Calculus II Name:____________________
Student ID:____________________

Quiz 5

Mar. 23, 2006

1. (8 pts) Given that

$$\frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k, \quad \text{for } -1 < x < 1,$$

find the Taylor series of $\ln(1+x)$ and $\frac{1}{1+x^2}$. Determine the corresponding radius and interval of convergence.

Ans:

(i)$\ln(1+x)$

Since $\int \frac{1}{1+x}\,dx = \ln(1+x) + C$, we can get the Taylor series of $\ln(1+x)$ from the series of $\frac{1}{1+x}$, (term-by-term integration)

$$\ln(1+x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{k+1}}{k+1} + C.$$

To determine the constant, C, we plug $x = 0$ into the equation,

$$0 = \ln(1) = \sum_{k=0}^{\infty}(-1)^k \frac{0^{k+1}}{k+1} + C.$$

Therefore,

$$\ln(1+x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{k+1}}{k+1}.$$

Since the radius of convergence of $\sum_{k=0}^{\infty}(-1)^k x^k$ is 1, the radius of convergence of $\sum_{k=0}^{\infty}(-1)^k \frac{x^{k+1}}{k+1}$ is also 1.

At $x = 1$, $\sum_{k=0}^{\infty}(-1)^k \frac{1}{k+1}$ is an Alternating Harmonic Series and the series converges.

At $x = -1$, $\sum_{k=0}^{\infty}(-1)^k \frac{(-1)^{k+1}}{k+1} = \sum_{k=0}^{\infty} \frac{-1}{k+1}$ is a Harmonic series and the series diverges.

Therefore the interval of convergence is $(-1, 1]$

(ii)$\frac{1}{1+x^2}$

Since $\frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k$, $\frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k (x^2)^k = \sum_{k=0}^{\infty}(-1)^k x^{2k}$.

The radius of convergence is the same.

At $x = \pm 1$, the series is $\sum_{k=0}^{\infty}(-1)^k$.

Since $\lim_{k\to\infty}(-1)^k \neq 0$ (diverge), by kth-term test for divergence, the series diverges at $x = \pm 1$.

Therefore the interval of convergence is $(-1, 1)$

2. (2pts)

$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{k+1} = ?$$

Since

$$\ln(1+x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{k+1}}{k+1} \quad \text{for } x \in (-1, 1],$$

$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{k+1} = \ln(1+1) = \ln 2.$$

(Note: $1 \in (-1, 1]$)

Write your solutions as complete as possible. Working time: 15 minutes.