac{d}{dx}(x^{\cos x})$.

$y = x^{\cos x}$, $\ln y = \cos x \ln x$,

$\frac{y'}{y} = -\sin x \ln x + \frac{\cos x}{x}$

$\frac{d}{dx}x^{\cos x} = y' = \boxed{\left(-\sin x \ln x + \frac{\cos x}{x}\right)\cdot x^{\cos x}}$

8. (10 pts) $f(x) = \frac{\ln x}{x}$, $x > 0$.

(a) Find the maximum value of $f(x)$.

(b) Prove that $\pi^e < e^\pi$

a) $f'(x) = -\frac{\ln x}{x^2} + \frac{1}{x^2} = (1-\ln x)/x^2$

$f'(e) = 0$

$f'(x) > 0$ on $(0, e)$ $f(x)$ increasing on $(0,e)$

$f'(x) < 0$ on (e, ∞) decreasing on (e,∞)

$\Rightarrow f(x)$ attains maximum at $x = e$

maximum of f $= f(e) = 1/e$

b) $\because$ $f(e)$ is the maximum value of f

$\therefore$ $f(e) > f(\pi)$ $\Rightarrow e^\pi > \pi^e$

or $\frac{\ln e}{e} > \frac{\ln \pi}{\pi}$ $\because$ ($\ln$ is a monotonic fun)

or $\pi \ln e > e \ln \pi$

or $\ln e^\pi > \ln \pi^e$

9. (5 pts) Evaluate $\int \frac{1}{x} e^{199+\ln x}\, dx$

$$\int \frac{1}{x} e^{199+\ln x} dx = \int \frac{1}{x} \cdot e^{199} \cdot x\, dx = e^{199} \int 1\, dx$$
$$= e^{199} x + C \;\#$$

10. (10 pts) Evaluate $\int_e^{e^2} \frac{\sqrt{\ln x}}{x}\, dx$

Let $t = \ln x$, $dt = \frac{1}{x} dx$

$$\int_e^{e^2} \frac{\sqrt{\ln x}}{x} dx = \int_1^2 t^{1/2} dt = \frac{2}{3} t^{3/2} \Big|_1^2 = \frac{2}{3}(2^{3/2} - 1)$$

11. (10 pts) Evaluate $\int \ln x\, dx$

$u = \ln x$, $dv = dx$

$du = \frac{1}{x} dx$, $v = x$

$$\int \ln x\, dx = x \ln x - \int x \cdot \frac{1}{x} dx$$
$$= x \ln x - \int dx$$
$$= x \ln x - x + C$$

$\#$

$= \frac{(A+B)t + 2A}{t(t+2)}$

12. (10 pts) Evaluate the indefinite integrals

(a) $\displaystyle\int \frac{2t+1}{t^2+2t}\,dt$

$\frac{2t+1}{t^2+2t} = \frac{A}{t} + \frac{B}{t+2} = \frac{A(t+2)+Bt}{t(t+2)}$

$\left(\begin{array}{lll} t=-2 & \Rightarrow & -2B = -3,\ B = 3/2 \\ t=0 & \Rightarrow & 2A = 1,\ A = 1/2 \end{array}\right)$

$\int \frac{2t+1}{t^2+2t}\,dt = \int \frac{1}{2}\cdot\frac{1}{t} + \frac{3}{2}\,\frac{1}{t+2}\,dt = \frac{1}{2}\ln|t| + \frac{3}{2}\ln|t+2| + C$

(b) $\displaystyle\int \frac{\cos^3 x}{\sin^2 x + 2\sin x}\,dx$

let ~~t~~ $t = \sin x$, $dt = \cos x\,dx$

$\int \frac{\cos^3 x}{\sin^2 x + 2\sin x}\,dx = \int \frac{1-t^2}{t^2+2t}\,dt = \int -1 + \frac{2t+1}{t^2+2t}\,dt$

$= -t + \frac{1}{2}\ln|t| + \frac{3}{2}\ln|t+2| + C$

$= -\sin x + \frac{1}{2}\ln|\sin x| + \frac{3}{2}\ln|\sin x + 2| + C$

13. (10 pts) Compute: (Be sure to check whether l'Hospital's rule can be applied before you use it.)

(a) $\displaystyle\lim_{x\to\infty} \frac{x}{e^x}$ $\left(\frac{\infty}{\infty}\right)$

$\overset{(L)}{=} \lim_{x\to\infty} \frac{1}{e^x}$

$= 0$

(b) $\displaystyle\lim_{x\to\infty} (1+\frac{2}{x})^x$

$(1+\frac{2}{x})^x = e^{\ln(1+\frac{2}{x})^x} = e^{x\ln(1+\frac{2}{x})}$

(e^x is a continuous fun)

$\lim_{x\to\infty} x\ln(1+\frac{2}{x}) = \lim_{x\to\infty} \frac{\ln(x+2)-\ln x}{1/x}$ $\left(\frac{0}{0}\right)$ $\left(\begin{array}{l} x\to\infty \\ x>0 \end{array}\right)$

$\overset{(L)}{=} \lim_{x\to\infty} \frac{\frac{1}{x+2} - \frac{1}{x}}{-1/x^2}$

$= \lim_{x\to\infty} \frac{x-(x+2)}{x(x+2)}(-x^2) = 2$

$\Rightarrow \lim_{x\to\infty} (1+\frac{2}{x})^x = \lim_{x\to\infty} e^{x\ln(1+\frac{2}{x})} = e^{\lim_{x\to\infty} x\ln(1+\frac{2}{x})} = e^2$ ⨳