The Three-Body Problem

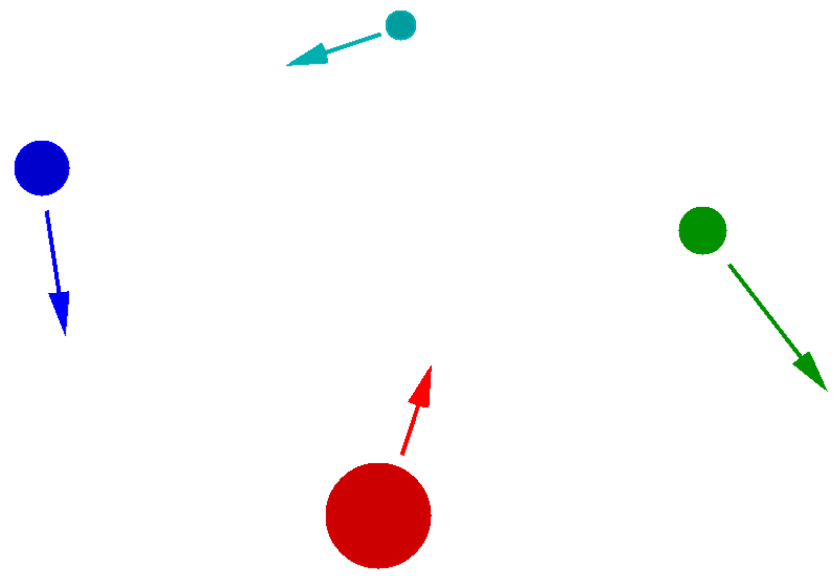
from a variational point of view

Kuo-Chang Chen
kchen@math.nthu.edu.tw

Celestial Mechanics – The N-Body Problem (NBD)

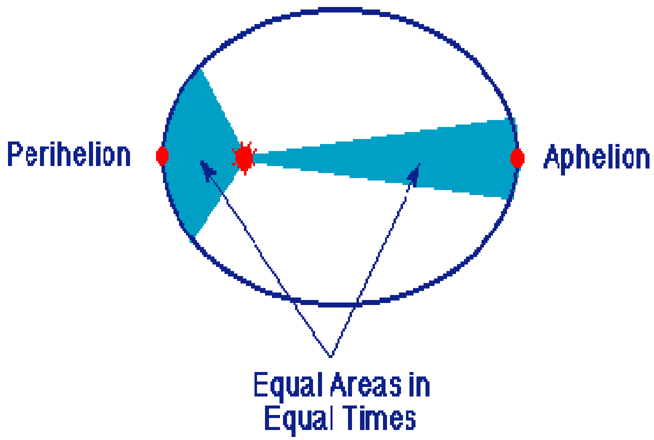
Ultimate Goal:

Understand the evolution of celestial bodies under the influence of gravitation.



Kepler (1609,1619)
Three Laws of Planetary Motions

- The orbit of Mars is an ellipse with the Sun in one of its foci
- The line joining the planet to the Sun swept out equal areas in equal times
- For any two planets the ratio of the squares of their periods is the same as the ratio of the cubes of the mean radii of their orbits



Newton (Principia 1687)
Law of Universal Gravitation

Everything happens as if matter attracts matter in direct proportion to the products of masses and in inverse proportion to the square of the distances.

Newton (Principia 1687)
Law of Universal Gravitation

Let $x_k \in \mathbb{R}^d$ be the position of mass $m_k > 0$
Equations of motions for the n-body problem:

$$m_k \ddot{x}_k = \sum_{i \neq k} \frac{m_i m_k (x_i - x_k)}{|x_i - x_k|^3}, \qquad k = 1, \ldots, n.$$

Kepler Problem versus the N-Body Problem

n=2: Newton's Law $\Rightarrow$ Kepler's Laws

$n \geq 3$?

Poincaré (1892):

The three-body problem is of such importance in astronomy, and is at the same time so difficult, that all efforts of geometers have long been directed toward it.

Major Questions on the N-Body Problem

- What kind of motions are possible and why? Does Newton's law along explains all astronomical phenomena when relativistic effects are negligible?

- According to Newton's law:
  Is the solar system stable?
  Is the Sun-Earth-Moon system stable?

- What kind of space mission designs can be realized, given the constraint of current technology? Will we ever be able to visit our neighboring star systems?

Hamiltonian Formulation

Let $$U(x) = U(x_1, \ldots, x_n) = \sum_{1 \le i < j \le n} \frac{m_i m_j}{|x_i - x_j|}$$

be the potential energy.

Newton's equations can be written

$$m_k \ddot{x}_k = \frac{\partial}{\partial x_k} U(x), \qquad k = 1, \ldots, n.$$

Let $$K(\dot{x}) = \frac{1}{2}\sum_{i=1}^{n} m_i|\dot{x}_i|^2$$

be the kinetic energy,

$$H(x, y) = K(M^{-1}y) - U(x)$$

be the Hamiltonian, where

$$M = \text{diag}[\underbrace{m_1, \cdots, m_1}_{d \text{ copies}}, \cdots\cdots, m_n, \cdots, m_n]$$

Then Newton's equations can be written

$$\begin{cases} \dot{x}_k = \dfrac{\partial}{\partial y_k} H(x, y) \\ \dot{y}_k = -\dfrac{\partial}{\partial x_k} H(x, y) \end{cases}$$

Lagrangian Formulation

Let $L(x, \dot{x}) = K(\dot{x}) + U(x)$ be the Lagrangian. Then Newton's equations are Euler-Lagrange equations of the action functional:

$$\mathcal{A}_{t_0,t_1}(x) = \int_{t_0}^{t_1} L(x, \dot{x})\, dt$$

Critical points of $\mathcal{A}_{t_0,t_1}$ in $H^1([t_0, t_1], (\mathbb{R}^d)^N)$ are solutions of the n-body problem (w/ or w/o collisions).

Why is the Three-Body Problem Difficult?

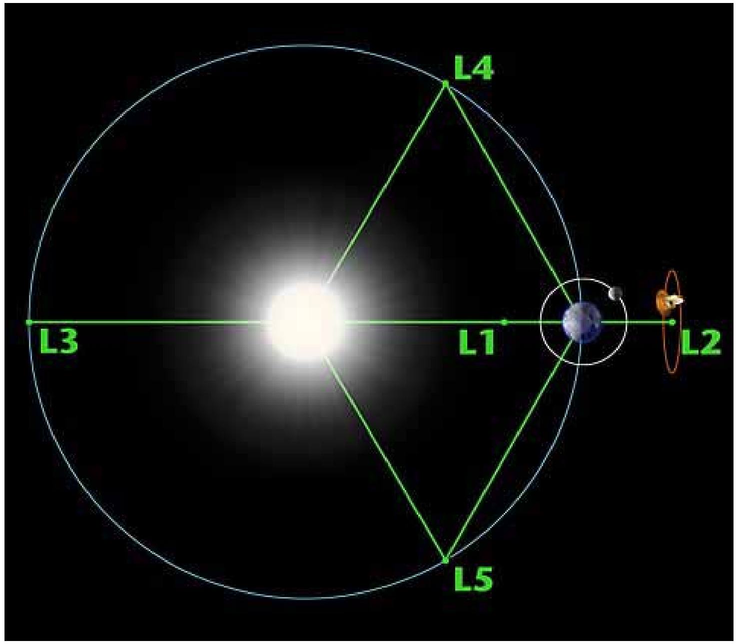
- The system of equations is singular.
  When $n \geq 3$, generally collision singularities
  can not be regularized.

- The dimension of the system is high.
  In the spatial (3BD), after reduction by
  integrals of motions, it is a dynamical system
  on an 8-dimensional integral manifold.

- The action functional does not discriminate
  collision solutions from classical solutions.

Classical Results

- Euler (1767), Lagrange (1772):
  Given any 3 masses, there are 5 classes of self-similar (homographic) solutions.



L1 ~ L5 are called Lagrange points or libration points

♦ Hill, Poincaré, Moulton, Birkhoff, Wintner, Hopf, Conley, … :
Existence of prograde, retrograde, and nearly circular solutions. Either one mass is infinitesimal or one binary is tight.

♦ Sundman (1913), McGehee (1974):
Dynamics near triple collisions.

♦ Chazy (1922), Sitnikov (1959):
Complete classification of asymptotic behavior.

♦ Smale (1970), Meyer-McCord-Wang (1998):
Birkhoff conjecture for the (3BD).

Action of Collision Orbits

♦ Poincaré (*C.R.Acad.Sci.* 1896):

Minimize $\mathcal{A}_T := \mathcal{A}_{0,T}$ *among planar loops in a homology class. Collision loops can have finite action.*

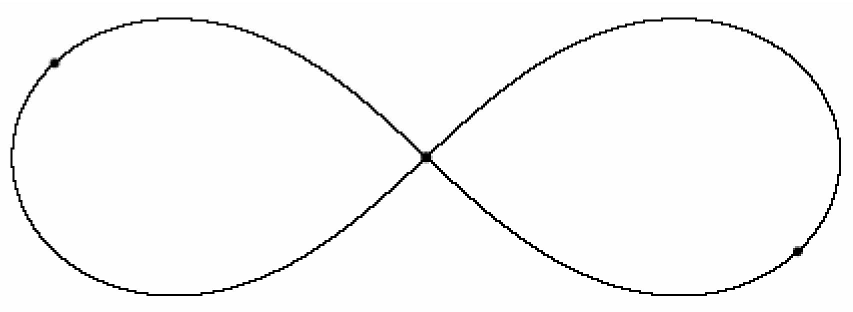
♦ Gordon (*Am. J. Math.* 1977):

Keplerian orbits with the same masses and least periods have the same action values.

♦ Venturelli (*C.R.Acad.Sci.* 2001):
Action minimizers for (3BD) with homology constraints are either Lagrangian solutions, collision solutions, or do not exist.

Recent Progress – Figure-8 Orbit

♦ Chenciner-Montgomery (*Ann.Math.*2000):
Existence of Figure-8 orbit for the (3BD) with equal masses.



Idea of proof:

Minimize $\mathcal{A}_T := \mathcal{A}_{0,T}$ among loops H^G in $H = H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^3)$ satisfying

$$\begin{aligned}(x_1, x_2, x_3)(t) &= -(\bar{x}_3, \bar{x}_1, \bar{x}_2)(t + \tfrac{T}{6}) \\ &= -(x_2, x_1, x_3)(-t).\end{aligned}$$

Principle of symmetric criticality (Palais 1979): Minimizers in H^G solve (3BD) provided all masses are equal.

Careful estimates show that

$$\inf_{x \in H^G} \mathcal{A}_T(x) < \inf_{\substack{x \in H^G \\ x \text{ has collision}}} \mathcal{A}_T(x)$$

Recent Progress – N-Body Problem

Chenciner-Venturelli (*Cel.Mech.Dyn.Ast.*2000)
K.C. (*Arch.Rat.Mech.Ana.*2001)
K.C. (*Erg.Thy.Dyn.Sys.*2003)
K.C. (*Arch.Rat.Mech.Ana.*2003)
Ferrario-Terracini (*Invent.Math.*2004)
Venturelli-Terracini (*Arch.Rat.Mech.Ana.*2007)
Barutello-Terracini (*Nonlinearity* 2005)
Ferrario (*Arch.Rat.Mech.Ana.*2006) …..

Existence of miscellaneous solutions with some equal masses.

Marchal, Chenciner (*Proc.I.C.M.* 2002)

Action minimizers for problems with fixed ends are free from interior collisions.

Ferrario-Terracini (*Invent.Math.*2004)

Action minimizers for some equivariant problems are free from collisions.

K.C. (*Arch.Rat.Mech.Ana.*2006)

Action minimizers for problems with free boundaries are free from collisions.

Question:

Are variational methods useful
only when some masses are equal?

Retrograde Orbits with Various Masses

♦ K.C. (*Annals Math.*, *2008*):

For "most" choices of masses, there exist infinitely many periodic and quasi-periodic retrograde solutions for the (3BD), none of which contain a tight binary.

Retrograde Orbits with Various Masses

♦ K.C. (*Annals Math., 2008*):

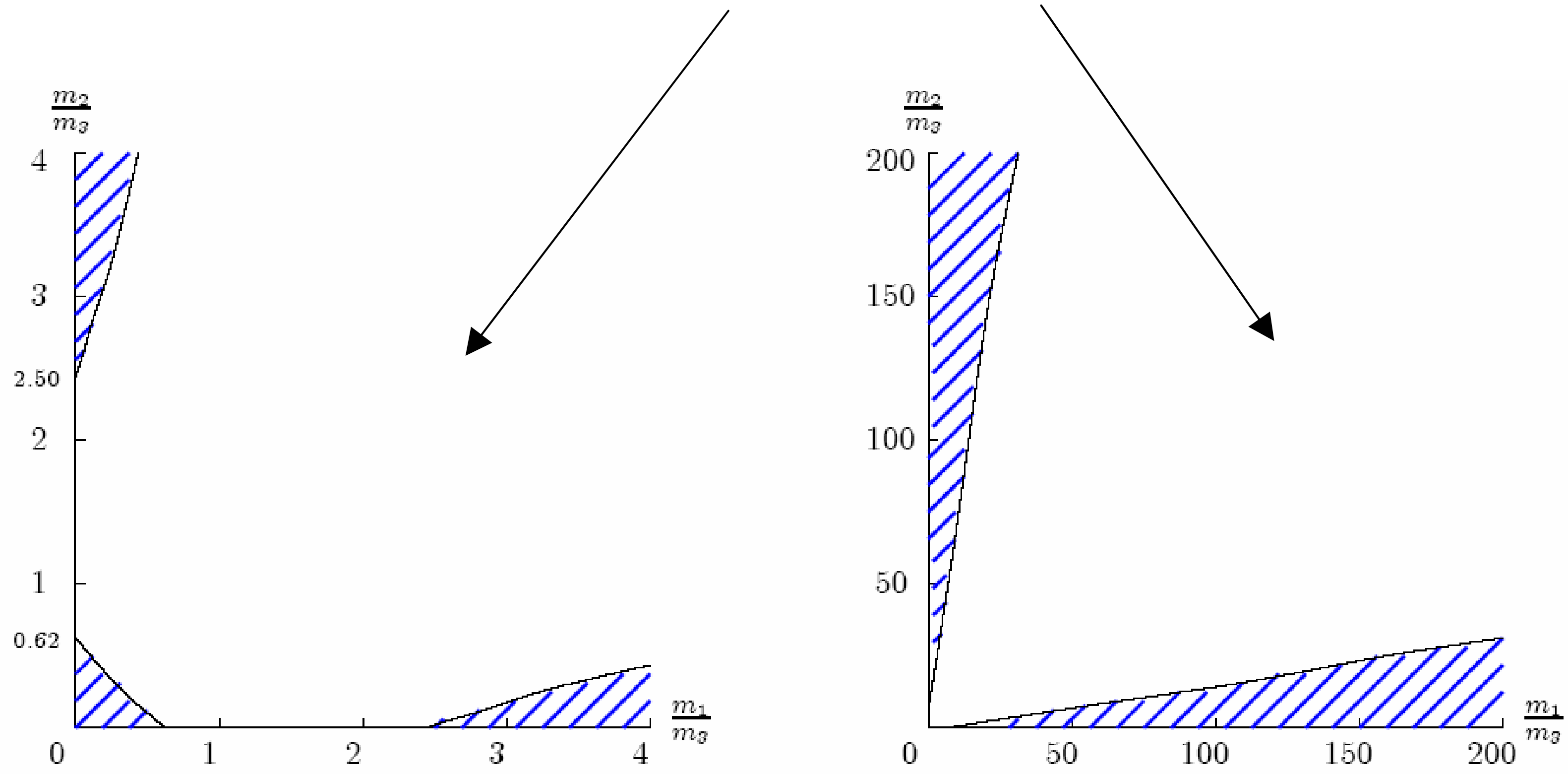
The (3BD) has infinitely many periodic and quasi-periodic retrograde solutions without tight binaries

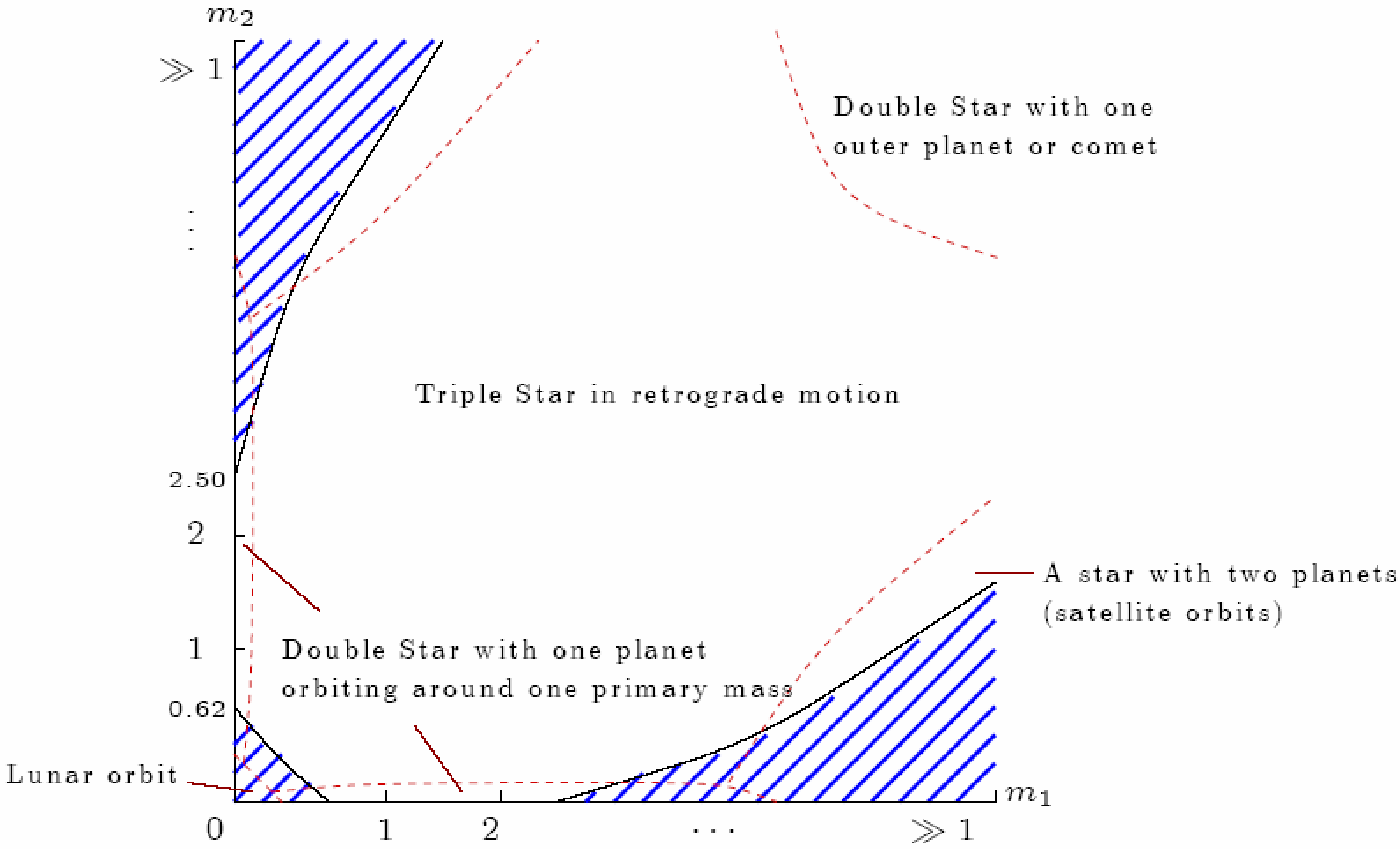
provided $$F(m_1, m_2) \;>\; G(m_1, m_2)$$

where

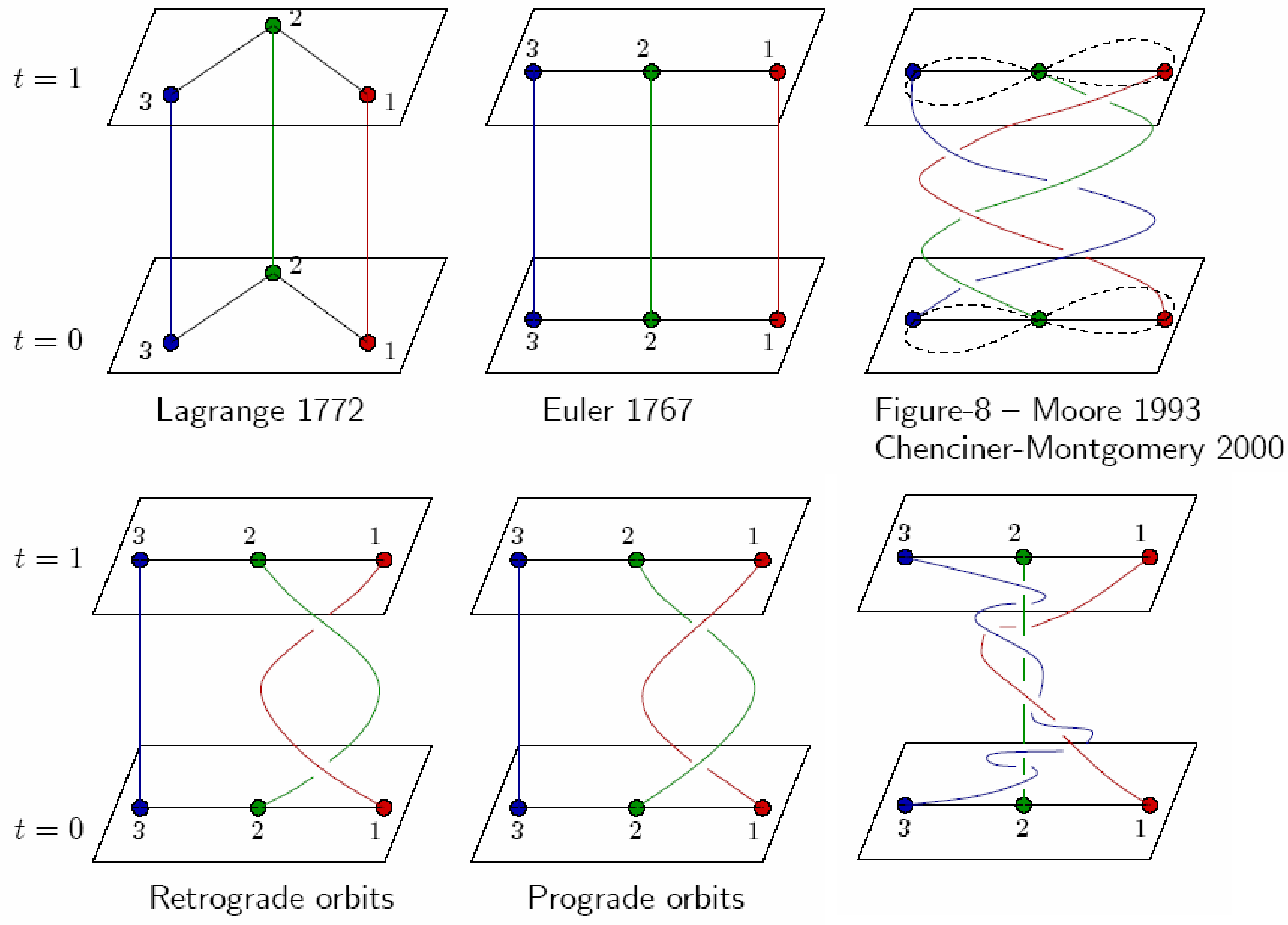
$$\begin{aligned}
J(s) &:= \int_0^1 \frac{1}{|1 - s\, e^{2\pi t i}|}\, dt \\
F(m_1, m_2) &:= \tfrac{3}{2}\left[\tfrac{2^{2/3}-1}{\max\{m_i\}} + 1 - \left(\tfrac{M}{m_1+m_2}\right)^{\frac{1}{3}}\right] \\
G(m_1, m_2) &:= \tfrac{1}{m_1}\left(J\left(\tfrac{m_1}{M^{1/3}(m_1+m_2)^{2/3}}\right) - 1\right) + \tfrac{1}{m_2}\left(J\left(\tfrac{m_2}{M^{1/3}(m_1+m_2)^{2/3}}\right) - 1\right).
\end{aligned}$$

Region of admissible masses





m_2
$\gg 1$
Double Star with one outer planet or comet
Triple Star in retrograde motion
2.50
2
A star with two planets (satellite orbits)
1
Double Star with one planet orbiting around one primary mass
0.62
Lunar orbit
m_1
0
1
2
$\gg 1$



$t = 1$
$t = 0$
3 2 1
Lagrange 1772
Euler 1767
Figure-8 – Moore 1993
Chenciner-Montgomery 2000
Retrograde orbits
Prograde orbits

Key idea of proof:

Given $\phi \in (0,\pi]$, $T>0$, define

$$\Gamma_{T,\phi} \ := \ \{\mathbf{r} \in H^1([0,T],\mathbb{C}) : \ \mathbf{r}(t) \neq 0, \ \langle \mathbf{r}(0), \mathbf{r}(T)\rangle = |\mathbf{r}(0)||\mathbf{r}(T)| \cos\phi\}$$

Let $\alpha > 0$, $\mu > 0$. Then

$$\begin{aligned} \inf_{\mathbf{r}\in\Gamma_{T,\phi}} \int_0^T \frac{\mu}{2}|\dot{\mathbf{r}}|^2 + \frac{\alpha}{|\mathbf{r}|}\,dt &= \frac{3}{2}(\mu\alpha^2\phi^2)^{\frac{1}{3}}T^{\frac{1}{3}}\,, \\ \inf_{\mathbf{r}\in\partial\Gamma_{T,\phi}} \int_0^T \frac{\mu}{2}|\dot{\mathbf{r}}|^2 + \frac{\alpha}{|\mathbf{r}|}\,dt &= \frac{3}{2}(\mu\alpha^2\pi^2)^{\frac{1}{3}}T^{\frac{1}{3}}\,. \end{aligned}$$

Let $V := \{x \in \mathbb{C}^3 : m_1x_1 + m_2x_2 + m_3x_3 = 0\}$

$H_\phi(V) := \{x \in H^1_{\text{loc}}(\mathbb{R}, V) : x(t+1) = e^{\phi i}x(t)\}$

On $\mathcal{X}_\phi = \left\{ x \in H_\phi(V) : \begin{array}{l} x(t) = \overline{x(-t)} \text{ for all } t \text{ and} \\ x \text{ is a retrograde loop} \end{array} \right\}$

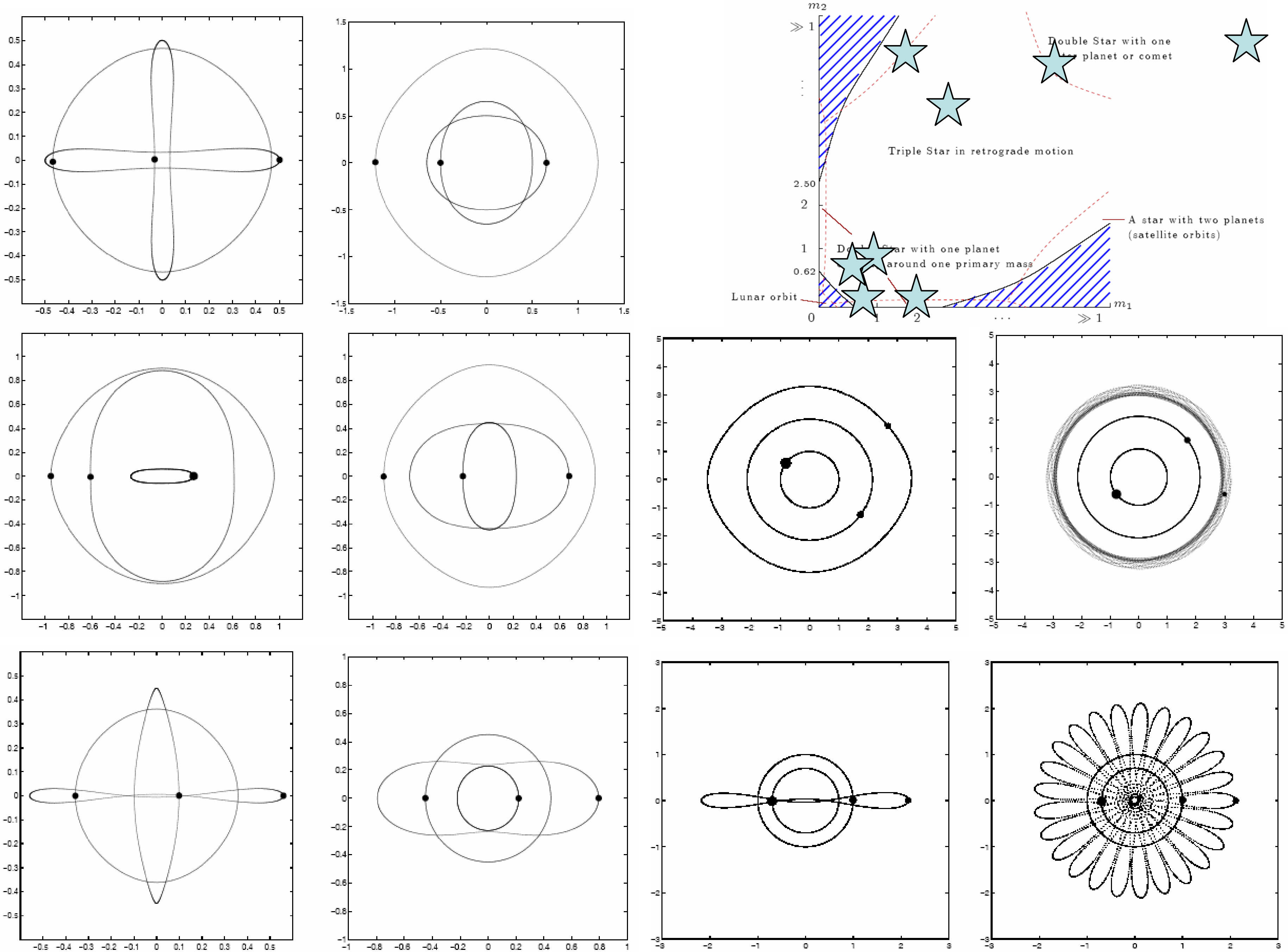
the action functional can be written

$$\mathcal{A}_T(x) = \frac{2}{M} \sum_{i<j} m_i m_j \int_0^{\frac{1}{2}} \frac{1}{2}|\dot{x}_i - \dot{x}_j|^2 + \frac{M}{|x_i - x_j|}\, dt.$$

Then

$$\inf_{x \in \partial\mathcal{X}_\phi} \mathcal{A}_T(x) \geq \frac{3}{2M^{1/3}} \min_{\sigma \in S_3} \left[m_{\sigma_1} m_{\sigma_2} (2\pi)^{\frac{2}{3}} + (m_{\sigma_1} m_{\sigma_3} + m_{\sigma_2} m_{\sigma_3}) \phi^{\frac{2}{3}} \right] .$$

Can show that this lower bound estimate is quite sharp and is (a little bit) higher than Kepler-like retrograde paths as long as $F(m_1, m_2) > G(m_1, m_2)$ which is valid for most choices of masses.



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THE END