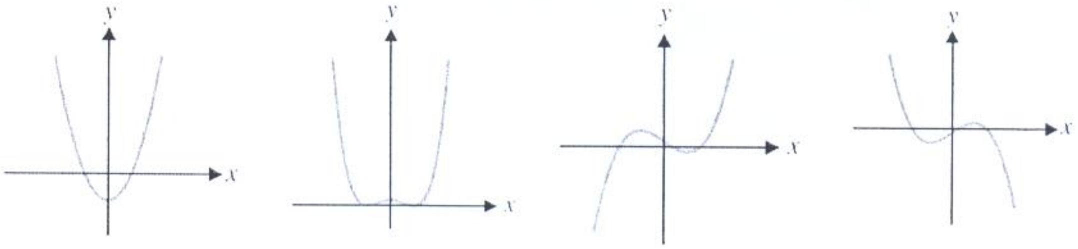
Calculus Student ID number: ____________

1. (5 pts) Given $f'(x)$, find the graph of $f(x)$?

B



$f'(x)$ (A) (B) (C)

2. (15 pts)

(a) Find the derivative of $f(x) = \ln\sqrt{e^{2x}(x^2+1)^{10}/(2x^3+2)}$, $x > 0$.

$f(x) = \frac{1}{2}\left(\ln e^{2x} + 10\ln(x^2+1) - \ln(2x^3+2)\right) = \frac{1}{2}\left(2x + 10\ln(x^2+1) - \ln(2x^3+2)\right)$

$\therefore f'(x) = \frac{1}{2}\left(2 + 10\cdot\frac{2x}{x^2+1} - \frac{6x^2}{2x^3+2}\right) = 1 + \frac{10x}{x^2+1} - \frac{3x^2}{2x^3+2}$

(b) Find the derivative of $f(x) = e^x\cos(x^3+x)$.

$f'(x) = e^x\cos(x^3+x) + e^x\left(-\sin(x^3+x)\right)\cdot(3x^2+1)$

$= e^x\left(\cos(x^3+x) - (3x^2+1)(\sin(x^3+x))\right)$

(c) Find $\frac{d}{dx}(x^2 - \frac{1}{x^2})^{10}$

$\frac{d}{dx}(x^2 - \frac{1}{x^2})^{10} = 10(x^2 - \frac{1}{x^2})^9\cdot(2x + 2\cdot\frac{1}{x^3}) = 20(x^2 - \frac{1}{x^2})^9(x + \frac{1}{x^3})$

3. (10 pts) Given the curve $x^2 + xy + y^2 = 3$.

(a) find $\frac{dy}{dx}$ implicitly;

$2x + y + x\cdot\frac{dy}{dx} + 2y\cdot\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{2x+y}{x+2y}$

(b) find the equation of the tangent line at $(1,1)$?

$\frac{dy}{dx}(1,1) = -\frac{2\cdot1+1}{1+2\cdot1} = -1$

$\therefore$ the equation of the tangent line at $(1,1)$

is $y - 1 = -(x-1) \Rightarrow x + y = 2$

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4. (20 pts) Compute: (Check whether l'Hospital's rule can be applied before you use it.)

   (a) $\lim_{x\to 9} \frac{9-x}{\sqrt{x}-3}$.

$\lim_{x\to 9} \frac{9-x}{\sqrt{x}-3} = \lim_{x\to 9} -(\sqrt{9}+3) = -6$

   (b) $\lim_{x\to 0^+} x\ln x$ $\quad \because \lim_{x\to 0^+} x = 0$ and $\lim_{x\to 0^+} \ln x = -\infty$

$\therefore$ by l'Hospital's rule

$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{L'}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$

   (c) $\lim_{x\to 0} xe^x$ $\quad \because \lim_{x\to 0} x = 0$ and $\lim_{x\to 0} e^x = 1$ (exist)

$\therefore \lim_{x\to 0} xe^x = (\lim_{x\to 0} x)(\lim_{x\to 0} e^x) = 0\cdot 1 = 0$

   (d) $\lim_{x\to 0^+} x^{2x}$ $\quad$ let $y = x^{2x} \Rightarrow \ln y = 2x\ln x$

$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} 2x\ln x = 2\lim_{x\to 0^+} x\ln x \overset{by (b)}{=} 0$

$\therefore \lim_{x\to 0^+} x^{2x} = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1$

5. (5 pts) Given $f(x) = \frac{x\cos x}{(x+1)(x+2)(x+3)\dots(x+100)}$, find $f'(0)$.

方法一.

$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{\cos x}{(x+1)(x+2)\cdots(x+100)} = \frac{1}{100!}$

方法二.

$f(x) = x\cdot\left(\frac{\cos x}{(x+1)\cdots(x+100)}\right)$

$f'(x) = \frac{\cos x}{(x+1)\cdots(x+100)} + x\left(\frac{\cos x}{(x+1)\cdots(x+100)}\right)'$

$\therefore f'(0) = \frac{1}{100!}$

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6. (5 pts) Given that $f(x) = \begin{cases} x^2, & x > 0 \\ -x^2, & x \le 0 \end{cases}$, compute $f'(0)$ by definition (limits).

$x>0.\ \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x^2-0}{x-0} = \lim_{x\to 0^+} x = 0$

$x\le 0.\ \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{-x^2-0}{x-0} = \lim_{x\to 0^-} -x = 0$

∵ $f'(x)$ 在 $x=0$ 時的左右極限相等

∴ $f'(0)$ 存在且等於左右極限值 ∴ $f'(0)=0$

7. (10 pts) Find the points on the ellipse $4x^2+y^2=4$ that are farthest away from the point $(1,0)$.

Let (a,b) be the points on the $4x^2+y^2=4$

and, d be the distance between (a,b) and $(1,0)$

$d=\sqrt{(a-1)^2+b^2}$, let $D=d^2$

$D=(a-1)^2+b^2=(a-1)^2+4-4a^2=-3a^2-2a+5$

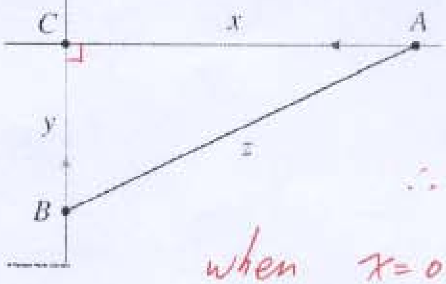
$D'=-6a-2 \Rightarrow D'=0,\ a=-\frac{1}{3}$ (Critical Number)

$D'(a)>0$ for $a<-1/3$, $D'(a)<0$ for $a>-1/3$ => D attains abs. max. at $a=-1/3$

∴ $b^2=4-4(-\frac{1}{3})^2 \Rightarrow b=\pm\frac{4\sqrt{2}}{3}$ ∴ the point is $(-\frac{1}{3}, \pm\frac{4\sqrt{2}}{3})$

8. (10 pts) Car A is traveling west at 90 **km/h** and car B is traveling north at 100 **km/h**. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 60 **m** and car B is 80 **m** from the intersection?

$\parallel$ 0.06 km $\quad\parallel$ 0.08 km



∵ $x^2+y^2=z^2$

∴ $2x\frac{dx}{dt}+2y\frac{dy}{dt}=2z\frac{dz}{dt}$ ——(*)

when $x=0.06$, $y=0.08 \Rightarrow z=0.1$

and $\frac{dx}{dt}=90$, $\frac{dy}{dt}=100$

代入(*), $\cancel{2}(0.06)\cdot 90+\cancel{2}(0.08)\cdot 100=\cancel{2}\cdot(0.1)\cdot\frac{dz}{dt}$

∴ $\frac{dz}{dt}=134$ (km/h)

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9. (total 20 points; (a)-(n) no partial credit) Study the function $f(x) = \dfrac{1}{x^2 - 9}$ and answer the following questions.

(a) (1 pt) Domain of f: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ or $\{x \mid x \in \mathbb{R}, x \neq \pm 3\}$.

(b) (1 pt) Horizontal Asymptote: $y = 0$.

(c) (1 pt) Vertical Asymptote: $x = 3$, $x = -3$.

(d) (1 pt) $f'(x) = \dfrac{-2x}{(x^2-9)^2}$.

(e) (1 pt) Intervals of increase of f: $(-\infty, -3) \cup (-3, 0)$ or $\{x \mid x \in \mathbb{R}, x < 0, x \neq -3\}$

(f) (1 pt) Intervals of decrease of f: $(0, 3) \cup (3, \infty)$ or $\{x \mid x \in \mathbb{R}, x > 0, x \neq 3\}$

(g) (1 pt) Local maxima of f: $-\dfrac{1}{9}$.

(h) (1 pt) Local minima of f: None.

(i) (1 pt) $f''(x) = \dfrac{6(x^2+3)}{(x^2-9)^3}$.

(j) (1 pt) Intervals of concave up: $(-\infty, -3) \cup (3, \infty)$ or $\{x \mid x \in \mathbb{R}, |x| > 3\}$

(k) (1 pt) Intervals of concave down: $(-3, 3)$ or $\{x \mid x \in \mathbb{R}, |x| < 3\}$

(l) (1 pt) Inflection point(s) of f: None.

(m) (1 pt) x-intercepts of f: None.

(n) (1 pt) y-intercepts of f: $-\dfrac{1}{9}$ or $\left(0, \dfrac{-1}{9}\right)$.

這也給分

(o) (6 pts) Sketch the graph of f showing all significant features.