1. (8 pts) Given that
\[ \frac{1}{1 + x} = \sum_{k=0}^{\infty} (-1)^k x^k, \quad \text{for } -1 < x < 1, \]
find the Taylor series of \( \ln (1 + x) \) and \( \frac{1}{1+x^2} \). Determine the corresponding radius and interval of convergence.

Ans:
(i) \( \ln (1 + x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} + C \).

To determine the constant, \( C \), we plug \( x = 0 \) into the equation,
\[ 0 = \ln(1) = \sum_{k=0}^{\infty} (-1)^k \frac{0^{k+1}}{k+1} + C. \]
Therefore,
\[ \ln (1 + x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}. \]

Since the radius of convergence of \( \sum_{k=0}^{\infty} (-1)^k x^k \) is 1, the radius of convergence of \( \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} \) is also 1.

At \( x = 1 \), \( \sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1} \) is an Alternating Harmonic Series and the series converges.

At \( x = -1 \), \( \sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1} = \sum_{k=0}^{\infty} \frac{1}{k+1} \) is a Harmonic series and the series diverges.

Therefore the interval of convergence is \((-1, 1]\)
(ii) \( \frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^k, \frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k (x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}. \)

The radius of convergence is the same.

At \( x = \pm 1 \), the series is \( \sum_{k=0}^{\infty} (-1)^k \).

Since \( \lim_{k \to \infty} (-1)^k \neq 0 \) (diverge), by kth-term test for divergence, the series diverges at \( x = \pm 1 \).
Therefore the interval of convergence is \((-1, 1)\)

2. (2pts)
\[ \sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1} = ? \]

Since
\[ \ln (1 + x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} \quad \text{for } x \in (-1, 1], \]
\[ \sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1} = \ln (1 + 1) = \ln 2. \]

(Note: \( 1 \in (-1, 1) \))

Write your solutions as complete as possible. Working time: 15 minutes.