Name: ______________________

Student ID number: ______________

TA/classroom:______________

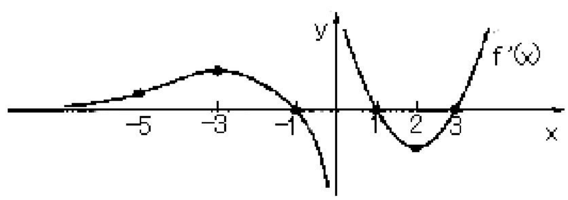
Guidelines for the test:

- Put your name or student ID number on every page.
- There are 11 problems
- The exam is closed book; calculators are not allowed.
- There is no partial credit for problem 1-3.
- For other problems,

  please show all work, unless instructed otherwise. Partial credit will be given only for work shown. Print as legibly as possible - correct answers may have points taken off, if they're illegible.
- **Mark the final answer**.

1. (2 pts each) $f(x)$ is a **continuous** function on $(-\infty, \infty)$ and the graph of its **derivative**, $f'(x)$, is shown in the figure below.

   (Note: $\lim_{x\to-\infty} f'(x) = 0$; $\lim_{x\to\infty} f'(x) = \infty$)



   Answer the following True/False questions (True $\Rightarrow$ ◯ ; False $\Rightarrow$ $\times$).

   - $\underline{\ \times\ }$ $(1, f(1))$ is an inflection point.
   - $\underline{\ \bigcirc\ }$ f has a local maximum at $x = -1$.
   - $\underline{\ \times\ }$ f has a local minimum at $x = 1$.
   - $\underline{\ \times\ }$ $f(x)$ has 3 critical numbers.

   Sol. ① Since the slope of $f'(x)$ at $x = 1$ is not zero, $f''(x) \neq 0$.
   ② $f'(-1) = 0$ with $f''(-1) < 0$.
   ③ $f'(1) = 0$ with $f''(1) < 0$. The point $x = 1$ should be a local maximum point of $f(x)$.

④ $f'(-1) = f'(1) = f'(3) = 0$, $f'(0)$ does not exist. There are 4 critical number of $f(x)$.

2. (2 pts each) Suppose $f(x)$ is a continuous function, and $F(x)$ is an antiderivative function of $f(x)$, i.e., $F'(x) = f(x)$. Answer the following True/False questions (True ⇒ ◯ ; False ⇒ ×).

- ___◯___ If $f(x)$ is an odd function, then $F(x)$ is an even function.
- ___×___ If $f(x)$ is an even function, then $F(x)$ is an odd function.
- ___×___ If $f(x)$ is a periodic function, then $F(x)$ is a periodic function.
- ___×___ If $f(x)$ is monotonically increasing, then $F(x)$ is monotonically increasing.

Note:

- The graph of an even function is symmetric with respect to the y-axis.
- The graph of an odd function is symmetric with respect to the origin.
- A function f is called monotonic increasing, if for all x and y such that $x \le y$ one has $f(x) \le f(y)$.

Sol. ① Let $G(x) = \int_0^x f(t)dt$ be an antiderivative function of $f(x)$, then $G(-x) = \int_0^{-x} f(t)dt$. By letting $u = -t$, $du = -dt$, we have

$$\begin{aligned} G(-x) &= \int_0^{-x} f(t)dt \\ &= \int_0^x -f(-u)du \\ &= \int_0^x f(u)du \quad \text{(since } f(x) \text{ is an odd function)} \\ &= G(x). \end{aligned}$$

Thus, $G(x)$ is an even function. By the fact that the difference of $F(x)$ and $G(x)$ is constant, we can conclude that any antiderivative of $f(x)$ is an even function.

② Choose $f(x) = x^2$ and $F(x) = \frac{1}{3}x^3 + 1$. We can check that $f(-x) = f(x)$, $\forall x \in \mathbb{R}$, but $F(-x) \neq -F(x)$.

③ Choose $f(x) = 1 + \cos x$ and $F(x) = x + \sin x$, the statement is false.

④ Choose $f(x) = x$ and $F(x) = \frac{1}{2}x^2, x \in \mathbb{R}$. We can check that $f(x)$ is strictly increasing on $\mathbb{R}$, but $F(x)$ is not.

3. (2 pts each) Answer the True/False questions (True ⇒ ◯ ; False ⇒ ×).

- ___◯___ $2 - \cos x$ is an antiderivative function of $\sin x$.

- $\underline{\bigcirc}$ $2\sin^2\dfrac{x}{2}$ is an antiderivative function of $\sin x$.

Sol. ① $\frac{d}{dx}(2-\cos x) = \sin x$.
② $\frac{d}{dx}\left(2\sin^2\frac{x}{2}\right) = \frac{d}{dx}\left(2\cdot\frac{1-\cos x}{2}\right) = \frac{d}{dx}(1-\cos x) = \sin x$.

4. Evaluate each of the following limits.

(a) (5 pts) $\lim\limits_{x\to 0^+} \sin x \,\ln x$.

Sol. $\lim\limits_{x\to 0^+}\sin x \ln x = \lim\limits_{x\to 0^+}\dfrac{\ln x}{\csc x}$. This limit is of $\frac{\infty}{\infty}$ form, by L' Hospital rule,

$$\begin{aligned}
\lim_{x\to 0^+}\frac{\ln x}{\csc x} &\overset{\textbf{L'}}{=} \lim_{x\to 0^+}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(\csc x)}\\
&= \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\csc x\cot x}\\
&= -\lim_{x\to 0^+}\left(\frac{\sin x}{x}\cdot\tan x\right)\\
&= -\left(\lim_{x\to 0^+}\frac{\sin x}{x}\right)\left(\lim_{x\to 0^+}\tan x\right)\\
&= -1\cdot 0\\
&= 0.
\end{aligned}$$

(b) (5 pts) $\lim\limits_{x\to 0^+} x^{\sin x}$.

Sol. Let $y(x) = x^{\sin x}$, we have $\ln y(x) = \sin x\ln x$. By (a), we have $\lim\limits_{x\to 0^+}\ln y(x) = \lim\limits_{x\to 0^+}\sin x\ln x = 0$. Thus,

$$\begin{aligned}
\lim_{x\to 0^+} x^{\sin x} &= \lim_{x\to 0^+} y(x)\\
&= \lim_{x\to 0^+} e^{\ln y(x)}\\
&= e^{\left(\lim\limits_{x\to 0^+}\ln y(x)\right)} \quad \text{(by the continuity of } e^x\text{)}\\
&= e^0\\
&= 1.
\end{aligned}$$

5. Find $\frac{dy}{dx}$ for each of the following.

(a) (5 pts) $y = x^{\sin x}$, $x>0$.

Sol. First we have $\ln y(x) = \ln\left(x^{\sin x}\right) = \sin x\ln x$. Then,

$$\begin{aligned}
&\tfrac{d}{dx}(\ln y(x)) = \tfrac{d}{dx}(\sin x\ln x)\\
\Longrightarrow\ &\tfrac{1}{y(x)}\cdot\left(\tfrac{d}{dx}y(x)\right) = \left(\tfrac{d}{dx}\sin x\right)\cdot\ln x + \sin x\cdot\left(\tfrac{d}{dx}\ln x\right)\\
\Longrightarrow\ &\tfrac{1}{y(x)}\left(\tfrac{d}{dx}y(x)\right) = \cos x\ln x + \tfrac{\sin x}{x}\\
\Longrightarrow\ &\tfrac{d}{dx}y(x) = y(x)\cdot\left(\cos x\ln x + \tfrac{\sin x}{x}\right) = x^{\sin x}\left(\cos x\ln x + \tfrac{\sin x}{x}\right).
\end{aligned}$$

(b) (5 pts) $y = e^{2x}\dfrac{\sqrt{x+1}}{x^2+2}(2x+1)^5, \quad x>0.$

Sol. First we have $\ln y(x) = \ln\left[e^{2x}\frac{\sqrt{x+1}}{x^2+2}(2x+1)^5\right] = \ln e^{2x} + \ln\sqrt{x+1} - \ln(x^2+2) + \ln(2x+1)^5 = 2x + \frac{1}{2}\ln(x+1) - \ln(x^2+2) + 5\ln(2x+1)$.
Then,

$$\begin{aligned}
&\tfrac{d}{dx}(\ln y(x)) = \tfrac{d}{dx}\left[2x + \tfrac{1}{2}\ln(x+1) - \ln(x^2+2) + 5\ln(2x+1)\right] \\
&\Longrightarrow \tfrac{1}{y(x)}\cdot\left(\tfrac{d}{dx}y(x)\right) = \left[2 + \tfrac{1}{2}\cdot\tfrac{1}{x+1} - \tfrac{1}{x^2+2}\cdot\left(\tfrac{d}{dx}(x^2+2)\right) + 5\cdot\tfrac{1}{2x+1}\cdot\left(\tfrac{d}{dx}(2x+1)\right)\right] \\
&\Longrightarrow \tfrac{1}{y(x)}\left(\tfrac{d}{dx}y(x)\right) = 2 + \tfrac{1}{2x+2} - \tfrac{2x}{x^2+2} + \tfrac{10}{2x+1} \\
&\Longrightarrow \tfrac{d}{dx}y(x) = y(x)\cdot\left(2 + \tfrac{1}{2x+2} - \tfrac{2x}{x^2+2} + \tfrac{10}{2x+1}\right) \\
&\qquad\qquad = \left[e^{2x}\tfrac{\sqrt{x+1}}{x^2+2}(2x+1)^5\right]\left(2 + \tfrac{1}{2x+2} - \tfrac{2x}{x^2+2} + \tfrac{10}{2x+1}\right).
\end{aligned}$$

6. (10 pts) Given that $F(x) = \displaystyle\int_1^{x^2} e^{t^2}\,dt$, for $x \geq 0$.

(a) Find $F'(x)$.

Sol. By Fundamental Theorem of Calculus, $F'(x) = e^{(x^2)^2}\cdot\left(\frac{d}{dx}x^2\right) = 2xe^{x^4}$.

(b) Find $(F^{-1})'(0)$.

Sol. Since $F(1) = 0$, we have $F^{-1}(0) = 1$. Hence,

$$(F^{-1})'(0) = \frac{1}{F'\left(F^{-1}(0)\right)} = \frac{1}{2F^{-1}(0)e^{(F^{-1}(0))^4}} = \frac{1}{2e}.$$

7. Evaluate the given integral.

(a) (5 pts) $\displaystyle\int e^{2x}\sin x\,dx$.

Sol.

$$\begin{aligned}
\int e^{2x}\sin x\,dx &= \tfrac{1}{2}e^{2x}\sin x - \int \frac{1}{2}e^{2x}\cos x\,dx \quad \text{(by integration by parts)} \\
&= \tfrac{1}{2}e^{2x}\sin x - \left(\tfrac{1}{4}e^{2x}\cos x + \int \frac{1}{4}e^{2x}\sin x\,dx\right) \quad \text{(by integration by parts)} \\
&= \tfrac{1}{2}e^{2x}\sin x - \tfrac{1}{4}e^{2x}\cos x - \tfrac{1}{4}\int e^{2x}\sin x\,dx.
\end{aligned}$$

We have $\frac{5}{4}\displaystyle\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x + C_1$. Thus, $\displaystyle\int e^{2x}\sin x\,dx = \frac{4}{5}\left(\frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x + C_1\right) = \frac{2}{5}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x + C$.

(b) (5 pts) $\displaystyle\int \frac{\sqrt{\ln x}}{x}dx$.

Sol. Let $u = \ln x$, $du = \frac{1}{x}dx$. Then $\displaystyle\int \frac{\sqrt{\ln x}}{x}dx = \int \sqrt{u}du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(\ln x)^{\frac{3}{2}} + C$.

(c) (5 pts) $\displaystyle\int \frac{3x}{(x+1)(x-4)}dx$.

Sol. $\int \frac{3x}{(x+1)(x-4)}dx = \int \left(\frac{\frac{3}{5}}{x+1} + \frac{\frac{12}{5}}{x-4} \right) dx = \frac{3}{5}\ln|x+1| + \frac{12}{5}\ln|x-4| + C.$

8. (10 pts) Evaluate the definite integrals $\int_1^4 e^{\sqrt{x}}dx$.

Sol. Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}dx$, which we have $dx = 2\sqrt{x}du = 2udu$. Then

$$\begin{aligned}
\int_1^4 e^{\sqrt{x}}dx &= \int_1^2 2ue^u du \\
&= 2ue^u\Big|_1^2 - \int_1^2 2e^u du \quad \text{(by integration by parts)} \\
&= 4e^2 - 2e - \left(2e^u\Big|_1^2 \right) \\
&= 4e^2 - 2e - (2e^2 - 2e) \\
&= 2e^2.
\end{aligned}$$

9. (a) (5 pts) Evaluate $\int \cos^2\theta d\theta$.

   Sol. $\int \cos^2\theta d\theta = \int \frac{1+\cos 2\theta}{2}d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C.$

   (b) (5 pts) Use the trigonometric substitution to evaluate $\int_0^1 \sqrt{1-x^2}dx$.

   Sol. Let $x = \sin\theta$, $dx = \cos\theta d\theta$. Then

$$\begin{aligned}
\int_0^1 \sqrt{1-x^2}dx &= \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\cdot\cos\theta d\theta \\
&= \int_0^{\frac{\pi}{2}} \cos\theta\cdot\cos\theta d\theta \\
&= \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta.
\end{aligned}$$

   By (a), we have

$$\int_0^1 \sqrt{1-x^2}dx = \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

10. (5 pts) Use formulas for indefinite integrals to evaluate $\int \frac{1}{x^2-4x+5}dx$.

Sol. Set $a = 1$, $b = -4$, $c = 5$, then $b^2 - 4ac = 16 - 20 = -4 < 0$. By using the integral formula

$$\int \frac{1}{ax^2+bx+c}dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}},$$

we have $\int \frac{1}{x^2-4x+5}dx = \frac{2}{\sqrt{4}}\tan^{-1}\frac{2x-4}{\sqrt{4}} + C = \tan^{-1}(x-2) + C.$

11. Evaluate the given integral.

(a) (5 pts) $\int_{-1}^{1} x^{-2}dx$.

Sol. Since $\int_0^1 x^{-2}dx = \lim_{t\to 0^+} \int_t^1 x^{-2}dx = \lim_{t\to 0^+} \left(-\frac{1}{x}\right)\Big|_t^1 = \lim_{t\to 0^+}\left(-1+\frac{1}{t}\right) = \infty$, the integral $\int_0^1 x^{-2}dx$ diverges. Hence, the integral $\int_{-1}^1 x^{-2}dx$ diverges.

(b) (5 pts) $\int_{-\infty}^{\infty} xdx$.

Sol. Since $\int_0^{\infty} xdx = \lim_{t\to+\infty}\int_0^t xdx = \lim_{t\to+\infty} \frac{1}{2}x^2\Big|_0^t = \lim_{t\to+\infty}\frac{1}{2}t^2 = +\infty$, the integral $\int_0^{\infty} xdx$ diverges. Hence, the integral $\int_{-\infty}^{\infty} xdx$ diverges.