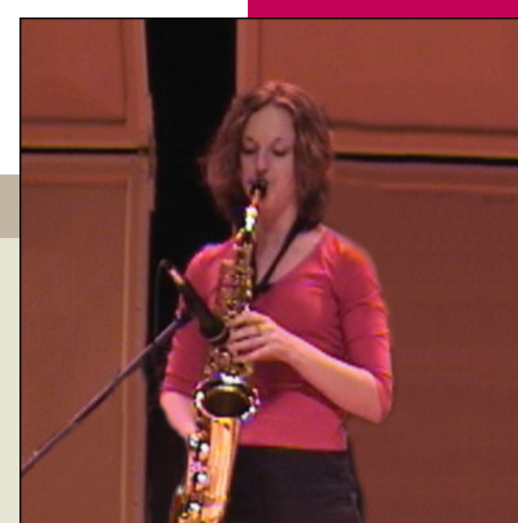
INFINITE SERIES

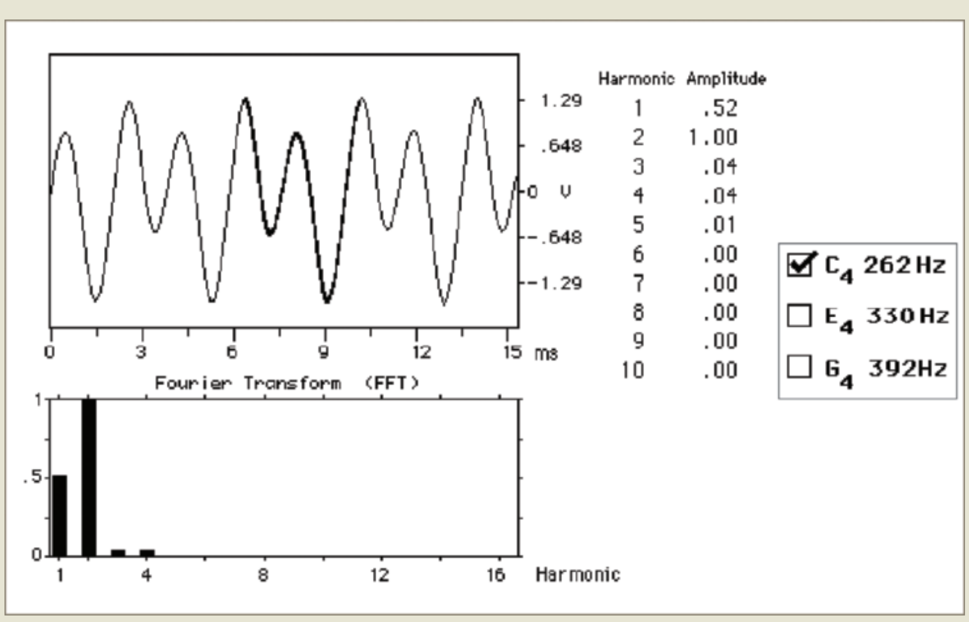


CHAPTER

7

In our daily lives, we are increasingly seeing the impact of digital technologies. You needn't go far to observe this phenomenon. For instance, the dominant media for the entertainment industry are now CDs and DVDs; we have digital video and still cameras and the Internet gives us easy access to a virtual world of digital information. An essential ingredient in this digital revolution is the use of Fourier analysis, a mathematical idea that is introduced in this chapter.

In this digital age, we have learned to represent information in a variety of ways. The ability to easily transform one representation into another gives us tremendous problem-solving powers. As an example, consider the music made by a saxophone. The music is initially represented as a series of notes on sheet music, but the musician brings her own special interpretation to the music. Such an individual performance can then be recorded, to be copied and replayed later. While this is easily accomplished with conventional analog technology, the advent of digital technology has allowed us to record the performance with a previously unknown



fidelity. The key to this is that the music is broken down into its component parts, which are individually recorded and then reassembled on demand to recreate the original sound. Think for a moment how spectacular this feat really is. The complex rhythms and intonations generated by the saxophone reed and body are somehow converted into a relatively small number of digital bits (zeroes and ones). The bits are then turned back into music by a CD player.

The basic idea behind any digital technology is to break down a complex whole into a set of component pieces. To digitally capture a saxophone note, all of the significant features of the saxophone waveform must be captured.

Done properly, the components can then be recombined to reproduce each original note. In this chapter, we learn how series of numbers combine and how functions can be broken down into a series of component functions. As part of this discussion, we will explore how music synthesizers work, but we will also see how calculators can quickly approximate a quantity like sin 1.23 and how equations can be solved using functions for which we don't even have names. This chapter opens up a new world of important applications.

7.1 SEQUENCES OF REAL NUMBERS

The mathematical notion of sequence is not much different from the common English usage of the word. For instance, if you were asked to describe the sequence of events that led up to a traffic accident, you'd not only need to list the events, but you'd need to do so in a specific *order* (hopefully, the order in which they actually occurred). In mathematics, we use the term sequence to mean an infinite collection of real numbers, written in a specific order.

We have already seen sequences several times now (although we have not formally introduced the notion). For instance, you have found approximate solutions to nonlinear equations like $\tan x - x = 0$, by first making an initial guess, x_0 and then using Newton's method to compute a sequence of successively improved approximations, $x_1, x_2, \ldots, x_n, \ldots$.

Definition of sequence

By **sequence,** we mean any function whose domain is the set of integers starting with some integer n_0 (often 0 or 1). For instance, the function $a(n) = \frac{1}{n}$, for $n = 1, 2, 3, \ldots$, defines the sequence

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

Here, $\frac{1}{1}$ is called the first **term,** $\frac{1}{2}$ is the second term and so on. We call $a(n) = \frac{1}{n}$ the **general term,** since it gives a (general) formula for computing all the terms of the sequence. Further, we usually use subscript notation instead of function notation and write a_n instead of $a(n)$.

EXAMPLE 1.1 The Terms of a Sequence

Write out the terms of the sequence whose general term is given by $a_n = \dfrac{n+1}{n}$, for $n = 1, 2, 3, \ldots$.

Solution We have the sequence

$$a_1 = \frac{1+1}{1} = \frac{2}{1},\ a_2 = \frac{2+1}{2} = \frac{3}{2},\ a_3 = \frac{4}{3},\ a_4 = \frac{5}{4}, \ldots.$$

We often use set notation to denote a sequence. For instance, the sequence with general term $a_n = \dfrac{1}{n^2}$, for $n = 1, 2, 3, \ldots$, is denoted by

$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n^2}\right\}_{n=1}^{\infty},$$

or equivalently, by listing the terms of the sequence:

$$\left\{\frac{1}{1}, \frac{1}{2^2}, \frac{1}{3^2}, \ldots, \frac{1}{n^2}, \ldots\right\}.$$

To graph this sequence, we plot a number of discrete points, since a sequence is a function defined only on the integers (see Figure 7.1). You have likely already noticed something about the sequence $\left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$. As n gets larger and larger, the terms of the sequence, $a_n = \frac{1}{n^2}$ get closer and closer to zero. In this case, we say that the sequence **converges** to 0 and write

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0.$$

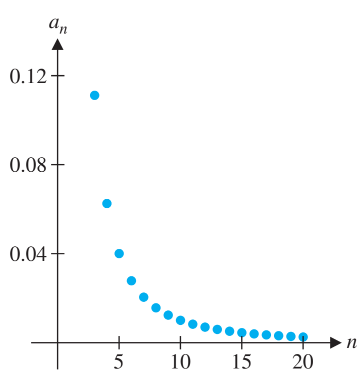


FIGURE 7.1
$a_n = \frac{1}{n^2}$.

In general, we say that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to L $\left(\text{i.e., } \lim_{n\to\infty} a_n = L\right)$ if we can make a_n as close to L as desired, simply by making n sufficiently large. We call L the *limit* of the sequence. You may notice that this language parallels that used in the definition of the limit

$$\lim_{x\to\infty} f(x) = L$$

for a function of a real variable x (given in Chapter 1). The only difference is that n can take on only integer values, while x can take on any real value (integer, rational or irrational).

Most of the usual rules for computing limits of functions of a real variable also apply to computing the limit of a sequence, as we see in the following result.

THEOREM 1.1

Suppose that $\{a_n\}_{n=n_0}^{\infty}$ and $\{b_n\}_{n=n_0}^{\infty}$ both converge. Then

(i) $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$,

(ii) $\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$,

(iii) $\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$ and

(iv) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ (assuming $\lim_{n\to\infty} b_n \neq 0$).

The proof of Theorem 1.1 is virtually identical to the proof of the corresponding theorem about limits of a function of a real variable (see Theorem 3.1 in section 1.3 and Appendix G) and is omitted.

NOTES

If you (incorrectly) apply l'Hôpital's Rule in example 1.2, you get the right answer. (Go ahead and try it; nobody's looking.) Unfortunately, you will not always be so lucky. It's a lot like trying to cross a busy highway: while there are times when you can successfully cross with your eyes closed, it's not generally recommended. Theorem 1.2 describes how you can safely use l'Hôpital's Rule.

REMARK 1.1

To find the limit of a sequence, you should work largely the same as when computing the limit of a function of a real variable, but keep in mind that sequences are defined *only* for integer values of the variable.

EXAMPLE 1.2 Finding the Limit of a Sequence

Evaluate $\lim_{n\to\infty} \frac{5n+7}{3n-5}$.

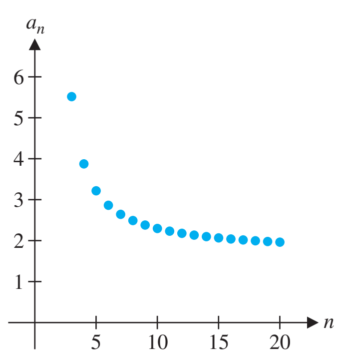


FIGURE 7.2
$a_n = \dfrac{5n+7}{3n-5}$.

Solution Of course, this has the indeterminate form $\frac{\infty}{\infty}$. From the graph in Figure 7.2, it looks like the sequence tends to some limit around 2. Note that we cannot apply l'Hôpital's Rule here, since the functions in the numerator and the denominator are not continuous. (They are only defined for integer values of n, even though you *could* define these expressions for any real values of n.) You can, of course, use the simpler method of dividing numerator and denominator by the highest power of n in the denominator. We have

$$\lim_{n\to\infty} \frac{5n+7}{3n-5} = \lim_{n\to\infty} \frac{(5n+7)\left(\frac{1}{n}\right)}{(3n-5)\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{5+\frac{7}{n}}{3-\frac{5}{n}} = \frac{5}{3}.$$ ■

In example 1.3, we see a sequence that diverges by virtue of its terms tending to $+\infty$.

EXAMPLE 1.3 A Divergent Sequence

Evaluate $\displaystyle\lim_{n\to\infty} \frac{n^2+1}{2n-3}$.

Solution Again, this has the indeterminate form $\frac{\infty}{\infty}$, but from the graph in Figure 7.3, the sequence appears to be increasing without bound. Dividing top and bottom by n (the highest power of n in the denominator) we have

$$\lim_{n\to\infty} \frac{n^2+1}{2n-3} = \lim_{n\to\infty} \frac{(n^2+1)\left(\frac{1}{n}\right)}{(2n-3)\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n+\frac{1}{n}}{2-\frac{3}{n}} = \infty$$

and so, the sequence $\left\{\dfrac{n^2+1}{2n-3}\right\}_{n=1}^{\infty}$ diverges. ■

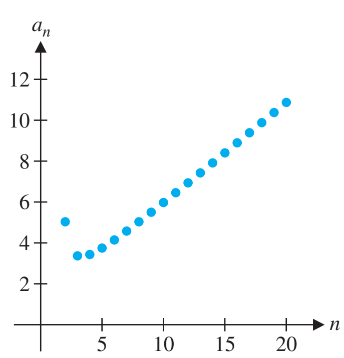


FIGURE 7.3
$a_n = \dfrac{n^2+1}{2n-3}$.

In example 1.4, we see that a sequence doesn't need to tend to $\pm\infty$ in order to diverge.

EXAMPLE 1.4 A Divergent Sequence Whose Terms Do Not Tend to ∞

Determine the convergence or divergence of the sequence $\{(-1)^n\}_{n=1}^{\infty}$.

Solution If we write out the terms of the sequence, we have

$$\{-1, 1, -1, 1, -1, 1, \ldots\}.$$

That is, the terms of the sequence alternate back and forth between -1 and 1 and so, the sequence diverges. To see this graphically, we plot the first few terms of the sequence in Figure 7.4. Notice that the points do not approach any limit (a horizontal line). ■

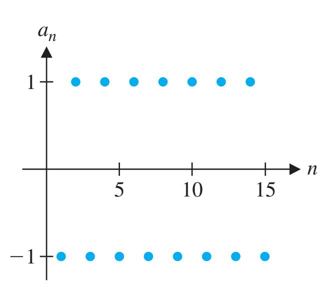


FIGURE 7.4
$a_n = (-1)^n$.

You can use an advanced tool like l'Hôpital's Rule to find the limit of a sequence, but you must be careful. Theorem 1.2 says that if $f(x) \to L$ as $x \to \infty$ through all real values, then $f(n)$ must approach L, too, as $n \to \infty$ through integer values. (See Figure 7.5 for a graphical representation of this.)

THEOREM 1.2

Suppose that $\displaystyle\lim_{x\to\infty} f(x) = L$. Then, $\displaystyle\lim_{n\to\infty} f(n) = L$, also.

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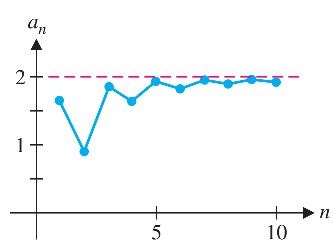

FIGURE 7.5
$a_n = f(n)$, where $f(x) \to 2$, as $x \to \infty$.

REMARK 1.2

The converse of Theorem 1.2 is false. That is, if $\lim_{n\to\infty} f(n) = L$, it need *not* be true that $\lim_{x\to\infty} f(x) = L$. This is clear from the following observation. Note that

$$\lim_{n\to\infty} \cos(2\pi n) = 1,$$

since $\cos(2\pi n) = 1$ for every *integer* n (see Figure 7.6).

However,

$$\lim_{x\to\infty} \cos(2\pi x) \text{ does not exist,}$$

since as $x \to \infty$, $\cos(2\pi x)$ oscillates between -1 and 1 (see Figure 7.7).

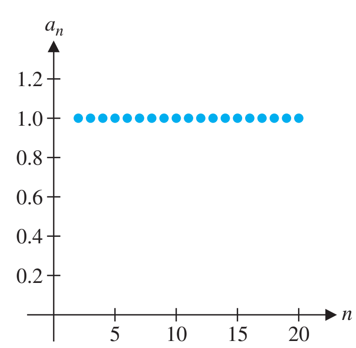

FIGURE 7.6
$a_n = \cos(2\pi n)$.

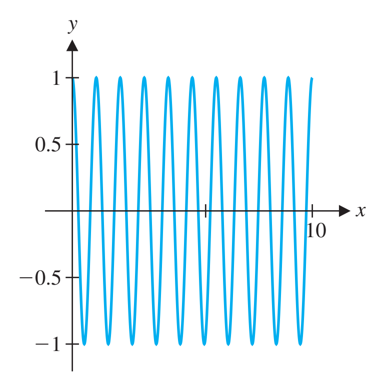

FIGURE 7.7
$y = \cos(2\pi x)$.

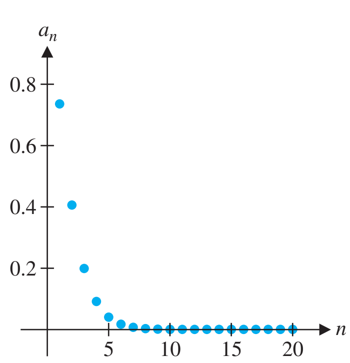

FIGURE 7.8
$a_n = \dfrac{n+1}{e^n}$.

EXAMPLE 1.5 Applying L'Hôpital's Rule to a Related Function

Evaluate $\lim_{n\to\infty} \dfrac{n+1}{e^n}$.

Solution This has the indeterminate form $\dfrac{\infty}{\infty}$, but from the graph in Figure 7.8, it appears that the sequence converges to 0. However, there is no obvious way to resolve this, except by l'Hôpital's Rule (which does *not* apply to limits of sequences). So, we instead consider the limit of the corresponding function of a real variable to which we may apply l'Hôpital's Rule. (Be sure you check the hypotheses.) We have

$$\lim_{x\to\infty} \frac{x+1}{e^x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(x+1)}{\frac{d}{dx}(e^x)} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

From Theorem 1.2, we now have

$$\lim_{n\to\infty} \frac{n+1}{e^n} = 0, \text{ also.}$$

Although we now have a few tools for computing the limit of a sequence, most interesting sequences resist our attempts to find their limit. In many cases (including infinite series, which we study throughout the remainder of this chapter), we don't even have an explicit formula for the general term. In such circumstances, we must test the sequence for convergence in some indirect way. The first indirect tool we present corresponds to the result (of the same name) for limits of functions of a real variable presented in section 1.3.

THEOREM 1.3 (Squeeze Theorem)

Suppose $\{a_n\}_{n=n_0}^{\infty}$ and $\{b_n\}_{n=n_0}^{\infty}$ are convergent sequences, both converging to the limit, L. If there is an integer $n_1 \geq n_0$ such that for all $n \geq n_1$, $a_n \leq c_n \leq b_n$, then $\{c_n\}_{n=n_0}^{\infty}$ converges to L, too.

In example 1.6, we demonstrate how to apply the Squeeze Theorem to a sequence. Observe that the trick here is to find two sequences, one on either side of the given sequence (i.e., one larger and one smaller) that have the same limit.

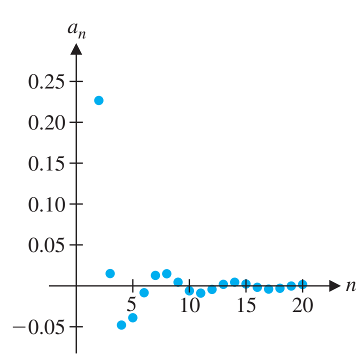

FIGURE 7.9
$a_n = \dfrac{\sin n}{n^2}$.

EXAMPLE 1.6 Applying the Squeeze Theorem to a Sequence

Determine the convergence or divergence of $\left\{\dfrac{\sin n}{n^2}\right\}_{n=1}^{\infty}$.

Solution From the graph in Figure 7.9, the sequence appears to converge to 0, despite the oscillation. Further, note that you cannot compute this limit using the rules we have established so far. (Try it!) However, recall that

$$-1 \leq \sin n \leq 1, \text{ for all } n.$$

Dividing through by n^2 gives us

$$\frac{-1}{n^2} \leq \frac{\sin n}{n^2} \leq \frac{1}{n^2}, \text{ for all } n \geq 1.$$

Finally, observe that

$$\lim_{n\to\infty} \frac{-1}{n^2} = 0 = \lim_{n\to\infty} \frac{1}{n^2}.$$

From the Squeeze Theorem, we now have that

$$\lim_{n\to\infty} \frac{\sin n}{n^2} = 0,$$

also. ■

The following useful result follows immediately from Theorem 1.3.

COROLLARY 1.1

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$, also.

PROOF

Notice that for all n,

$$-|a_n| \leq a_n \leq |a_n|$$

and

$$\lim_{n\to\infty} |a_n| = 0 \quad \text{and} \quad \lim_{n\to\infty} (-|a_n|) = -\lim_{n\to\infty} |a_n| = 0.$$

So, from the Squeeze Theorem, $\lim_{n\to\infty} a_n = 0$, too. ■

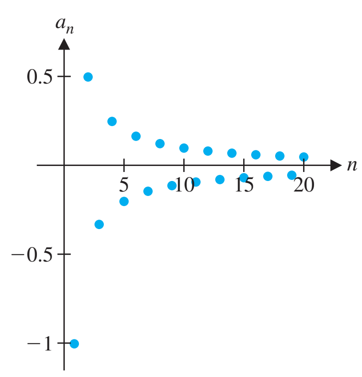

FIGURE 7.10
$a_n = \dfrac{(-1)^n}{n}$.

Corollary 1.1 is particularly useful for sequences with both positive and negative terms, as in example 1.7.

EXAMPLE 1.7 A Sequence with Terms of Alternating Signs

Determine the convergence or divergence of $\left\{\dfrac{(-1)^n}{n}\right\}_{n=1}^{\infty}$.

Solution From the graph of the sequence in Figure 7.10, it seems that the sequence oscillates but still may be converging to 0. Since $(-1)^n$ oscillates back and forth between -1 and 1, we cannot compute $\lim\limits_{n\to\infty} \dfrac{(-1)^n}{n}$ directly. However, notice that

$$\left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$$

and

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

From Corollary 1.1, we get that $\lim\limits_{n\to\infty} \dfrac{(-1)^n}{n} = 0$, too. ■

We remind you of Definition 1.1, which we use throughout the chapter.

DEFINITION 1.1

For any integer $n \geq 1$, the **factorial,** $n!$ is defined as the product of the first n positive integers,

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n.$$

We define $0! = 1$.

Example 1.8 shows a sequence whose limit would be extremely difficult to find without the Squeeze Theorem.

EXAMPLE 1.8 An Indirect Proof of Convergence

Investigate the convergence of $\left\{\dfrac{n!}{n^n}\right\}_{n=1}^{\infty}$.

Solution First, notice that we have no means of computing $\lim\limits_{n\to\infty} \dfrac{n!}{n^n}$ directly. (Try this!) From the graph of the sequence in Figure 7.11, it appears that the sequence is converging to 0. Notice that the general term of the sequence satisfies

$$0 < \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{\underbrace{n \cdot n \cdot n \cdot \cdots \cdot n}_{n \text{ factors}}}$$

$$= \left(\frac{1}{n}\right)\frac{2 \cdot 3 \cdot \cdots \cdot n}{\underbrace{n \cdot n \cdot \cdots \cdot n}_{n-1 \text{ factors}}} \leq \left(\frac{1}{n}\right)(1) = \frac{1}{n}. \quad (1.1)$$

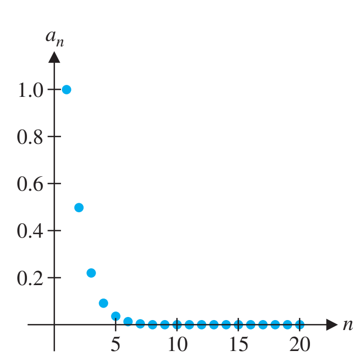

FIGURE 7.11
$a_n = \dfrac{n!}{n^n}$.

From the Squeeze Theorem and (1.1), we have that since

$$\lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{and} \quad \lim_{n\to\infty} 0 = 0,$$

then

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0, \text{ also.}$$

Just as we did with functions of a real variable, we need to distinguish between sequences that are increasing and decreasing. The definitions are quite straightforward.

The sequence $\{a_n\}_{n=1}^{\infty}$ is **increasing** if

$$a_1 \le a_2 \le \cdots \le a_n \le a_{n+1} \le \cdots.$$

The sequence $\{a_n\}_{n=1}^{\infty}$ is **decreasing** if

$$a_1 \ge a_2 \ge \cdots \ge a_n \ge a_{n+1} \ge \cdots.$$

If a sequence is either increasing or decreasing, it is called **monotonic.**

There are any number of ways to show that a sequence is monotonic. Regardless of which method you use, you will need to show that either $a_n \le a_{n+1}$ for all n (increasing) or $a_{n+1} \le a_n$ for all n (decreasing). One very useful method is to look at the ratio of the two successive terms a_n and a_{n+1}. We illustrate this in example 1.9.

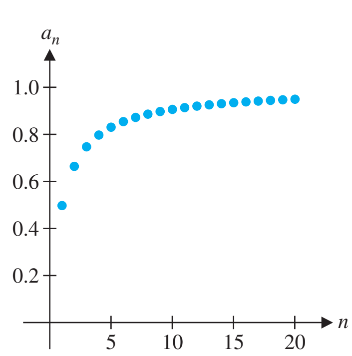

FIGURE 7.12
$a_n = \dfrac{n}{n+1}$.

EXAMPLE 1.9 An Increasing Sequence

Investigate whether the sequence $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ is increasing, decreasing or neither.

Solution From the graph in Figure 7.12, it appears that the sequence is increasing. However, you should not be deceived by looking at the first few terms of a sequence. Instead, we look at the ratio of two successive terms. So, if we define $a_n = \dfrac{n}{n+1}$, we have $a_{n+1} = \dfrac{n+1}{n+2}$ and

$$\frac{a_{n+1}}{a_n} = \frac{\left(\dfrac{n+1}{n+2}\right)}{\left(\dfrac{n}{n+1}\right)} = \left(\frac{n+1}{n+2}\right)\left(\frac{n+1}{n}\right)$$

$$= \frac{n^2+2n+1}{n^2+2n} = 1 + \frac{1}{n^2+2n} > 1. \qquad (1.2)$$

Since $a_n > 0$, notice that we can multiply both sides of (1.2) by a_n, to obtain

$$a_{n+1} > a_n,$$

for all n and so, the sequence is increasing. As an alternative, notice that you can always consider the function $f(x) = \dfrac{x}{x+1}$ (of the real variable x) corresponding to the sequence. Observe that

$$f'(x) = \frac{(x+1)-x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0,$$

which says that the function $f(x)$ is increasing. From this, it follows that the corresponding sequence $a_n = \frac{n}{n+1}$ is also increasing. ■

EXAMPLE 1.10 A Sequence That Is Increasing for $n \geq 2$

Investigate whether the sequence $\left\{\frac{n!}{e^n}\right\}_{n=1}^{\infty}$ is increasing, decreasing or neither.

Solution From the graph of the sequence in Figure 7.13, it appears that the sequence is increasing (and rather rapidly, at that). Here, for $a_n = \frac{n!}{e^n}$, we have $a_{n+1} = \frac{(n+1)!}{e^{n+1}}$, so that

$$\frac{a_{n+1}}{a_n} = \frac{\left[\frac{(n+1)!}{e^{n+1}}\right]}{\left(\frac{n!}{e^n}\right)} = \frac{(n+1)!}{e^{n+1}} \frac{e^n}{n!}$$

$$= \frac{(n+1)n!e^n}{e(e^n)n!} = \frac{n+1}{e} > 1, \text{ for } n \geq 2. \quad (1.3)$$

Since $(n+1)! = (n+1) \cdot n!$ and $e^{n+1} = e \cdot e^n$.

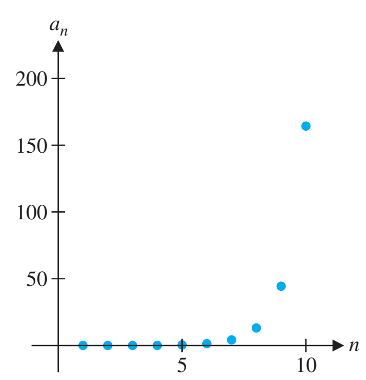

FIGURE 7.13
$a_n = \frac{n!}{e^n}$.

Multiplying both sides of (1.3) by $a_n > 0$, we get

$$a_{n+1} > a_n, \text{ for } n \geq 2.$$

Notice that in this case, although the sequence is not increasing for all n, it is increasing for $n \geq 2$. Keep in mind that it doesn't really matter what the first few terms do, anyway. We are only concerned with the behavior of a sequence as $n \to \infty$. ■

We need to define one additional property of sequences. We say that the sequence $\{a_n\}_{n=n_0}^{\infty}$ is **bounded** if there is a number $M > 0$ (called a **bound**) for which $|a_n| \leq M$, for all n.

There is often some slight confusion here. A bound is *not* the same as a limit, although the two may coincide. The limit of a convergent sequence is the value that the terms of the sequence are approaching as $n \to \infty$. On the other hand, a bound is any number that is greater than or equal to the absolute value of every term. This says that a given sequence may have any number of bounds (e.g., if $|a_n| \leq 10$ for all n, then $|a_n| \leq 20$, for all n, too). However, a sequence may have only *one* limit (or no limit, in the case of a divergent sequence).

EXAMPLE 1.11 A Bounded Sequence

Show that the sequence $\left\{\frac{3-4n^2}{n^2+1}\right\}_{n=1}^{\infty}$ is bounded.

Solution We use the fact that $4n^2 - 3 > 0$, for all $n \geq 1$, to get

$$|a_n| = \left|\frac{3-4n^2}{n^2+1}\right| = \frac{4n^2-3}{n^2+1} < \frac{4n^2}{n^2+1} < \frac{4n^2}{n^2} = 4.$$

So, this sequence is bounded by 4. (We might also say in this case that the sequence is bounded between −4 and 4.) Further, note that we could also use any number greater than 4 as a bound. ■

We should emphasize that the reason we are considering whether a sequence is monotonic or bounded is that very often we cannot compute the limit of a given sequence directly and must rely on indirect means to determine whether or not the sequence is convergent. Theorem 1.4 provides a powerful tool for the investigation of sequences.

THEOREM 1.4

Every bounded, monotonic sequence converges.

A typical bounded and increasing sequence is illustrated in Figure 7.14a, while a bounded and decreasing sequence is illustrated in Figure 7.14b. In both figures, notice that a bounded and monotonic sequence has nowhere to go and consequently, must converge. The proof of Theorem 1.4 is rather involved and can be found in a more advanced text.

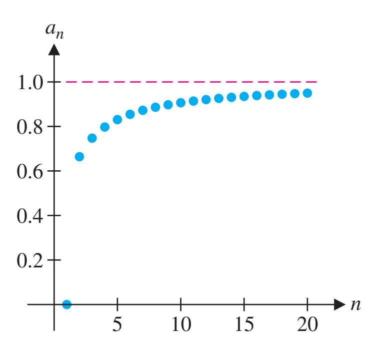

FIGURE 7.14a
A bounded and increasing sequence.

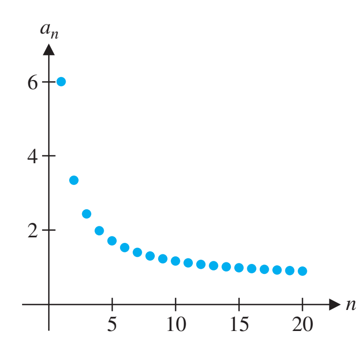

FIGURE 7.14b
A bounded and decreasing sequence.

In the very common case where we do not know how to compute the limit of a sequence, this theorem says that if we can show that a sequence is bounded and monotonic, then it must also be convergent, although we may have little idea of what its limit might be. Once we establish that a sequence converges, we can approximate its limit by computing a sufficient number of terms, as in example 1.12.

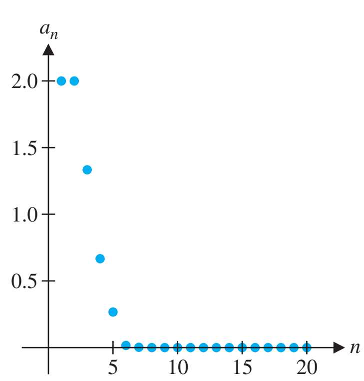

FIGURE 7.15
$a_n = \frac{2^n}{n!}$.

n	$a_n = \frac{2^n}{n!}$
2	2
4	0.666667
6	0.088889
8	0.006349
10	0.000282
12	0.0000086
14	1.88×10^{-7}
16	3.13×10^{-9}
18	4.09×10^{-11}
20	4.31×10^{-13}

EXAMPLE 1.12 An Indirect Proof of Convergence

Investigate the convergence of the sequence $\left\{\frac{2^n}{n!}\right\}_{n=1}^{\infty}$.

Solution First, note that we do not know how to compute $\lim_{n\to\infty}\frac{2^n}{n!}$. This has the indeterminate form $\frac{\infty}{\infty}$, but we cannot apply l'Hôpital's Rule to it. (Why not?) The graph in Figure 7.15 suggests that the sequence converges to some number close to 0. To confirm this suspicion, we first show that the sequence is monotonic. We have

$$\frac{a_{n+1}}{a_n} = \frac{\left[\frac{2^{n+1}}{(n+1)!}\right]}{\left(\frac{2^n}{n!}\right)} = \frac{2^{n+1}}{(n+1)!}\frac{n!}{2^n}$$

$$= \frac{2(2^n)\,n!}{(n+1)\,n!2^n} = \frac{2}{n+1} \le 1, \text{ for } n \ge 1. \tag{1.4}$$

Since $2^{n+1} = 2 \cdot 2^n$ and $(n+1)! = (n+1) \cdot n!$

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Multiplying both sides of (1.4) by a_n gives us $a_{n+1} \le a_n$, for all n and so, the sequence is decreasing. Next, since we have already shown that the sequence is decreasing, observe that

$$0 < \frac{2^n}{n!} \le \frac{2^1}{1!} = 2,$$

for $n \ge 1$ (i.e., the sequence is bounded by 2). Since the sequence is both bounded and monotonic, it must be convergent, by Theorem 1.4. To get an approximation of the limit of the sequence, we display a number of terms of the sequence in the accompanying table.

From the table, it appears that the sequence is converging to approximately 0. We can make a slightly stronger statement, though. Since we have established that the sequence is *decreasing* and convergent, we have from our computations that

$$0 \le a_n \le a_{20} \approx 4.31 \times 10^{-13}, \quad \text{for } n \ge 20.$$

Further, the limit L must also satisfy the inequality

$$0 \le L \le 4.31 \times 10^{-13}.$$

So, even if we can't conclude that the sequence converges to 0, we can conclude that it converges to some number extremely close to zero. For most purposes, such an estimate is entirely adequate. Of course, if you need greater precision, you can always compute a few more terms. ■

REMARK 1.3

Do not underestimate the importance of Theorem 1.4. This indirect way of testing a sequence for convergence takes on additional significance when we study infinite series (a special type of sequence that is the topic of the remainder of this chapter).

EXERCISES 7.1

WRITING EXERCISES

1. Compare and contrast $\lim_{x\to\infty} \sin \pi x$ and $\lim_{n\to\infty} \sin \pi n$. Indicate the domains of the two functions and how this affects the limits.
2. Explain why Theorem 1.2 should be true, taking into account the respective domains and their effect on the limits.
3. In words, explain why a decreasing bounded sequence must converge.
4. A sequence is said to diverge if it does not converge. The word "diverge" is well chosen for sequences that diverge to ∞, but is less descriptive of sequences such as $\{1, 2, 1, 2, 1, 2, \ldots\}$ and $\{1, 2, 3, 1, 2, 3, \ldots\}$. Briefly describe the limiting behavior of these sequences and discuss other possible limiting behaviors of divergent sequences.

In exercises 1–4, write out the terms $a_1, a_2, \ldots, a_6$ of the given sequence.

1. $a_n = \dfrac{2n-1}{n^2}$
2. $a_n = \dfrac{3}{n+4}$
3. $a_n = \dfrac{4}{n!}$
4. $a_n = (-1)^n \dfrac{n}{n+1}$

In exercises 5–10, (a) find the limit of each sequence and (b) plot the sequence on a calculator or CAS.

5. $a_n = \dfrac{1}{n^3}$
6. $a_n = \dfrac{2}{n^2}$
7. $a_n = \dfrac{n}{n+1}$
8. $a_n = \dfrac{2n+1}{n}$
9. $a_n = \dfrac{2}{\sqrt{n}}$
10. $a_n = \dfrac{4}{\sqrt{n+1}}$

In exercises 11–30, determine whether the sequence converges or diverges. If it converges, determine the limit.

11. $a_n = \dfrac{3n^2+1}{2n^2-1}$
12. $a_n = \dfrac{5n^3-1}{2n^3+1}$
13. $a_n = \dfrac{n^2+1}{n+1}$
14. $a_n = \dfrac{n^2+1}{n^3+1}$
15. $a_n = \dfrac{n+2}{3n-1}$
16. $a_n = \dfrac{n+4}{n+1}$
17. $a_n = (-1)^n \dfrac{n+2}{3n-1}$
18. $a_n = (-1)^n \dfrac{n+4}{n+1}$
19. $a_n = (-1)^n \dfrac{n+2}{n^2+4}$
20. $a_n = (-1)^n \dfrac{4}{n+1}$

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21. $a_n = \cos \pi n$

22. $a_n = \sin \pi n$

23. $a_n = ne^{-n}$

24. $a_n = \dfrac{\cos n}{e^n}$

25. $a_n = \dfrac{e^n + 2}{e^{2n} - 1}$

26. $a_n = \dfrac{e^{2n}}{e^n + 1}$

27. $a_n = \dfrac{3^n}{e^n + 1}$

28. $a_n = \dfrac{n2^n}{3^n}$

29. $a_n = \dfrac{\cos n}{n!}$

30. $a_n = \dfrac{n!}{2^n}$

In exercises 31–34, use the Squeeze Theorem and Corollary 1.1 to prove that the sequence converges to 0 (given that $\lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^2} = 0$).

31. $a_n = \dfrac{\cos n}{n^2}$

32. $a_n = \dfrac{\cos n\pi}{n^2}$

33. $a_n = (-1)^n \dfrac{e^{-n}}{n}$

34. $a_n = (-1)^n \dfrac{\ln n}{n^2}$

In exercises 35–42, determine whether the sequence is increasing, decreasing or neither.

35. $a_n = \dfrac{n+3}{n+2}$

36. $a_n = \dfrac{n-1}{n+1}$

37. $a_n = \dfrac{e^n}{n}$

38. $a_n = \dfrac{n}{2^n}$

39. $a_n = \dfrac{2^n}{(n+1)!}$

40. $a_n = \dfrac{3^n}{(n+2)!}$

41. $a_n = \dfrac{10^n}{n!}$

42. $a_n = \dfrac{n!}{5^n}$

In exercises 43–46, show that the sequence is bounded.

43. $a_n = \dfrac{3n^2 - 2}{n^2 + 1}$

44. $a_n = \dfrac{6n - 1}{n + 3}$

45. $a_n = \dfrac{\sin(n^2)}{n+1}$

46. $a_n = e^{1/n}$

T **47.** Numerically estimate the limits of the sequences $a_n = \left(1 + \frac{1}{n}\right)^n$ and $b_n = \left(1 - \frac{1}{n}\right)^n$. Compare the answers to e and e^{-1}.

T **48.** Numerically estimate the limits of the sequences $a_n = \left(1 + \frac{2}{n}\right)^n$ and $b_n = \left(1 - \frac{2}{n}\right)^n$. Compare the answers to e^2 and e^{-2}.

49. Given that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$, show that $\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n = e^r$ for any constant r. (Hint: Make the substitution $n = rm$.)

50. Evaluate $\lim_{n\to\infty} \dfrac{(n+1)^n}{n^n}$ and $\lim_{n\to\infty} \dfrac{(n+1)^{n+1}}{n^n}$.

T **51.** Numerically estimate the limit of the sequence defined by $a_1 = \sqrt{2}$, $a_2 = \sqrt{2\sqrt{2}}$, $a_3 = \sqrt{2\sqrt{2\sqrt{2}}}$, and so on.

52. To verify your conjecture from exercise 51, note that the terms form the pattern $a_2 = \sqrt{2a_1}$, $a_3 = \sqrt{2a_2}$, and so on. If the sequence converges, then $a_{n+1} \approx a_n$ or $\sqrt{2a_n} \approx a_n$. Solve the equation $\sqrt{2a} = a$ to determine the values at which this can happen.

53. Find all values of p such that the sequence $a_n = \dfrac{1}{p^n}$ converges.

54. Find all values of p such that the sequence $a_n = \dfrac{1}{n^p}$ converges.

55. A packing company works with 12″ square boxes. Show that for $n = 1, 2, 3, \ldots$, a total of n^2 disks of diameter $\frac{12}{n}$″ fit into a box. Let a_n be the wasted area in a box with n^2 disks. Compute a_n.

56. The pattern of a sequence can't always be determined from the first few terms. Start with a circle, pick two points on the circle and connect them with a line segment. The circle is divided into $a_1 = 2$ regions. Add a third point, connect all points and show that there are now $a_2 = 4$ regions. Add a fourth point, connect all points and show that there are $a_3 = 8$ regions. Is the pattern clear? Show that $a_4 = 16$ and then compute a_5 for a surprise!

T **57.** You have heard about the "population explosion." The following dramatic warning is adapted from the article "Doomsday: Friday 13 November 2026" by Foerster, Mora and Amiot in *Science* (Nov. 1960). Start with $a_0 = 3.049$ to indicate that the world population in 1960 was approximately 3.049 billion. Then compute $a_1 = a_0 + 0.005a_0^{2.01}$ to estimate the population in 1961. Compute $a_2 = a_1 + 0.005a_1^{2.01}$ to estimate the population in 1962, then $a_3 = a_2 + 0.005a_2^{2.01}$ for 1963, and so on. Continue iterating and compare your calculations to the actual populations in 1970 (3.721 billion), 1980 (4.473 billion) and 1990 (5.333 billion). Then project ahead to the year 2035. Frightening, isn't it?

58. The so-called **hailstone sequence** is defined by

$$x_k = \begin{cases} 3x_{k-1} + 1 & \text{if } x_{k-1} \text{ is odd} \\ \frac{1}{2}x_{k-1} & \text{if } x_{k-1} \text{ is even} \end{cases}.$$

If you start with $x_1 = 2^n$ for some positive integer n, show that $x_{n+1} = 1$. The question (an unsolved research problem) is whether you eventually reach 1 from *any* starting value. Try several odd values for x_1 and show that you always reach 1.

59. A different population model was studied by Fibonacci, an Italian mathematician of the thirteenth century. He imagined a population of rabbits starting with a pair of newborns. For one month, they grow and mature. The second month, they have a pair of newborn baby rabbits. We count the number of pairs of rabbits. Thus far, $a_0 = 1$, $a_1 = 1$ and $a_2 = 2$. The rules are: adult rabbit pairs give birth to a pair of newborns every month, newborns take one month to mature and no rabbits die. Show that $a_3 = 3$, $a_4 = 5$ and in general $a_n = a_{n-1} + a_{n-2}$. This sequence of numbers, known as the **Fibonacci sequence,** occurs in an amazing number of applications.

60. In this exercise, we visualize the Fibonacci sequence. Start with two squares of side 1 placed next to each other (see Figure A).

Place a square on the long side of the resulting rectangle (see Figure B); this square has side 2. Continue placing squares on the long sides of the rectangles: a square of side 3 is added in Figure C, then a square of side 5 is added to the bottom of Figure C, and so on.

FIGURE A

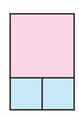


FIGURE B

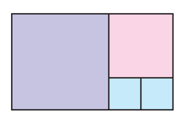
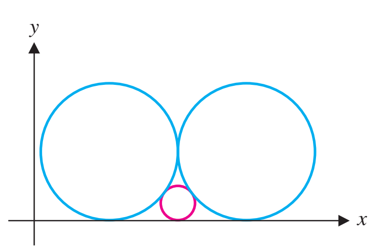


FIGURE C

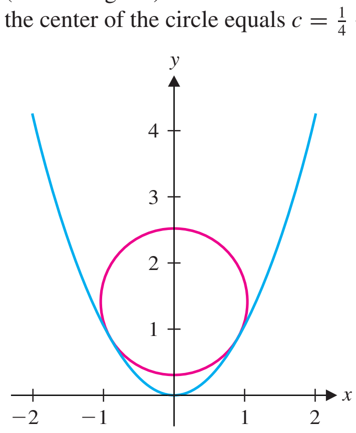
Argue that the sides of the squares are determined by the Fibonacci sequence of exercise 59.

61. Start with two circles C_1 and C_2 of radii r_1 and r_2, respectively, that are tangent to each other and each tangent to the x-axis. Construct the circle C_3 that is tangent to C_1, C_2 and the x-axis. (See the figure.) If the centers of C_1 and C_2 are (c_1, r_1) and (c_2, r_2), respectively, show that $(c_2 - c_1)^2 + (r_2 - r_1)^2 = (r_1 + r_2)^2$ and then $|c_2 - c_1| = 2\sqrt{r_1 r_2}$. Find similar relationships for circles C_1 and C_3 and for circles C_2 and C_3. Show that the radius r_3 of C_3 is given by $\sqrt{r_3} = \dfrac{\sqrt{r_1 r_2}}{\sqrt{r_1} + \sqrt{r_2}}$.



62. In exercise 61, construct a sequence of circles where C_4 is tangent to C_2, C_3 and the x-axis. Then, C_5 is tangent to C_3, C_4 and the x-axis. If you start with unit circles $r_1 = r_2 = 1$, find a formula for the radius r_n in terms of F_n, the nth term in the Fibonacci sequence of exercises 59 and 60.

63. Let C be the circle of radius r inscribed in the parabola $y = x^2$. (See the figure.) Show that for $r > {}^1/_2$ the y-coordinate c of the center of the circle equals $c = \frac{1}{4} + r^2$.



64. In exercise 63, let C_1 be the circle of radius $r_1 = 1$ inscribed in $y = x^2$. Construct a sequence of circles C_2, C_3 and so on, where each circle C_n rests on top of the previous circle C_{n-1} (that is, C_n is tangent to C_{n-1}) and is inscribed in the parabola. If r_n is the radius of circle C_n, find a (simple) formula for r_n.

65. Determine whether the sequence $a_n = \sqrt[n]{n}$ converges or diverges. (Hint: Use $n^{1/n} = e^{(1/n)\ln n}$.)

66. Suppose that $a_1 = 1$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{4}{a_n}\right)$. Show numerically that the sequence converges to 2. To find this limit analytically, let $L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$ and solve the equation $L = \frac{1}{2}\left(L + \frac{4}{L}\right)$.

67. As in exercise 66, determine the limit of the sequence defined by $a_1 = 1$, $a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right)$ for $c > 0$ and $a_n > 0$.

EXPLORATORY EXERCISES

1. Suppose that a ball is launched from the ground with initial velocity v. Ignoring air resistance, it will rise to a height of $v^2/(2g)$ and fall back to the ground at time $t = 2v/g$. Depending on how "lively" the ball is, the next bounce will only rise to a fraction of the previous height. The **coefficient of restitution** r, defined as the ratio of landing velocity to rebound velocity, measures the liveliness of the ball. The second bounce has launch velocity rv, the third bounce has launch velocity $r^2 v$ and so on. It might seem that the ball will bounce forever. To see that it does not, argue that the time to complete two bounces is $a_2 = \frac{2v}{g}(1 + r)$, the time to complete three bounces is $a_3 = \frac{2v}{g}(1 + r + r^2)$, and so on. Take $r = 0.5$ and numerically determine the limit of this sequence. (We study this type of sequence in detail in section 7.2.) In particular, show that $(1 + 0.5) = \frac{3}{2}$, $(1 + 0.5 + 0.5^2) = \frac{7}{4}$ and $(1 + 0.5 + 0.5^2 + 0.5^3) = \frac{15}{8}$, find a general expression for a_n and determine the limit of the sequence. Argue that at the end of this amount of time, the ball has stopped bouncing.

2. A surprising follow-up to the bouncing ball problem of exercise 1 is found in *An Experimental Approach to Nonlinear Dynamics and Chaos* by Tufillaro, Abbott and Reilly. Suppose the ball is bouncing on a moving table that oscillates up and down according to the equation $A\cos\omega t$ for some amplitude A and frequency ω. Without the motion of the table, the ball will quickly reach a height of 0 as in exercise 1. For different values of A and ω, however, the ball can settle into an amazing variety of patterns. To understand this, explain why the collision between table and ball could subtract or add velocity to the ball (What happens if the table is going up? down?). A simplified model of the velocity of the ball at successive collisions with the table is $v_{n+1} = 0.8v_n - 10\cos(v_0 + v_1 + \cdots + v_n)$. Starting with $v_0 = 5$, compute $v_1, v_2, \ldots, v_{15}$. In this case, the ball never settles into a pattern; its motion is chaotic.

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7.2 INFINITE SERIES

Recall that we write the decimal expansion of $\frac{1}{3}$ as the repeating decimal

$$\frac{1}{3} = 0.3333333\overline{3},$$

where we understand that the 3s in this expansion go on for ever and ever. An alternative way to think of this is as

$$\begin{aligned} \frac{1}{3} &= 0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 + \cdots \\ &= 3(0.1) + 3(0.1)^2 + 3(0.1)^3 + 3(0.1)^4 + \cdots + 3(0.1)^k + \cdots. \end{aligned} \tag{2.1}$$

For convenience, we write (2.1) using summation notation as

$$\frac{1}{3} = \sum_{k=1}^{\infty} [3(0.1)^k]. \tag{2.2}$$

But, what exactly could we mean by the *infinite sum* indicated in (2.2)? Of course, you can't add infinitely many things together. (You can only add two things at a time.) By this expression, we mean that as you add together more and more terms, the sum gets closer and closer to $\frac{1}{3}$.

In general, for any sequence $\{a_k\}_{k=1}^{\infty}$, suppose we start adding the terms together. We define the individual sums by

$$\begin{aligned} S_1 &= a_1, \\ S_2 &= a_1 + a_2 = S_1 + a_2, \\ S_3 &= \underbrace{a_1 + a_2}_{S_2} + a_3 = S_2 + a_3, \\ S_4 &= \underbrace{a_1 + a_2 + a_3}_{S_3} + a_4 = S_3 + a_4, \qquad (2.3) \\ &\vdots \\ S_n &= \underbrace{a_1 + a_2 + \cdots + a_{n-1}}_{S_{n-1}} + a_n = S_{n-1} + a_n \qquad (2.4) \end{aligned}$$

and so on. We refer to S_n as the nth **partial sum.** Note that we can compute any one of these as the sum of two numbers: the nth term, a_n and the previous partial sum, S_{n-1}, as indicated in (2.4).

For instance, for the sequence $\left\{\frac{1}{2^k}\right\}_{k=1}^{\infty}$, consider the partial sums

$$\begin{aligned} S_1 &= \frac{1}{2}, \\ S_2 &= \frac{1}{2} + \frac{1}{2^2} = \frac{3}{4}, \\ S_3 &= \frac{3}{4} + \frac{1}{2^3} = \frac{7}{8}, \\ S_4 &= \frac{7}{8} + \frac{1}{2^4} = \frac{15}{16}, \end{aligned}$$

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and so on. Look at these carefully and you might notice that $S_2 = \frac{3}{4} = 1 - \frac{1}{2^2}$, $S_3 = \frac{7}{8} = 1 - \frac{1}{2^3}$, $S_4 = \frac{15}{16} = 1 - \frac{1}{2^4}$ and so on, so that $S_n = 1 - \frac{1}{2^n}$, for each $n = 1, 2, \ldots$. If we were to consider the convergence or divergence of the sequence $\{S_n\}_{n=1}^{\infty}$ of partial sums, observe that we now have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) = 1.$$

Think about what this says: as we add together more and more terms of the sequence $\left\{\frac{1}{2^k}\right\}_{k=1}^{\infty}$, the partial sums are drawing closer and closer to 1. In this instance, we write

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1. \tag{2.5}$$

It's very important to understand what's going on here. This new mathematical object, $\sum_{k=1}^{\infty} \frac{1}{2^k}$ is called a **series** (or **infinite series**). It is *not a sum* in the usual sense of the word, but rather, the *limit* of the sequence of partial sums. Equation (2.5) says that as we add together more and more terms, the sums are approaching the limit of 1.

In general, for any sequence, $\{a_k\}_{k=1}^{\infty}$, we can write down the series

$$a_1 + a_2 + \cdots + a_k + \cdots = \sum_{k=1}^{\infty} a_k.$$

If the sequence of partial sums $S_n = \sum_{k=1}^{n} a_k$ converges (to some number S), then we say that the series $\sum_{k=1}^{\infty} a_k$ **converges** (to S). We write

Definition of infinite series

$$\boxed{\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = S.}$$

In this case, we call S the **sum** of the series. On the other hand, if the sequence of partial sums, $\{S_n\}_{n=1}^{\infty}$ diverges (i.e., $\lim_{n\to\infty} S_n$ does not exist), then we say that the series **diverges.**

EXAMPLE 2.1 A Convergent Series

Determine if the series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges or diverges.

Solution From our work on the introductory example, observe that

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) = 1.$$

In this case, we say that the series converges to 1. ■

In example 2.2, we examine a simple divergent series.

EXAMPLE 2.2 A Divergent Series

Investigate the convergence or divergence of the series $\sum_{k=1}^{\infty} k^2$.

Solution Here, we have the nth partial sum

$$S_n = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2$$

and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (1^2 + 2^2 + \cdots + n^2) = \infty.$$

Since the sequence of partial sums diverges, the series diverges also. ■

Determining the convergence or divergence of a series is only rarely as simple as it was in examples 2.1 and 2.2.

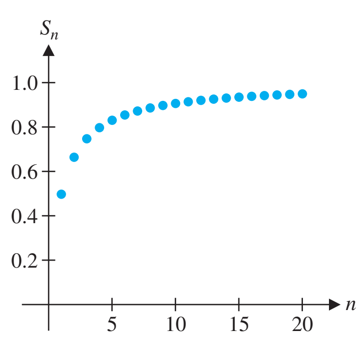

FIGURE 7.16

$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$.

n	$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$
10	0.90909091
100	0.99009901
1000	0.999001
10,000	0.99990001
100,000	0.99999
1×10^6	0.999999
1×10^7	0.9999999

EXAMPLE 2.3 A Series with a Simple Expression for the Partial Sums

Investigate the convergence or divergence of the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$.

Solution In Figure 7.16, we have plotted the first 20 partial sums. In the accompanying table, we list a number of partial sums of the series.

From both the graph and the table, it appears that the partial sums are approaching 1, as $n \to \infty$. However, we must urge caution. It is extremely difficult to look at a graph or a table of any finite number of the partial sums and decide whether a given series is converging or diverging. In the present case, we are fortunate that we can find a simple expression for the partial sums. We leave it as an exercise to find the partial fractions decomposition of the general term of the series

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}. \tag{2.6}$$

Now, consider the nth partial sum. From (2.6), we have

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$
$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Notice how nearly every term in the partial sum is canceled by another term in the sum (the next term). For this reason, such a sum is referred to as a **telescoping** (or **collapsing**) sum. We now have

$$S_n = 1 - \frac{1}{n+1}$$

and so,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

This says that the series converges to 1, as conjectured from the graph and the table. ■

It is relatively rare that we know the sum of a convergent series exactly. Usually, we must test a series for convergence using some indirect method and then approximate the sum by calculating some partial sums. We now consider one additional type of series whose sum is known exactly. The series we considered in example 2.1, $\sum_{k=1}^{\infty} \frac{1}{2^k}$, is an example of a type of series called *geometric series.* We prove a general result in Theorem 2.1.

THEOREM 2.1

For $a \neq 0$, the **geometric series** $\sum_{k=0}^{\infty} ar^k$ converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$. (Here, r is referred to as the **ratio.**)

PROOF

The proof relies on a clever observation. Note that the first term of the series corresponds to $k = 0$ and so, the nth partial sum (the sum of the first n terms) is

$$S_n = a + ar^1 + ar^2 + \cdots + ar^{n-1}. \tag{2.7}$$

Multiplying (2.7) by r, we get

$$rS_n = ar^1 + ar^2 + ar^3 + \cdots + ar^n. \tag{2.8}$$

Subtracting (2.8) from (2.7), we get

$$\begin{aligned}(1-r)S_n &= (a + ar^1 + ar^2 + \cdots + ar^{n-1}) - (ar^1 + ar^2 + ar^3 + \cdots + ar^n)\\ &= a - ar^n = a(1 - r^n).\end{aligned}$$

Dividing both sides by $(1 - r)$ gives us

$$S_n = \frac{a(1-r^n)}{1-r}.$$

If $|r| < 1$, notice that $r^n \to 0$ as $n \to \infty$ and so,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}.$$

We leave it as an exercise to show that if $|r| \geq 1$, $\lim_{n\to\infty} S_n$ does not exist. ■

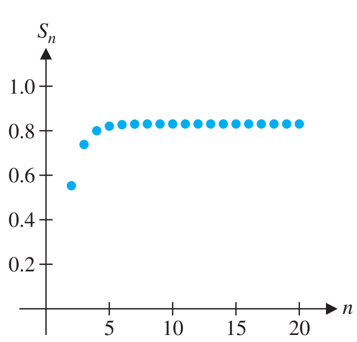

FIGURE 7.17
$S_n = \sum_{k=2}^{n+1} 5 \cdot \left(\frac{1}{3}\right)^k.$

EXAMPLE 2.4 A Convergent Geometric Series

Investigate the convergence or divergence of the series $\sum_{k=2}^{\infty} 5\left(\frac{1}{3}\right)^k$.

Solution The first 20 partial sums are plotted in Figure 7.17. It appears from the graph that the sequence of partial sums is converging to some number around 0.8. Further evidence is found in the following table, showing a number of partial sums.

The table suggests that the series converges to approximately 0.83333333. Again, we must urge caution. Graphical and numerical evidence can be very misleading when examining series. Some sequences and series converge (or diverge) far too slowly to

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n	$S_n = \sum_{k=2}^{n+1} 5\left(\frac{1}{3}\right)^k$	n	$S_n = \sum_{k=2}^{n+1} 5\left(\frac{1}{3}\right)^k$
1	0.55555556	10	0.83331922
2	0.74074074	13	0.83333281
3	0.80246914	16	0.83333331
4	0.82304527	19	0.83333333
5	0.82990398	20	0.83333333

observe graphically or numerically. You must *always* confirm your suspicions with careful mathematical analysis. In the present case, note that while the series is not quite written in the usual form, it is a geometric series, as follows:

$$\sum_{k=2}^{\infty} 5\left(\frac{1}{3}\right)^k = 5\left(\frac{1}{3}\right)^2 + 5\left(\frac{1}{3}\right)^3 + 5\left(\frac{1}{3}\right)^4 + \cdots + 5\left(\frac{1}{3}\right)^n + \cdots$$

$$= 5\left(\frac{1}{3}\right)^2\left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots\right]$$

$$= \sum_{k=0}^{\infty}\left\{5\left(\frac{1}{3}\right)^2\left(\frac{1}{3}\right)^k\right\}.$$

You can now see that this is a geometric series with ratio $r = \frac{1}{3}$ and $a = 5\left(\frac{1}{3}\right)^2$. Further, since

$$|r| = \frac{1}{3} < 1,$$

we have from Theorem 2.1 that the series converges to

$$\frac{a}{1-r} = \frac{5\left(\frac{1}{3}\right)^2}{1-\left(\frac{1}{3}\right)} = \frac{\left(\frac{5}{9}\right)}{\left(\frac{2}{3}\right)} = \frac{5}{6} = 0.8333333\bar{3},$$

which is consistent with the graph and the table of partial sums. ■

EXAMPLE 2.5 A Divergent Geometric Series

Investigate the convergence or divergence of the series $\sum_{k=0}^{\infty} 6\left(-\frac{7}{2}\right)^k$.

Solution A graph showing the first 20 partial sums (see Figure 7.18) is not particularly helpful, until you look at the vertical scale. The following table showing the values of the first 10 partial sums is more revealing.

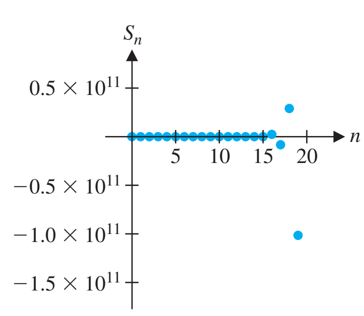

FIGURE 7.18

$S_n = \sum_{k=0}^{n-1} 6 \cdot \left(-\frac{7}{2}\right)^k$.

n	$S_n = \sum_{k=0}^{n-1} 6\left(-\frac{7}{2}\right)^k$	n	$S_n = \sum_{k=0}^{n-1} 6\left(-\frac{7}{2}\right)^k$
1	6	6	−2449.7
2	−15	7	8579.9
3	58.5	8	−30,024
4	−198.75	9	1.05×10^5
5	701.63	10	-3.68×10^5

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Note that while the partial sums are oscillating back and forth between positive and negative values, they are growing larger and larger in absolute value. We can confirm our suspicions by observing that this is a geometric series with ratio $r = -\frac{7}{2}$. Since

$$|r| = \left|-\frac{7}{2}\right| = \frac{7}{2} \geq 1,$$

the series is divergent, as we suspected. ■

You will find that determining whether a series is convergent or divergent usually involves a lot of hard work. The simple observation in Theorem 2.2 provides us with a very useful test.

THEOREM 2.2

If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

PROOF

Suppose that $\sum_{k=1}^{\infty} a_k$ converges to some number L. This means that the sequence of partial sums defined by $S_n = \sum_{k=1}^{n} a_k$ also converges to L. Notice that

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n-1} a_k + a_n = S_{n-1} + a_n.$$

Subtracting S_{n-1} from both sides, we have

$$a_n = S_n - S_{n-1}.$$

This gives us

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = L - L = 0,$$

as desired. ■

The following very useful test follows directly from Theorem 2.2.

***k*TH-TERM TEST FOR DIVERGENCE**

If $\lim_{k\to\infty} a_k \neq 0$, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

The *k*th-term test is so simple, you should use it to test every series you consider. It says that if the terms don't tend to zero, the series is divergent and there's nothing more to do. However, as we'll soon see, if the terms *do* tend to zero, there is no guarantee that the series converges and additional testing is needed.

EXAMPLE 2.6 A Series Whose Terms Do Not Tend to Zero

Investigate the convergence or divergence of the series $\sum_{k=1}^{\infty} \frac{k}{k+1}$.

Solution A graph showing the first 20 partial sums is shown in Figure 7.19. The partial sums appear to be increasing without bound as n increases. A table of values would indicate the same sort of growth. (Try this!) We can resolve the question of convergence quickly by observing that

$$\lim_{k\to\infty} \frac{k}{k+1} = 1 \neq 0.$$

From the kth-term test for divergence, the series must diverge. ■

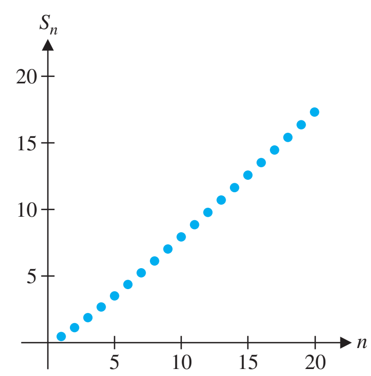


FIGURE 7.19
$S_n = \sum_{k=1}^{n} \frac{k}{k+1}$.

Example 2.7 shows an important series whose terms tend to 0 as $k \to \infty$, but that diverges, nonetheless.

EXAMPLE 2.7 The Harmonic Series

Investigate the convergence or divergence of the **harmonic series:** $\sum_{k=1}^{\infty} \frac{1}{k}$.

Solution In Figure 7.20, we see the first 20 partial sums of the series. In the following table, we display the first 20 partial sums. (You could do the same with the first 200 or 2000 partial sums and there would be little difference in your conclusion.) The table and the graph suggest that the series might converge to a number around 3.6. As always with sequences and series, we need to confirm this suspicion. From our test for divergence, we have

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k} = 0.$$

n	$S_n = \sum_{k=1}^{n} \frac{1}{k}$	n	$S_n = \sum_{k=1}^{n} \frac{1}{k}$
1	1	11	3.01988
2	1.5	12	3.10321
3	1.83333	13	3.18013
4	2.08333	14	3.25156
5	2.28333	15	3.31823
6	2.45	16	3.38073
7	2.59286	17	3.43955
8	2.71786	18	3.49511
9	2.82897	19	3.54774
10	2.92897	20	3.59774

CAUTION

The converse of Theorem 2.2 is *false*. That is, having $\lim_{k\to\infty} a_k = 0$ does *not* guarantee that the series $\sum_{k=1}^{\infty} a_k$ converges. *Be very clear about this point.* This is a very common misconception.

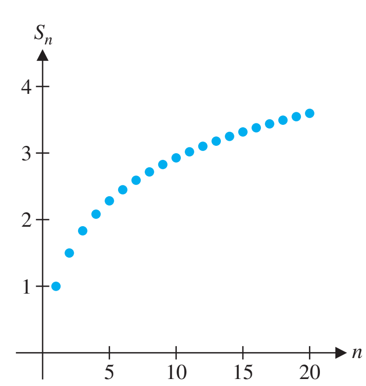


FIGURE 7.20
$S_n = \sum_{k=1}^{n} \frac{1}{k}$.

Be careful: once again, this does *not* say that the series converges. If the limit had been nonzero, we would have stopped and concluded that the series diverges. In the present case, where the limit is 0, we can only conclude that further study is needed. (That is, the series *may* converge, but we will need to investigate further.)

The following clever proof provides a preview of things to come. Consider the nth partial sum

$$S_n = \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Note that S_n corresponds to the sum of the areas of the n rectangles superimposed on the graph of $y = \frac{1}{x}$ shown in Figure 7.21 for the case where $n = 7$.

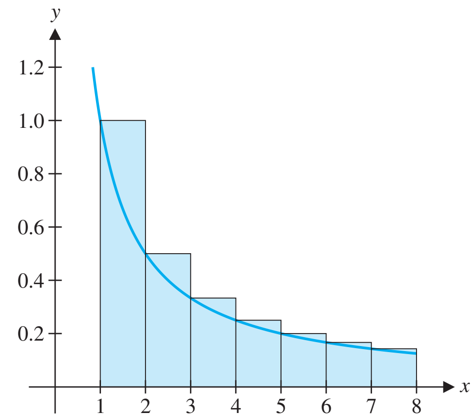


FIGURE 7.21
$y = \dfrac{1}{x}$.

Notice that since each of the indicated rectangles lies partly above the curve, we have

$$\begin{aligned} S_n &= \text{Sum of areas of } n \text{ rectangles} \\ &\geq \text{Area under the curve} = \int_1^{n+1} \frac{1}{x}\,dx \\ &= \ln|x|\Big|_1^{n+1} = \ln(n+1). \end{aligned} \tag{2.9}$$

However, the sequence $\{\ln(n+1)\}_{n=1}^{\infty}$ diverges, as

$$\lim_{n\to\infty} \ln(n+1) = \infty.$$

Since $S_n \geq \ln(n+1)$, for all n [from (2.9)], we must also have that $\lim_{n\to\infty} S_n = \infty$. From the definition of convergence of a series, we now have that $\sum_{k=1}^{\infty} \frac{1}{k}$ *diverges,* too, even though $\lim_{k\to\infty} \frac{1}{k} = 0$. ■

We complete this section with several unsurprising results.

THEOREM 2.3

(i) If $\sum_{k=1}^{\infty} a_k$ converges to A and $\sum_{k=1}^{\infty} b_k$ converges to B, then the series $\sum_{k=1}^{\infty} (a_k \pm b_k)$ converges to $A \pm B$ and $\sum_{k=1}^{\infty} (ca_k)$ converges to cA, for any constant, c.

(ii) If $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty} (a_k \pm b_k)$ diverges.

The proof of the theorem is left as an exercise.

EXERCISES 7.2

WRITING EXERCISES

1. Suppose that your friend is confused about the difference between the convergence of a sequence and the convergence of a series. Carefully explain the difference between convergence or divergence of the sequence $a_k = \frac{k}{k+1}$ and the series $\sum_{k=1}^{\infty} \frac{k}{k+1}$.
2. Explain in words why the kth term test for divergence is true. Explain why it is *not* true that if $\lim_{k\to\infty} a_k = 0$ then $\sum_{k=1}^{\infty} a_k$ necessarily converges. In your explanation, include an important example that proves that this is not true and comment on the fact that the convergence of a_k to 0 can be slow or fast.
3. In Theorems 2.2 and 2.3, the series start at $k = 1$, as in $\sum_{k=1}^{\infty} a_k$. Explain why the conclusions of the theorems hold if the series starts at $k = 2$, or $k = 3$ or at any positive integer.
4. We emphasized in the text that numerical and graphical evidence for the convergence of a series can be misleading. Suppose your calculator carries 14 digits in its calculations. Explain why for large enough values of n, the term $\frac{1}{n}$ will be too small to change the partial sum $\sum_{k=1}^{n} \frac{1}{k}$. Thus, the calculator would incorrectly indicate that the harmonic series converges.

In exercises 1–24, determine if the series converges or diverges. For convergent series, find the sum of the series.

1. $\sum_{k=0}^{\infty} 3\left(\frac{1}{5}\right)^k$
2. $\sum_{k=0}^{\infty} \frac{1}{3}(5)^k$
3. $\sum_{k=0}^{\infty} \frac{1}{2}\left(-\frac{1}{3}\right)^k$
4. $\sum_{k=0}^{\infty} 4\left(\frac{1}{2}\right)^k$
5. $\sum_{k=0}^{\infty} \frac{1}{2}(3)^k$
6. $\sum_{k=0}^{\infty} 5\left(-\frac{1}{3}\right)^k$
7. $\sum_{k=1}^{\infty} \frac{4}{k(k+2)}$
8. $\sum_{k=1}^{\infty} \frac{4k}{k+2}$
9. $\sum_{k=1}^{\infty} \frac{3k}{k+4}$
10. $\sum_{k=1}^{\infty} \frac{9}{k(k+3)}$
11. $\sum_{k=1}^{\infty} \frac{2}{k}$
12. $\sum_{k=0}^{\infty} \frac{4}{k+1}$
13. $\sum_{k=1}^{\infty} \frac{2k+1}{k^2(k+1)^2}$
14. $\sum_{k=0}^{\infty} [e^{-k} - e^{-(k+1)}]$
15. $\sum_{k=0}^{\infty} 3^{-k}$
16. $\sum_{k=0}^{\infty} 2e^{-k}$
17. $\sum_{k=0}^{\infty} \left(\frac{1}{2^k} - \frac{1}{k+1}\right)$
18. $\sum_{k=0}^{\infty} \left(\frac{1}{2^k} - \frac{1}{3^k}\right)$
19. $\sum_{k=0}^{\infty} \left(\frac{2}{3^k} + \frac{1}{2^k}\right)$
20. $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{4^k}\right)$
21. $\sum_{k=0}^{\infty} (-1)^k \frac{3}{2^k}$
22. $\sum_{k=0}^{\infty} (-1)^{k+1} \frac{4}{3^k}$
23. $\sum_{k=0}^{\infty} (-1)^{k+1} \frac{3k}{k+1}$
24. $\sum_{k=0}^{\infty} (-1)^k \frac{k^3}{k^2+1}$

T In exercises 25–30, use graphical and numerical evidence to conjecture the convergence or divergence of the series.

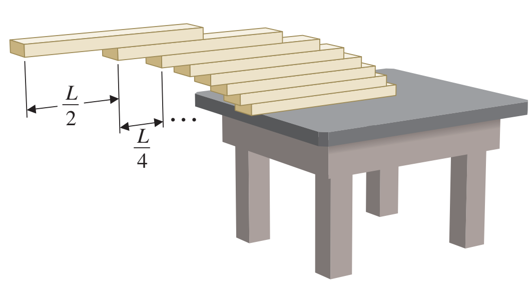
25. $\sum_{k=1}^{\infty} \frac{1}{k^2}$
26. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$
27. $\sum_{k=1}^{\infty} \frac{3}{k!}$
28. $\sum_{k=1}^{\infty} \frac{2^k}{k}$
29. $\sum_{k=1}^{\infty} \frac{4^k}{k^2}$
30. $\sum_{k=1}^{\infty} \frac{2^k}{k!}$
31. Prove that if $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=m}^{\infty} a_k$ converges for any positive integer m. In particular, if $\sum_{k=1}^{\infty} a_k$ converges to L, what does $\sum_{k=m}^{\infty} a_k$ converge to?
32. Prove that if $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=m}^{\infty} a_k$ diverges for any positive integer m.
33. Prove Theorem 2.3 (i).
34. Prove Theorem 2.3 (ii).
35. Prove that if the series $\sum_{k=0}^{\infty} a_k$ converges, then the series $\sum_{k=0}^{\infty} \frac{1}{a_k}$ diverges.
36. Prove that the partial sum $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ does not equal an integer for any prime $n > 1$. Is the statement true for all integers $n > 1$?
37. The harmonic series is probably the single most important series to understand. In this exercise, we guide you through another proof of the divergence of this series. Let $S_n = \sum_{k=1}^{n} \frac{1}{k}$. Show that $S_1 = 1$ and $S_2 = \frac{3}{2}$. Since $\frac{1}{3} > \frac{1}{4}$, we have $\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Therefore, $S_4 > \frac{3}{2} + \frac{1}{2} = 2$. Similarly, $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$, so $S_8 > \frac{5}{2}$. Show that $S_{16} > 3$ and $S_{32} > \frac{7}{2}$. For which n can you guarantee that $S_n > 4$? $S_n > 5$? For any positive integer m, determine n such that $S_n > m$. Conclude that the harmonic series diverges.
38. Compute several partial sums of the series $1 - 1 + 1 - 1 + 1 - 1 + \cdots$. Argue that the limit of the sequence of partial sums

does not exist, so that the series diverges. Also, write this series as a geometric series and use Theorem 2.1 to conclude that the series diverges. Finally, use the kth-term test for divergence to conclude that the series diverges.

39. Write $0.9999\bar{9} = 0.9 + 0.09 + 0.009 + \cdots$ and sum the geometric series to prove that $0.9999\bar{9} = 1$.

40. As in exercise 39, prove that $0.19999\bar{9} = 0.2$.

41. Suppose you have n boards of length L. Place the first board with length $\frac{L}{2n}$ hanging over the edge of the table. Place the next board with length $\frac{L}{2(n-1)}$ hanging over the edge of the first board. The next board should hang $\frac{L}{2(n-2)}$ over the edge of the second board. Continuing on until the last board hangs $\frac{L}{2}$ over the edge of the $(n-1)$st board. Theoretically, this stack will balance. (In practice, don't use quite as much overhang.) With $n = 8$, compute the total overhang of the stack. Determine the number of boards n such that the total overhang is greater than L. This means that the last board is entirely beyond the edge of the table. What is the limit of the total overhang as $n \to \infty$?



42. Have you ever felt that the line you're standing in moves more slowly than the other lines? In *An Introduction to Probability Theory and Its Applications,* William Feller proved just how bad your luck is. Let N be the number of people who get in line until someone waits longer than you do. (You're the first, so $N \geq 2$.) The probability that $N = k$ is given by $p(k) = \dfrac{1}{k(k-1)}$. Prove that the total probability equals 1; that is, $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(k-1)} = 1$. From probability theory, the average (mean) number of people who must get in line before someone has waited longer than you is given by $\displaystyle\sum_{k=2}^{\infty} k\frac{1}{k(k-1)}$. Prove that this diverges to ∞. Talk about bad luck!

43. If $0 < r < \frac{1}{2}$, show that $1 + 2r + 4r^2 + \cdots + (2r)^n + \cdots = \dfrac{1}{1-2r}$. Replace r with $\dfrac{1}{1000}$ and discuss what's interesting about the decimal representation of $\dfrac{500}{499}$.

44. In exploratory exercise 1 of section 7.1, you showed that a particular bouncing ball takes 2 seconds to complete its infinite number of bounces. In general, the total time it takes for a ball to complete its bounces is $\dfrac{2v}{g}\displaystyle\sum_{k=0}^{\infty} r^k$ and the total distance the ball moves is $\dfrac{v^2}{g}\displaystyle\sum_{k=0}^{\infty} r^{2k}$. Assuming $0 < r < 1$, find the sums of these geometric series.

45. To win a deuce tennis game, one player or the other must win the next two points. If each player wins one point, the deuce starts over. If you win each point with probability p, the probability that you win the next two points is p^2. The probability that you win one of the next two points is $2p(1-p)$. The probability that you win a deuce game is then $p^2 + 2p(1-p)p^2 + [2p(1-p)]^2p^2 + [2p(1-p)]^3p^2 + \cdots$. Explain what each term represents, explain why the geometric series converges and find the sum of the series. If $p = 0.6$, you're a better player than your opponent. Show that you are more likely to win a deuce game than you are a single point. The slightly strange scoring rules in tennis make it more likely that the better player wins.

46. On an analog clock, at 1:00 the minute hand points to 12 and the hour hand points to 1. When the minute hand reaches 1, the hour hand has progressed to $1 + \frac{1}{12}$. When the minute hand reaches $1 + \frac{1}{12}$, the hour hand has moved to $1 + \frac{1}{12} + \frac{1}{12^2}$. Find the sum of a geometric series to determine the time at which the minute hand and hour hand are in the same location.

47. Two bicyclists are 40 miles apart, riding toward each other at 20 mph (each). A fly starts at one bicyclist and flies toward the other bicyclist at 60 mph. When it reaches the bike, it turns around and flies back to the first bike. It continues flying back and forth until the bikes meet. Determine the distance flown on each leg of the fly's journey and find the sum of the geometric series to get the total distance flown. Verify that this is the right answer by solving the problem the easy way.

48. In this exercise, we will find the **present value** of a plot of farmland. Assume that a crop of value $\$c$ will be planted in years 1, 2, 3, and so on, and the yearly inflation rate is r. The present value is given by

$$P = ce^{-r} + ce^{-2r} + ce^{-3r} + \cdots.$$

Find the sum of the geometric series to compute the present value.

49. Suppose \$100,000 of counterfeit money is introduced into the economy. Each time the money is used, 25% of the remaining money is identified as counterfeit and removed from circulation. Determine the total amount of counterfeit money successfully used in transactions. This is an example of the **multiplier effect** in economics.

50. In exercise 49, suppose that a new marking scheme on dollar bills helps raise the detection rate to 40%. Determine the reduction in the total amount of counterfeit money successfully spent.

51. A dosage d of a drug is given at times $t = 0, 1, 2, \ldots$. The drug decays exponentially with rate r in the bloodstream. The amount in the bloodstream after $n + 1$ doses is $d + de^{-r} + de^{-2r} + \cdots + de^{-nr}$. Show that the eventual level of the drug (after an "infinite" number of doses) is $\dfrac{d}{1-e^{-r}}$. If $r = 0.1$, find the dosage needed to maintain a drug level of 2.

52. The **Cantor set** is one of the most famous sets in mathematics. To construct the Cantor set, start with the interval [0, 1]. Then remove the middle third, $\left(\frac{1}{3}, \frac{2}{3}\right)$. This leaves the set $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. For each of the two subintervals, remove the middle third; in this case, remove the intervals $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$. Continue in this way, removing the middle thirds of each remaining interval. The Cantor set is all points in [0, 1] that are **not** removed. Argue that 0, 1, $\frac{1}{3}$ and $\frac{2}{3}$ are in the Cantor set, and identify four more points in the set. It can be shown that there are an infinite number of points in the Cantor set. On the other hand, the total length of the subintervals removed is $\frac{1}{3} + 2\left(\frac{1}{9}\right) + \cdots$. Find the third term in this series, identify the series as a convergent geometric series and find the sum of the series. Given that you started with an interval of length 1, how much "length" does the Cantor set have?

53. Give an example where $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both diverge but $\sum_{k=1}^{\infty} (a_k + b_k)$ converges.

54. If $\sum_{k=0}^{\infty} a_k$ converges and $\sum_{k=0}^{\infty} b_k$ diverges, is it necessarily true that $\sum_{k=0}^{\infty} (a_k + b_k)$ diverges?

EXPLORATORY EXERCISES

1. **Infinite products** are also of great interest to mathematicians. Numerically explore the convergence or divergence of the infinite product $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{25}\right)\left(1-\frac{1}{49}\right)\cdots = \prod_{p=\text{prime}} \left(1 - \frac{1}{p^2}\right)$. Note that the product is taken over the prime numbers, not all integers. Compare your results to the number $\dfrac{6}{\pi^2}$.

2. In example 2.7, we showed that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln(n+1)$. Superimpose the graph of $f(x) = \frac{1}{x-1}$ onto Figure 7.21 and show that $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln(n)$. Conclude that $\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \ln(n)$. **Euler's constant** is defined by

$$\gamma = \lim_{n\to\infty}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n)\right].$$

Look up the value of γ. (Hint: Use your CAS.) Use γ to estimate $\sum_{i=1}^{n} \frac{1}{i}$ for $n = 10{,}000$ and $n = 100{,}000$. Investigate whether or not the sequence $a_n = \sum_{k=n}^{2n} \frac{1}{k}$ converges.

7.3 THE INTEGRAL TEST AND COMPARISON TESTS

As we have observed a number of times now, we are usually unable to determine the sum of a convergent series. In fact, for most series we cannot determine whether they converge or diverge by simply looking at the sequence of partial sums. Most of the time, we will need to test a series for convergence in some indirect way. If we find that the series is convergent, we can then approximate its sum by numerically computing some partial sums. In this section, we will develop additional tests for convergence of series. The first of these is a generalization of the method we used to show that the harmonic series was divergent in section 7.2.

For a given series $\sum_{k=1}^{\infty} a_k$, suppose that there is a function f for which

$$f(k) = a_k, \text{ for } k = 1, 2, \ldots,$$

where f is continuous, decreasing and $f(x) \geq 0$ for all $x \geq 1$. We consider the nth partial sum

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

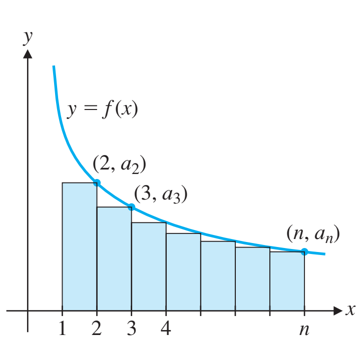


FIGURE 7.22
$(n-1)$ rectangles, lying beneath the curve.

Look carefully at Figure 7.22. We have constructed $(n-1)$ rectangles on the interval $[1, n]$, each of width 1 and with height equal to the value of the function at the right-hand endpoint of the subinterval on which the rectangle is constructed. Notice that since each rectangle lies completely beneath the curve, the sum of the areas of the $(n-1)$ rectangles shown is less than the area under the curve from $x = 1$ to $x = n$. That is,

$$0 \le \text{ Sum of areas of } (n-1) \text{ rectangles } \le \text{ Area under the curve } = \int_1^n f(x)\,dx. \tag{3.1}$$

Note that the area of the first rectangle is length × width = $(1)(a_2)$, the area of the second rectangle is $(1)(a_3)$ and so on. We get that the sum of the areas of the $(n-1)$ rectangles indicated in Figure 7.22 is

$$a_2 + a_3 + a_4 + \cdots + a_n = S_n - a_1,$$

since

$$S_n = a_1 + a_2 + \cdots + a_n.$$

Together with (3.1), this gives us

$$\begin{aligned} 0 \le \ & \text{Sum of areas of } (n-1) \text{ rectangles} \\ &= S_n - a_1 \le \text{ Area under the curve } = \int_1^n f(x)\,dx. \end{aligned} \tag{3.2}$$

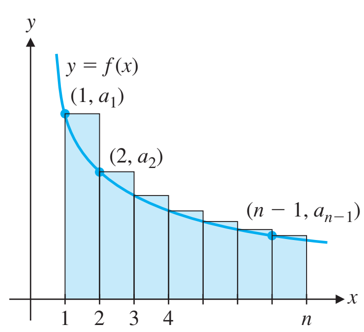


FIGURE 7.23
$(n-1)$ rectangles, partially above the curve.

Now, suppose that the improper integral $\int_1^\infty f(x)\,dx$ converges. Then, from (3.2), we have

$$0 \le S_n - a_1 \le \int_1^n f(x)\,dx \le \int_1^\infty f(x)\,dx.$$

Adding a_1 to all the terms gives us

$$a_1 \le S_n \le a_1 + \int_1^\infty f(x)\,dx.$$

This says that the sequence of partial sums $\{S_n\}_{n=1}^\infty$ is bounded. Since $\{S_n\}_{n=1}^\infty$ is also monotonic (Why is that?), $\{S_n\}_{n=1}^\infty$ is convergent by Theorem 1.4 and so, the series $\sum_{k=1}^\infty a_k$ is also convergent.

In Figure 7.23, we have constructed $(n-1)$ rectangles on the interval $[1, n]$, each of width 1, but with height equal to the value of the function at the left-hand endpoint of the subinterval on which the rectangle is constructed. In this case, the sum of the areas of the $(n-1)$ rectangles shown is greater than the area under the curve. That is,

$$\begin{aligned} 0 \le \text{ Area under the curve } &= \int_1^n f(x)\,dx \\ &\le \text{ Sum of areas of } (n-1) \text{ rectangles.} \end{aligned} \tag{3.3}$$

Further, note that the area of the first rectangle is length × width = $(1)(a_1)$, the area of the second rectangle is $(1)(a_2)$ and so on. We get that the sum of the areas of the $(n-1)$ rectangles indicated in Figure 7.23 is

$$a_1 + a_2 + \cdots + a_{n-1} = S_{n-1}.$$

Together with (3.3), this gives us

$$\begin{aligned} 0 \le \ \text{Area under the curve } &= \int_1^n f(x)\,dx \\ &\le \text{ Sum of areas of } (n-1) \text{ rectangles} = S_{n-1}. \end{aligned} \tag{3.4}$$

Now, suppose that the improper integral $\int_1^\infty f(x)\,dx$ diverges. Since $f(x) \ge 0$, this says that $\lim_{n\to\infty} \int_1^n f(x)\,dx = \infty$. From (3.4), we have that

$$\int_1^n f(x)\,dx \le S_{n-1}.$$

HISTORICAL NOTES

Colin Maclaurin (1698–1746) Scottish mathematician who discovered the Integral Test. Maclaurin was one of the founders of the Royal Society of Edinburgh and was a pioneer in the mathematics of actuarial studies. The Integral Test was introduced in a highly influential book that also included a new treatment of an important method for finding series of functions. Maclaurin series, as we now call them, are developed in section 7.7.

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This says that

$$\lim_{n\to\infty} S_{n-1} = \infty,$$

also. So, the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ diverges and hence, the series $\sum_{k=1}^{\infty} a_k$ diverges, too.

We summarize the results of this analysis in Theorem 3.1.

THEOREM 3.1 (Integral Test)

If $f(k) = a_k$ for each $k = 1, 2, \ldots$ and f is continuous, decreasing and $f(x) \geq 0$, for $x \geq 1$, then $\int_1^{\infty} f(x)\,dx$ and $\sum_{k=1}^{\infty} a_k$ either *both* converge or *both* diverge.

It is important to recognize that while the Integral Test might say that a given series and improper integral both converge, it does *not* say that they will converge to the same thing. In fact, this is generally not true, as we see in example 3.1.

EXAMPLE 3.1 Using the Integral Test

Investigate the convergence or divergence of the series $\sum_{k=0}^{\infty} \frac{1}{k^2+1}$.

Solution The graph of the first 20 partial sums shown in Figure 7.24 suggests that the series converges to some value around 2. In the accompanying table (found in the margin), you can find some selected partial sums. We have shown so many partial sums due to the very slow rate of convergence of this series.

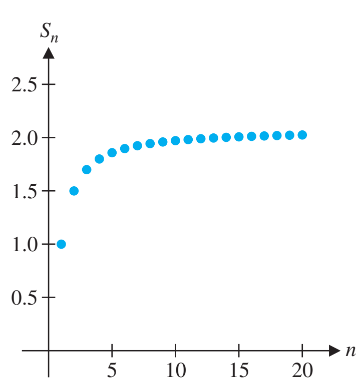

FIGURE 7.24

$S_n = \sum_{k=0}^{n-1} \frac{1}{k^2+1}$.

n	$S_n = \sum_{k=0}^{n-1} \frac{1}{k^2+1}$
10	1.97189
50	2.05648
100	2.06662
200	2.07166
500	2.07467
1000	2.07567
2000	2.07617

Based on our computations, we cannot say whether the series is converging very slowly to a limit around 2.076 or whether the series is instead diverging very slowly, as we saw earlier for the harmonic series. To determine which is the case, we must test the series further. Define $f(x) = \frac{1}{x^2+1}$. Note that f is continuous and positive everywhere and $f(k) = \frac{1}{k^2+1} = a_k$, for all $k \geq 1$. Further,

$$f'(x) = (-1)(x^2+1)^{-2}(2x) < 0,$$

for $x \in (0, \infty)$ and so, f is decreasing. This says that the Integral Test applies to this series. So, we consider the improper integral

$$\int_0^{\infty} \frac{1}{x^2+1}\,dx = \lim_{R\to\infty} \int_0^R \frac{1}{x^2+1}\,dx = \lim_{R\to\infty} \tan^{-1} x \Big|_0^R$$

$$= \lim_{R\to\infty} (\tan^{-1} R - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

By the Integral Test, we have that since the improper integral converges, the series must converge, also. Since we have now established that the series is convergent, we can use our earlier calculations to arrive at the estimated sum 2.076. Notice that this is *not* the same as the value of the corresponding improper integral, which is $\frac{\pi}{2} \approx 1.5708$. ■

In example 3.2, we discuss an important type of series.

EXAMPLE 3.2 The p-Series

Determine for which values of p the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ (a ***p*-series**) converges.

Solution First, notice that for $p = 1$, this is the harmonic series, which diverges. For $p > 1$, define $f(x) = \frac{1}{x^p} = x^{-p}$. Notice that for $x \geq 1$, f is continuous and positive. Further,

$$f'(x) = -px^{-p-1} < 0,$$

so that f is decreasing for $x \geq 1$. This says that the Integral Test applies. We now consider

$$\int_1^{\infty} x^{-p}\,dx = \lim_{R\to\infty} \int_1^R x^{-p}\,dx = \lim_{R\to\infty} \frac{x^{-p+1}}{-p+1}\bigg|_1^R$$

$$= \lim_{R\to\infty} \left(\frac{R^{-p+1}}{-p+1} - \frac{1}{-p+1} \right) = \frac{-1}{-p+1}.$$

Since $p > 1$ implies that $-p + 1 < 0$.

In this case, the improper integral converges and so too, must the series. In the case where $p < 1$, we leave it as an exercise to show that the series diverges. ■

We summarize the result of example 3.2 as follows.

p-series

> The p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

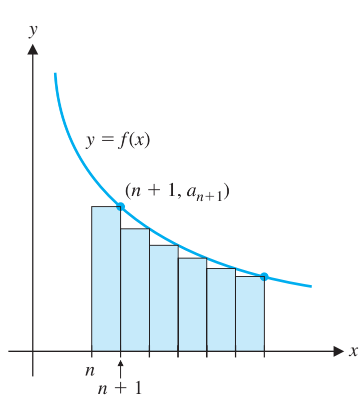

FIGURE 7.25
Estimate of the remainder.

Notice that in examples 3.1 and 3.2, we were able to use the Integral Test to establish the convergence of several series. So, now what? We have observed that you can use the partial sums of a series to estimate its value, but just how precise is a given estimate? We answer this question with the following result. First, if we estimate the sum S of the series $\sum_{k=1}^{\infty} a_k$ by the nth partial sum $S_n = \sum_{k=1}^{n} a_k$, we define the **remainder** R_n to be

$$R_n = S - S_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k = \sum_{k=n+1}^{\infty} a_k.$$

Notice that this says that the remainder is the error in approximating S by S_n. For any series shown to be convergent by the Integral Test, we can estimate the size of the remainder, as follows. From Figure 7.25, observe that the remainder R_n corresponds to the sum of the areas of the indicated rectangles. Further, under the conditions of the Integral Test, this is less than the area under the curve $y = f(x)$. (Recall that this area is finite, as $\int_1^{\infty} f(x)\,dx$ converges.) This gives us the result in Theorem 3.2.

> **THEOREM 3.2** (Error Estimate for the Integral Test)
>
> Suppose that $f(k) = a_k$ for all $k = 1, 2, \ldots$, where f is continuous, decreasing and $f(x) \geq 0$ for all $x \geq 1$. Further, suppose that $\int_1^{\infty} f(x)\,dx$ converges. Then, the remainder R_n satisfies
>
> $$0 \leq R_n = \sum_{k=n+1}^{\infty} a_k \leq \int_n^{\infty} f(x)\,dx.$$

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We can use Theorem 3.2 to estimate the error in using a partial sum to approximate the sum of a series.

EXAMPLE 3.3 Estimating the Error in a Partial Sum

Estimate the error in using the partial sum S_{100} to approximate the sum of the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$.

Solution First, recall that in example 3.2, we had shown that this series (a p-series, with $p = 3$) is convergent, by the Integral Test. From Theorem 3.2 then, the remainder term satisfies

$$0 \le R_{100} \le \int_{100}^{\infty} \frac{1}{x^3}\,dx = \lim_{R\to\infty} \int_{100}^{R} \frac{1}{x^3}\,dx = \lim_{R\to\infty} \left(-\frac{1}{2x^2}\right)\Bigg|_{100}^{R}$$

$$= \lim_{R\to\infty} \left(\frac{-1}{2R^2} + \frac{1}{2(100)^2}\right) = 5 \times 10^{-5}.$$

A more interesting and far more practical question related to example 3.3 is to use Theorem 3.2 to help us determine the number of terms of the series necessary to obtain a given accuracy.

EXAMPLE 3.4 Finding the Number of Terms Needed for a Given Accuracy

Determine the number of terms needed to obtain an approximation to the sum of the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$ correct to within 10^{-5}.

Solution Again, we already used the Integral Test to show that the series in question converges. Then, by Theorem 3.2, we have that the remainder satisfies

$$0 \le R_n \le \int_{n}^{\infty} \frac{1}{x^3}\,dx = \lim_{R\to\infty} \int_{n}^{R} \frac{1}{x^3}\,dx = \lim_{R\to\infty} \left(-\frac{1}{2x^2}\right)\Bigg|_{n}^{R}$$

$$= \lim_{R\to\infty} \left(\frac{-1}{2R^2} + \frac{1}{2n^2}\right) = \frac{1}{2n^2}.$$

So, to ensure that the remainder is less than 10^{-5}, we require

$$0 \le R_n \le \frac{1}{2n^2} \le 10^{-5}.$$

Solving this last inequality for n yields

$$n^2 \ge \frac{10^5}{2} \quad \text{or} \quad n \ge \sqrt{\frac{10^5}{2}} = 100\sqrt{5} \approx 223.6.$$

So, taking $n \ge 224$ will guarantee the required accuracy and consequently, we have $\sum_{k=1}^{\infty} \frac{1}{k^3} \approx \sum_{k=1}^{224} \frac{1}{k^3} \approx 1.202047$, which is correct to within 10^{-5}, as desired.

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Comparison Tests

We next present two results that will allow us to compare a given series with one that is already known to be convergent or divergent, much as we did with improper integrals in section 4.10.

THEOREM 3.3 (Comparison Test)

Suppose that $0 \leq a_k \leq b_k$, for all k.

(i) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges, too.

(ii) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges, too.

Intuitively, this theorem should make abundant sense: if the "larger" series converges, then the "smaller" one must also converge. Likewise, if the "smaller" series diverges, then the "larger" one must diverge, too.

PROOF

Given that $0 \leq a_k \leq b_k$ for all k, observe that the nth partial sums of the two series satisfy

$$0 \leq S_n = a_1 + a_2 + \cdots + a_n \leq b_1 + b_2 + \cdots + b_n.$$

(i) If $\sum_{k=1}^{\infty} b_k$ converges (say to B), this says that

$$0 \leq S_n \leq a_1 + a_2 + \cdots + a_n \leq b_1 + b_2 + \cdots + b_n \leq \sum_{k=1}^{\infty} b_k = B, \tag{3.5}$$

for all $n \geq 1$. From (3.5), the sequence $\{S_n\}_{n=1}^{\infty}$ of partial sums of $\sum_{k=1}^{\infty} a_k$ is bounded. Notice that $\{S_n\}_{n=1}^{\infty}$ is also increasing. (Why?) Since every bounded, monotonic sequence is convergent (see Theorem 1.4), we get that $\sum_{k=1}^{\infty} a_k$ is convergent, too.

(ii) If $\sum_{k=1}^{\infty} a_k$ is divergent, we have (since all of the terms of the series are nonnegative) that

$$\lim_{n\to\infty} (b_1 + b_2 + \cdots + b_n) \geq \lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n) = \infty.$$

Thus, $\sum_{k=1}^{\infty} b_k$ must be divergent, also. ■

You can use the Comparison Test to test the convergence of series that look similar to series that you already know are convergent or divergent (notably, geometric series or p-series).

EXAMPLE 3.5 Using the Comparison Test for a Convergent Series

Investigate the convergence or divergence of $\sum_{k=1}^{\infty} \frac{1}{k^3 + 5k}$.

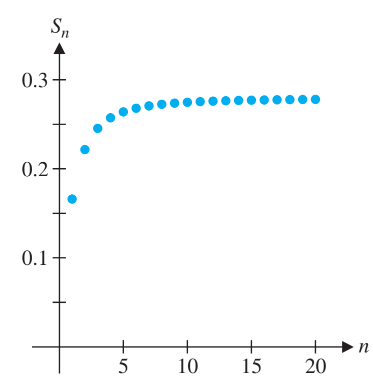

FIGURE 7.26
$S_n = \sum_{k=1}^{n} \frac{1}{k^3 + 5k}$.

Solution From the graph of the first 20 partial sums shown in Figure 7.26, it appears that the series converges to some value near 0.3. To confirm such a conjecture, we must carefully test the series. Note that for large values of k, the general term of the series looks like $\frac{1}{k^3}$, since when k is large, k^3 is much larger than $5k$. This observation is significant, since we already know that $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series ($p = 3 > 1$). Further, observe that

$$\frac{1}{k^3 + 5k} \le \frac{1}{k^3},$$

for all $k \ge 1$. Since $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges, the Comparison Test says that $\sum_{k=1}^{\infty} \frac{1}{k^3 + 5k}$ converges, too. As with the Integral Test, although the Comparison Test tells us that both series converge, the two series *need not* converge to the same sum. A quick calculation of a few partial sums should convince you that $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges to approximately 1.202, while $\sum_{k=1}^{\infty} \frac{1}{k^3 + 5k}$ converges to approximately 0.2798. (Note that this is consistent with what we saw in Figure 7.26.) ■

EXAMPLE 3.6 Using the Comparison Test for a Divergent Series

Investigate the convergence or divergence of $\sum_{k=1}^{\infty} \frac{5^k + 1}{2^k - 1}$.

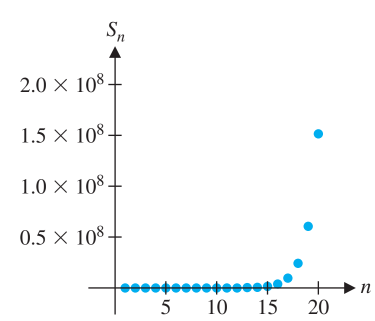

FIGURE 7.27
$S_n = \sum_{k=1}^{n} \frac{5^k + 1}{2^k - 1}$.

Solution From the graph of the first 20 partial sums seen in Figure 7.27, it appears that the partial sums are growing very rapidly. On this basis, we would conjecture that the series diverges. Of course, to verify this, we need further testing. Notice that for k large, the general term looks like $\frac{5^k}{2^k} = \left(\frac{5}{2}\right)^k$ and we know that $\sum_{k=1}^{\infty} \left(\frac{5}{2}\right)^k$ is a divergent geometric series $\left(|r| = \frac{5}{2} > 1\right)$. Further,

$$\frac{5^k + 1}{2^k - 1} \ge \frac{5^k}{2^k - 1} \ge \frac{5^k}{2^k} = \left(\frac{5}{2}\right)^k.$$

By the Comparison Test, $\sum_{k=1}^{\infty} \frac{5^k + 1}{2^k - 1}$ diverges, too. ■

There are plenty of series whose general term looks like the general term of a familiar series, but for which it is unclear how to get the inequality required for the Comparison Test to go in the right direction.

EXAMPLE 3.7 A Comparison That Does Not Work

Investigate the convergence or divergence of the series $\sum_{k=3}^{\infty} \frac{1}{k^3 - 5k}$.

Solution Note that this is nearly identical to example 3.5, except that there is a "−" sign in the denominator instead of a "+" sign. The graph of the first 20 partial sums

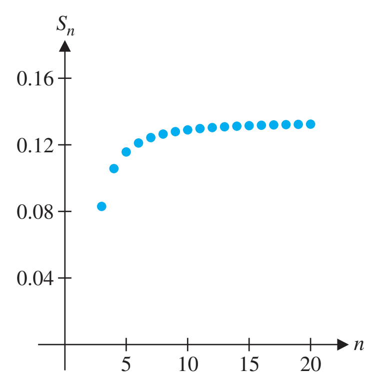

FIGURE 7.28

$S_n = \sum_{k=3}^{n} \frac{1}{k^3 - 5k}$.

seen in Figure 7.28 looks somewhat similar to the graph in Figure 7.26, except that the series appears to be converging to about 0.12. In this case, however, we have the inequality

$$\frac{1}{k^3 - 5k} \geq \frac{1}{k^3}, \text{ for all } k \geq 3.$$

Unfortunately, this inequality goes the *wrong way:* we know that $\sum_{k=3}^{\infty} \frac{1}{k^3}$ is a convergent p-series, but since $\sum_{k=3}^{\infty} \frac{1}{k^3 - 5k}$ is "larger" than this convergent series, the Comparison Test says nothing. ■

Think about what happened in example 3.7 this way: while you might observe that

$$k^2 \geq \frac{1}{k^3}, \text{ for all } k \geq 1,$$

and you know that $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is convergent, the Comparison Test says nothing about the "larger" series $\sum_{k=1}^{\infty} k^2$. In fact, we know that this last series is divergent (by the kth-term test for divergence, since $\lim_{k \to \infty} k^2 = \infty \neq 0$). To resolve this difficulty for the present problem, we will need to either make a more appropriate comparison or use the Limit Comparison Test, shown in Theorem 3.4.

NOTES

When we say $\lim_{k \to \infty} \frac{a_k}{b_k} = L > 0$, we mean that the limit exists and is positive. In particular, we mean that $\lim_{k \to \infty} \frac{a_k}{b_k} \neq \infty$.

THEOREM 3.4 (Limit Comparison Test)

Suppose that $a_k, b_k > 0$ and that for some (finite) value, L, $\lim_{k \to \infty} \frac{a_k}{b_k} = L > 0$. Then, either $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ *both* converge or they *both* diverge.

The proof of the Limit Comparison Test can be found in a more advanced text.

We can now use the Limit Comparison Test to test the series from example 3.7 whose convergence we have so far been unable to confirm.

EXAMPLE 3.8 Using the Limit Comparison Test

Investigate the convergence or divergence of the series $\sum_{k=3}^{\infty} \frac{1}{k^3 - 5k}$.

Solution Recall that we had already observed in example 3.7 that the general term $a_k = \frac{1}{k^3 - 5k}$ "looks like" $b_k = \frac{1}{k^3}$, for k large. We then consider the limit

$$\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \left(a_k \frac{1}{b_k} \right) = \lim_{k \to \infty} \frac{1}{(k^3 - 5k)} \frac{1}{\left(\frac{1}{k^3}\right)} = \lim_{k \to \infty} \frac{1}{1 - \frac{5}{k^2}} = 1 > 0.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series ($p = 3 > 1$), the Limit Comparison Test says that $\sum_{k=1}^{\infty} \frac{1}{k^3 - 5k}$ is also convergent, as we had originally suspected. ■

The Limit Comparison Test can be used to resolve convergence questions for a great many series. The first step in using this (like the Comparison Test) is to find another series (whose convergence or divergence is known) that "looks like" the series in question.

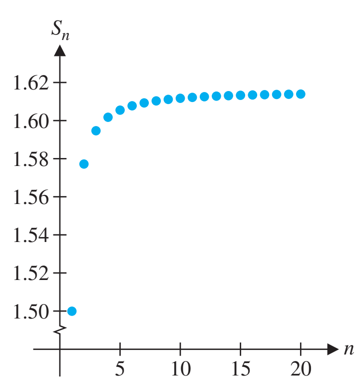

FIGURE 7.29

$S_n = \sum_{k=1}^{n} \frac{k^2 - 2k + 7}{k^5 + 5k^4 - 3k^3 + 2k - 1}$.

EXAMPLE 3.9 Using the Limit Comparison Test

Investigate the convergence or divergence of the series

$$\sum_{k=1}^{\infty} \frac{k^2 - 2k + 7}{k^5 + 5k^4 - 3k^3 + 2k - 1}.$$

Solution The graph of the first 20 partial sums in Figure 7.29 suggests that the series converges to a limit of about 1.61. The following table of partial sums supports this conjecture.

n	$S_n = \sum_{k=1}^{n} \frac{k^2 - 2k + 7}{k^5 + 5k^4 - 3k^3 + 2k - 1}$
5	1.60522
10	1.61145
20	1.61365
50	1.61444
75	1.61453
100	1.61457

Notice that for k large, the general term looks like $\frac{k^2}{k^5} = \frac{1}{k^3}$ (since the terms with the largest exponents tend to dominate the expression, for large values of k). From the Limit Comparison Test, we have

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_k}{b_k} &= \lim_{k\to\infty} \frac{k^2 - 2k + 7}{k^5 + 5k^4 - 3k^3 + 2k - 1} \frac{1}{\left(\frac{1}{k^3}\right)} \\
&= \lim_{k\to\infty} \frac{(k^2 - 2k + 7)}{(k^5 + 5k^4 - 3k^3 + 2k - 1)} \frac{k^3}{1} \\
&= \lim_{k\to\infty} \frac{(k^5 - 2k^4 + 7k^3)}{(k^5 + 5k^4 - 3k^3 + 2k - 1)} \frac{\left(\frac{1}{k^5}\right)}{\left(\frac{1}{k^5}\right)} \\
&= \lim_{k\to\infty} \frac{1 - \frac{2}{k} + \frac{7}{k^2}}{1 + \frac{5}{k} - \frac{3}{k^2} + \frac{2}{k^4} - \frac{1}{k^5}} = 1 > 0.
\end{aligned}$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series ($p = 3 > 1$), the Limit Comparison Test says that $\sum_{k=1}^{\infty} \frac{k^2 - 2k + 7}{k^5 + 5k^4 - 3k^3 + 2k - 1}$ converges, also. Finally, now that we have established that the series is in fact, convergent, we can use our table of computed partial sums to approximate the sum of the series as 1.61457. ■

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EXERCISES 7.3

WRITING EXERCISES

1. Notice that the Comparison Test doesn't always give us information about convergence or divergence. If $a_k \le b_k$ for each k and $\sum_{k=1}^{\infty} b_k$ diverges, explain why you can't tell whether or not $\sum_{k=1}^{\infty} a_k$ diverges.
2. Explain why the Limit Comparison Test works. In particular, if $\lim_{k\to\infty} \frac{a_k}{b_k} = 1$, explain how a_k and b_k compare and conclude that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ either both converge or both diverge.
3. In the Limit Comparison Test, if $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\sum_{k=1}^{\infty} a_k$ converges, explain why you can't tell whether or not $\sum_{k=1}^{\infty} b_k$ converges.
4. A p-series converges if $p > 1$ and diverges if $p < 1$. What happens for $p = 1$? If your friend knows that the harmonic series diverges, explain an easy way to remember the rest of the conclusion of the p-series test.

In exercises 1–36, determine convergence or divergence of the series.

1. $\sum_{k=1}^{\infty} \frac{4}{\sqrt[3]{k}}$
2. $\sum_{k=1}^{\infty} k^{-9/10}$
3. $\sum_{k=1}^{\infty} k^{-11/10}$
4. $\sum_{k=1}^{\infty} \frac{4}{\sqrt{k}}$
5. $\sum_{k=0}^{\infty} \frac{k+1}{k^2+2k+3}$
6. $\sum_{k=0}^{\infty} \frac{k^2+1}{k^3+3k+2}$
7. $\sum_{k=1}^{\infty} \frac{4}{2+4k}$
8. $\sum_{k=1}^{\infty} \frac{4}{(2+4k)^2}$
9. $\sum_{k=2}^{\infty} \frac{2}{k \ln k}$
10. $\sum_{k=2}^{\infty} \frac{3}{k(\ln k)^2}$
11. $\sum_{k=1}^{\infty} \frac{2k}{k^3+1}$
12. $\sum_{k=0}^{\infty} \frac{\sqrt{k}}{k^2+1}$
13. $\sum_{k=1}^{\infty} \frac{e^{1/k}}{k^2}$
14. $\sum_{k=1}^{\infty} \frac{\sqrt{1+1/k}}{k^2}$
15. $\sum_{k=1}^{\infty} \frac{e^{-\sqrt{k}}}{\sqrt{k}}$
16. $\sum_{k=1}^{\infty} \frac{ke^{-k^2}}{4+e^{-k}}$
17. $\sum_{k=1}^{\infty} \frac{3k}{k^{3/2}+2}$
18. $\sum_{k=1}^{\infty} \frac{2k^2}{k^{5/2}+2}$
19. $\sum_{k=0}^{\infty} \frac{2}{\sqrt{k^2+4}}$
20. $\sum_{k=0}^{\infty} \frac{4}{\sqrt{k^3+1}}$
21. $\sum_{k=0}^{\infty} \frac{k^2+1}{\sqrt{k^5+1}}$
22. $\sum_{k=0}^{\infty} \frac{k+2}{\sqrt[3]{k^5+4}}$
23. $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{1+k^2}$
24. $\sum_{k=1}^{\infty} \frac{\sin^{-1}(1/k)}{k^2}$
25. $\sum_{k=1}^{\infty} \frac{1}{\cos^2 k}$
26. $\sum_{k=1}^{\infty} \frac{1}{\sin^2 k}$
27. $\sum_{k=1}^{\infty} \frac{\sin k + 2}{k^2}$
28. $\sum_{k=1}^{\infty} \frac{e^{1/k}+1}{k^3}$
29. $\sum_{k=2}^{\infty} \frac{\ln k}{k}$
30. $\sum_{k=1}^{\infty} \frac{2+\cos k}{k}$
31. $\sum_{k=1}^{\infty} \frac{k^4+2k-1}{k^5+3k^2+1}$
32. $\sum_{k=0}^{\infty} \frac{k^3+2k+3}{k^4+2k^2+4}$
33. $\sum_{k=1}^{\infty} \frac{k+1}{k+2}$
34. $\sum_{k=1}^{\infty} \frac{k+1}{k^2+2}$
35. $\sum_{k=1}^{\infty} \frac{k+1}{k^3+2}$
36. $\sum_{k=1}^{\infty} \frac{k+1}{k^4+2}$
37. In our statement of the Comparison Test, we required that $a_k \le b_k$ for all k. Explain why the conclusion would remain true if $a_k \le b_k$ for $k \ge 100$.
38. If $a_k > 0$ and $\sum_{k=1}^{\infty} a_k$ converges, prove that $\sum_{k=1}^{\infty} a_k^2$ converges.
39. Prove the following extension of the Limit Comparison Test: if $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.
40. Prove the following extension of the Limit Comparison Test: if $\lim_{k\to\infty} \frac{a_k}{b_k} = \infty$ and $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.

In exercises 41–44, determine all values of p for which the series converges.

41. $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$
42. $\sum_{k=0}^{\infty} \frac{1}{(a+bk)^p}$, $a > 0, b > 0$
43. $\sum_{k=2}^{\infty} \frac{\ln k}{k^p}$
44. $\sum_{k=1}^{\infty} k^{p-1} e^{kp}$

In exercises 45–50, estimate the error in using the indicated partial sum S_n to approximate the sum of the series.

45. S_{100}, $\sum_{k=1}^{\infty} \frac{1}{k^4}$
46. S_{100}, $\sum_{k=1}^{\infty} \frac{4}{k^2}$

47. $S_{50}, \sum_{k=1}^{\infty} \frac{6}{k^8}$

48. $S_{80}, \sum_{k=1}^{\infty} \frac{2}{k^2+1}$

49. $S_{40}, \sum_{k=1}^{\infty} ke^{-k^2}$

50. $S_{200}, \sum_{k=1}^{\infty} \frac{\tan^{-1} k}{1+k^2}$

In exercises 51–54, determine the number of terms needed to obtain an approximation of the sum accurate to within 10^{-6}.

51. $\sum_{k=1}^{\infty} \frac{3}{k^4}$

52. $\sum_{k=1}^{\infty} \frac{2}{k^2}$

53. $\sum_{k=1}^{\infty} ke^{-k^2}$

54. $\sum_{k=1}^{\infty} \frac{4}{k^5}$

In exercises 55 and 56, answer with "converges" or "diverges" or "can't tell." Assume that $a_k > 0$ and $b_k > 0$.

55. Assume that $\sum_{k=1}^{\infty} a_k$ converges and fill in the blanks.

a. If $b_k \geq a_k$ for $k \geq 10$, then $\sum_{k=1}^{\infty} b_k$ ______.

b. If $\lim_{k\to\infty} \frac{b_k}{a_k} = 0$, then $\sum_{k=1}^{\infty} b_k$ ______.

c. If $b_k \leq a_k$ for $k \geq 6$, then $\sum_{k=1}^{\infty} b_k$ ______.

d. If $\lim_{k\to\infty} \frac{b_k}{a_k} = \infty$, then $\sum_{k=1}^{\infty} b_k$ ______.

56. Assume that $\sum_{k=1}^{\infty} a_k$ diverges and fill in the blanks.

a. If $b_k \geq a_k$ for $k \geq 10$, then $\sum_{k=1}^{\infty} b_k$ ______.

b. If $\lim_{k\to\infty} \frac{b_k}{a_k} = 0$, then $\sum_{k=1}^{\infty} b_k$ ______.

c. If $b_k \leq a_k$ for $k \geq 6$, then $\sum_{k=1}^{\infty} b_k$ ______.

d. If $\lim_{k\to\infty} \frac{b_k}{a_k} = \infty$, then $\sum_{k=1}^{\infty} b_k$ ______.

57. Prove that the every-other-term harmonic series $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ diverges. (Hint: Write the series as $\sum_{k=0}^{\infty} \frac{1}{2k+1}$ and use the Limit Comparison Test.)

58. Would the every-third-term harmonic series $1 + \frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \cdots$ diverge? How about the every-fourth-term harmonic series $1 + \frac{1}{5} + \frac{1}{9} + \frac{1}{13} + \cdots$? Make as general a statement as possible about such series.

59. The **Riemann-zeta function** is defined by $\zeta(x) = \sum_{k=1}^{\infty} \frac{1}{k^x}$ for $x > 1$. Explain why the restriction $x > 1$ is necessary. Leonhard Euler, considered to be one of the greatest and most prolific mathematicians ever, proved the remarkable result that $\zeta(x) = \prod_{p \text{ prime}} \frac{1}{\left(1 - \frac{1}{p^x}\right)}$.

60. Estimate $\zeta(2)$ numerically. Compare your result with that of exploratory exercise 1 of section 7.2.

In exercises 61–64, use your CAS or graphing calculator to numerically estimate the sum of the convergent p-series and identify x such that the sum equals $\zeta(x)$ for the Riemann-zeta function of exercise 59.

61. $\sum_{k=1}^{\infty} \frac{1}{k^4}$

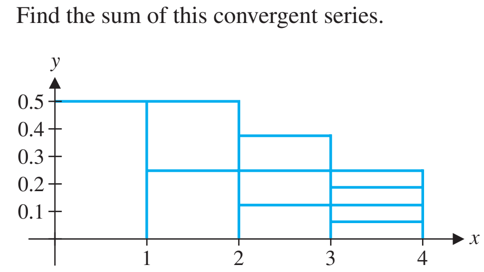
62. $\sum_{k=1}^{\infty} \frac{1}{k^6}$

63. $\sum_{k=1}^{\infty} \frac{1}{k^8}$

64. $\sum_{k=1}^{\infty} \frac{1}{k^{10}}$

65. Suppose that you toss a fair coin until you get heads. How many times would you expect to toss the coin? To answer this, notice that the probability of getting heads on the first toss is $\frac{1}{2}$, getting tails then heads is $\left(\frac{1}{2}\right)^2$, getting two tails then heads is $\left(\frac{1}{2}\right)^3$ and so on. The mean number of tosses is $\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k$. Use the Integral Test to prove that this series converges and estimate the sum numerically.

66. A clever trick can be used to sum the series in exercise 65. The series $\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k$ can be visualized as the area of the figure shown below. In columns of width one, we see one rectangle of height $\frac{1}{2}$, two rectangles of height $\frac{1}{4}$, three rectangles of height $\frac{1}{8}$ and so on. Start the sum by taking one rectangle from each column. The combined area of the first rectangles is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$. Show that this is a convergent series with sum 1. Next, take the second rectangle from each column that has at least two rectangles. The combined area of the second rectangles is $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$. Show that this is a convergent series with sum $\frac{1}{2}$. Next, take the third rectangle from each column that has at least three rectangles. The combined area from the third rectangles is $\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$. Show that this is a convergent series with sum $\frac{1}{4}$. Continue this process and show that the total area of all rectangles is $1 + \frac{1}{2} + \frac{1}{4} + \cdots$. Find the sum of this convergent series.



67. This problem is sometimes called the **coupon collector's problem.** The problem is faced by collectors of trading cards. If there are n different cards that make a complete set and you randomly obtain one at a time, how many cards would you expect to obtain before having a complete set? (By random, we mean that each different card has the same probability of $\frac{1}{n}$ of being the next card obtained.) Here and in exercises 68–70,

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we find the answer for $n = 10$. The first step is simple; to collect one card you need to obtain one card. Now, given that you have one card, how many cards do you need to obtain to get a second (different) card? If you're lucky, the next card is it (this has probability $\frac{9}{10}$). But your next card might be a duplicate and then you get a new card (this has probability $\frac{1}{10}\frac{9}{10}$). Or you might get two duplicates and then a new card (this has probability $\frac{1}{10}\frac{1}{10}\frac{9}{10}$) and so on. The mean is $1\frac{9}{10} + 2\frac{1}{10}\frac{9}{10} + 3\frac{1}{10}\frac{1}{10}\frac{9}{10} + \cdots$ or $\sum_{k=1}^{\infty} k\left(\frac{1}{10}\right)^{k-1}\left(\frac{9}{10}\right) = \sum_{k=1}^{\infty} \frac{9k}{10^k}$. Using the same trick as in exercise 66, show that this is a convergent series with sum $\frac{10}{9}$.

68. In the situation of exercise 67, if you have two different cards out of 10, the average number of cards to get a third distinct card is $\sum_{k=1}^{\infty} \frac{8k}{10^k}$; show that this is a convergent series with sum $\frac{10}{8}$.

69. Use the results of exercises 67 and 68 to find the average numbers of cards you need to obtain to complete the set of 10 different cards.

70. Compute the ratio of cards obtained to cards in the set in exercise 69. That is, for a set of 10 cards, on the average you need to obtain ___ times 10 cards to complete the set.

71. Generalize exercises 69 and 70 in the case of n cards in the set ($n > 2$).

72. Use the divergence of the harmonic series to state the unfortunate fact about the ratio of cards obtained to cards in the set as n increases.

EXPLORATORY EXERCISES

1. Numerically investigate the p-series $\sum_{k=1}^{\infty} \frac{1}{k^{0.9}}$ and $\sum_{k=1}^{\infty} \frac{1}{k^{1.1}}$ and for other values of p close to 1. Can you distinguish convergent from divergent series numerically?

2. You know that $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges. This is the "smallest" p-series that diverges, in the sense that $\frac{1}{k} < \frac{1}{k^p}$ for $p < 1$. Show that $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ diverges and $\frac{1}{k \ln k} < \frac{1}{k}$. Show that $\sum_{k=2}^{\infty} \frac{1}{k \ln k \ln(\ln k)}$ diverges and $\frac{1}{k \ln k \ln(\ln k)} < \frac{1}{k \ln k}$. Find a series such that $\sum_{k=2}^{\infty} a_k$ diverges and $a_k < \frac{1}{k \ln k \ln(\ln k)}$. Is there a smallest divergent series?

7.4 ALTERNATING SERIES

So far, we have focused our attention on positive term series, that is, series all of whose terms are positive. Before we consider the general case, we spend some time in this section examining *alternating series,* that is, series whose terms alternate back and forth from positive to negative. There are several reasons for doing this. First, alternating series appear frequently in applications. Second, alternating series are surprisingly simple to deal with and studying them will yield significant insight into how series work.

An **alternating series** is any series of the form

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots,$$

where $a_k > 0$, for all k.

EXAMPLE 4.1 The Alternating Harmonic Series

Investigate the convergence or divergence of the **alternating harmonic series**

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

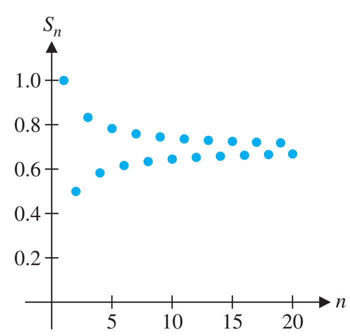

FIGURE 7.30

$S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$.

Solution The graph of the first 20 partial sums seen in Figure 7.30 suggests that the series might converge to about 0.7. We now calculate the first few partial sums by hand. Note that

$$S_1 = 1, \qquad S_5 = \frac{7}{12} + \frac{1}{5} = \frac{47}{60},$$

$$S_2 = 1 - \frac{1}{2} = \frac{1}{2}, \qquad S_6 = \frac{47}{60} - \frac{1}{6} = \frac{37}{60},$$

$$S_3 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}, \qquad S_7 = \frac{37}{60} + \frac{1}{7} = \frac{319}{420},$$

$$S_4 = \frac{5}{6} - \frac{1}{4} = \frac{7}{12}, \qquad S_8 = \frac{319}{420} - \frac{1}{8} = \frac{533}{840}$$

and so on. We have plotted these first 8 partial sums on the number line shown in Figure 7.31.

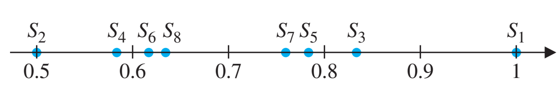

FIGURE 7.31

Partial sums of $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$.

n	$S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$
1	1
2	0.5
3	0.83333
4	0.58333
5	0.78333
6	0.61667
7	0.75952
8	0.63452
9	0.74563
10	0.64563
11	0.73654
12	0.65321
13	0.73013
14	0.65871
15	0.72537
16	0.66287
17	0.7217
18	0.66614
19	0.71877
20	0.66877

You should notice that the partial sums are bouncing back and forth, but seem to be zeroing-in on some value. (Could it be the sum of the series?) This should not be surprising, since as each new term is added or subtracted, we are adding or subtracting less than we *subtracted or added* (Why did we reverse the order here?) to get the previous partial sum. You should notice this same zeroing-in in the accompanying table displaying the first 20 partial sums of the series. Based on the behavior of the partial sums, it is reasonable to conjecture that the series converges to some value between 0.66877 and 0.71877. We can resolve the question of convergence definitively with Theorem 4.1. ■

THEOREM 4.1 (Alternating Series Test)

Suppose that $\lim_{k\to\infty} a_k = 0$ and $0 < a_{k+1} \le a_k$ for all $k \ge 1$. Then, the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges.

Make sure that you have a clear idea of what it is saying. In the case of an alternating series satisfying the hypotheses of the theorem, we start with 0 and add $a_1 > 0$ to get the first partial sum S_1. To get the next partial sum, S_2, we subtract a_2 from S_1, where $a_2 < a_1$. This says that S_2 will be between 0 and S_1. We illustrate this situation in Figure 7.32.

Continuing in this fashion, we add a_3 to S_2 to get S_3. Since $a_3 < a_2$, we must have that $S_2 < S_3 < S_1$. Referring to Figure 7.32, notice that

$$S_2 < S_4 < S_6 < \cdots < S_5 < S_3 < S_1.$$

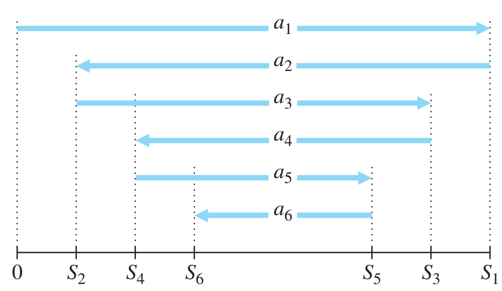

FIGURE 7.32
Convergence of the partial sums of an alternating series.

In particular, this says that *all* of the odd-indexed partial sums (i.e., S_{2n+1}, for $n = 0, 1, 2, \ldots$) are larger than *all* of the even-indexed partial sums (i.e., S_{2n}, for $n = 1, 2, \ldots$). As the partial sums oscillate back and forth, they should be drawing closer and closer to some limit S, somewhere between all of the even-indexed partial sums and the odd-indexed partial sums,

$$S_2 < S_4 < S_6 < \cdots < S < \cdots < S_5 < S_3 < S_1. \tag{4.1}$$

We illustrate the use of this new test in example 4.2.

EXAMPLE 4.2 Using the Alternating Series Test

Reconsider the convergence of the alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$.

Solution Notice that

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k} = 0.$$

Further,

$$0 < a_{k+1} = \frac{1}{k+1} \le \frac{1}{k} = a_k, \text{ for all } k \ge 1.$$

By the Alternating Series Test, the series converges. (You can use the calculations from example 4.1 to arrive at an approximate sum.) ■

The Alternating Series Test is certainly the easiest test we've discussed so far for determining the convergence of a series. It's straightforward, but you will sometimes need to work a bit to verify the hypotheses.

EXAMPLE 4.3 Using the Alternating Series Test

Investigate the convergence or divergence of the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^k (k+3)}{k(k+1)}$.

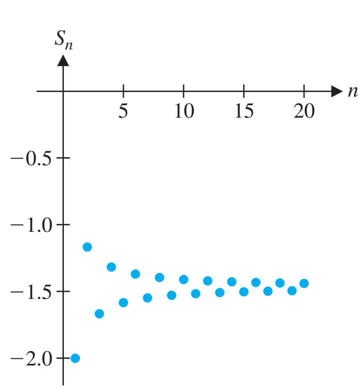

FIGURE 7.33
$S_n = \sum_{k=1}^{n} \frac{(-1)^k (k+3)}{k(k+1)}$.

Solution The graph of the first 20 partial sums seen in Figure 7.33 suggests that the series converges to some value around -1.5. The following table showing some select partial sums suggests the same conclusion.

n	$S_n = \sum_{k=1}^{n} \frac{(-1)^k(k+3)}{k(k+1)}$	n	$S_n = \sum_{k=1}^{n} \frac{(-1)^k(k+3)}{k(k+1)}$
50	−1.45545	51	−1.47581
100	−1.46066	101	−1.47076
200	−1.46322	201	−1.46824
300	−1.46406	301	−1.46741
400	−1.46448	401	−1.46699

We can verify that the series converges by first checking that

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{(k+3)}{k(k+1)} \frac{\frac{1}{k^2}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{\frac{1}{k}+\frac{3}{k^2}}{1+\frac{1}{k}} = 0.$$

Next, consider the ratio of the absolute value of two consecutive terms:

$$\frac{a_{k+1}}{a_k} = \frac{(k+4)}{(k+1)(k+2)} \frac{k(k+1)}{(k+3)} = \frac{k^2+4k}{k^2+5k+6} < 1,$$

for all $k \geq 1$. From this, it follows that $a_{k+1} < a_k$, for all $k \geq 1$ and so, by the Alternating Series Test, the series converges. Finally, from the preceding table, we can see that the series converges to a sum between −1.46448 and −1.46699. (How can you be sure that the sum is in this interval?) ■

EXAMPLE 4.4 A Divergent Alternating Series

Determine whether the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^k k}{k+2}$ converges or diverges.

Solution First, notice that

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{k+2} = 1 \neq 0.$$

So, this alternating series is divergent, since by the kth-term test for divergence, the terms must tend to zero in order for the series to be convergent. ■

☐ Estimating the Sum of an Alternating Series

We have repeatedly remarked that once you know that a series converges, you can always approximate the sum of the series by computing some partial sums. However, in finding an approximate sum of a convergent series, how close is close enough? Realize that answers to such questions of accuracy are not "one size fits all," but rather, are highly context-sensitive. For instance, if the sum of the series is to be used to find the angle from the ground at which you throw a ball to a friend, you might accept one answer. On the other hand, if the sum of that same series is to be used to find the angle at which to aim your spacecraft to ensure a safe reentry into the earth's atmosphere, you would likely insist on greater precision (at least, we would).

So far, we have calculated approximate sums of series by observing that a number of successive partial sums of the series are within a given distance of one another. The underlying assumption here is that when this happens, the partial sums are also within that same distance of the sum of the series. Unfortunately, this is simply not true, in general (although it is true for some series). What's a mathematician to do? For the case of alternating

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series, we are quite fortunate to have available a simple way to get a handle on the accuracy. Note that the error in approximating the sum S by the nth partial sum S_n is $S - S_n$.

Take a look back at Figure 7.32. Recall that we had observed from the figure that all of the even-indexed partial sums S_n of the convergent alternating series, $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ lie below the sum S, while all of the odd-indexed partial sums lie above S. That is, [as in (4.1)],

$$S_2 < S_4 < S_6 < \cdots < S < \cdots < S_5 < S_3 < S_1.$$

This says that for n even,

$$S_n \leq S \leq S_{n+1}.$$

Subtracting S_n from all terms, we get

$$0 \leq S - S_n \leq S_{n+1} - S_n = a_{n+1}.$$

Since $a_{n+1} > 0$, we have

$$-a_{n+1} \leq 0 \leq S - S_n \leq a_{n+1},$$

or

$$|S - S_n| \leq a_{n+1}, \text{ for } n \text{ even.} \tag{4.2}$$

Similarly, for n odd, we have that

$$S_{n+1} \leq S \leq S_n.$$

Again subtracting S_n, we get

$$-a_{n+1} = S_{n+1} - S_n \leq S - S_n \leq 0 \leq a_{n+1}$$

or

$$|S - S_n| \leq a_{n+1}, \text{ for } n \text{ odd.} \tag{4.3}$$

Since (4.2) and (4.3) (these are called **error bounds**) are the same, we have the same error bound whether n is even or odd. This establishes the result stated in Theorem 4.2.

THEOREM 4.2

Suppose that $\lim_{k\to\infty} a_k = 0$ and $0 < a_{k+1} \leq a_k$ for all $k \geq 1$. Then, the alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges to some number S and the error in approximating S by the nth partial sum S_n satisfies

$$|S - S_n| \leq a_{n+1}. \tag{4.4}$$

Notice that this says that the absolute value of the error in approximating S by S_n does not exceed a_{n+1} (the absolute value of the first neglected term).

EXAMPLE 4.5 Estimating the Sum of an Alternating Series

Approximate the sum of the alternating series $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^4}$ by the 40th partial sum and estimate the error in this approximation.

Solution We leave it as an exercise to show that this series is convergent. We then approximate the sum by

$$S \approx S_{40} = 0.9470326439.$$

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From our error estimate (4.4), we have

$$|S - S_{40}| \le a_{41} = \frac{1}{41^4} \approx 3.54 \times 10^{-7}.$$

This says that our approximation $S \approx 0.9470326439$ is off by no more than $\pm 3.54 \times 10^{-7}$. ■

A much more interesting question than the one asked in example 4.5 is the following. For a given convergent alternating series, how many terms must we take in order to guarantee that our approximation is accurate to a given level? We use the same estimate of error from (4.4) to answer this question, as in example 4.6.

EXAMPLE 4.6 Finding the Number of Terms Needed for a Given Accuracy

For the convergent alternating series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4}$, how many terms are needed to guarantee that S_n is within 1×10^{-10} of the actual sum S?

Solution In this case, we want to find the number of terms n for which

$$|S - S_n| \le 1 \times 10^{-10}.$$

From (4.4), we have that

$$|S - S_n| \le a_{n+1} = \frac{1}{(n+1)^4}.$$

So, we look for n such that

$$\frac{1}{(n+1)^4} \le 1 \times 10^{-10}.$$

Solving for n, we get

$$10^{10} \le (n+1)^4,$$

so that

$$\sqrt[4]{10^{10}} \le n + 1$$

or

$$n \ge \sqrt[4]{10^{10}} - 1 \approx 315.2.$$

So, taking $n \ge 316$ will guarantee an error of no more than 1×10^{-10}. Using the suggested number of terms, we get the approximate sum

$$S \approx S_{316} \approx 0.947032829447,$$

which we now know to be correct to within 1×10^{-10}. ■

EXERCISES 7.4

WRITING EXERCISES

1. If $a_k \ge 0$, explain in terms of partial sums why $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ is more likely to converge than $\sum_{k=1}^{\infty} a_k$.

2. Explain why in Theorem 4.1 we need the assumption that $a_{k+1} \le a_k$. That is, what would go wrong with the proof if $a_{k+1} > a_k$?

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3. The Alternating Series Test was stated for the series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$. Explain the difference between $\sum_{k=1}^{\infty}(-1)^{k}a_k$ and $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ and explain why we could have stated the theorem for $\sum_{k=1}^{\infty}(-1)^{k}a_k$.

4. A common mistake is to think that if $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=1}^{\infty} a_k$ converges. Explain why this is not true for positive-term series. This is also not true for alternating series *unless* you add one more hypothesis. State the extra hypothesis and explain why it's needed.

In exercises 1–26, determine if the series is convergent or divergent.

1. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{3}{k}$ **2.** $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{4}{k+1}$

3. $\sum_{k=1}^{\infty}(-1)^{k}\frac{2}{k^2}$ **4.** $\sum_{k=1}^{\infty}(-1)^{k}\frac{4}{\sqrt{k}}$

5. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k^2}{k+1}$ **6.** $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2k^2-1}{k}$

7. $\sum_{k=1}^{\infty}\frac{k}{k^2+2}$ **8.** $\sum_{k=1}^{\infty}\frac{2k-1}{k^3}$

9. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k}{2^k}$ **10.** $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{3^k}{k}$

11. $\sum_{k=1}^{\infty}(-1)^{k}\frac{4^k}{k^2}$ **12.** $\sum_{k=1}^{\infty}(-1)^{k}\frac{k+2}{4^k}$

13. $\sum_{k=1}^{\infty}\frac{2k}{k+1}$ **14.** $\sum_{k=1}^{\infty}\frac{4k^2}{k^2+2k+2}$

15. $\sum_{k=1}^{\infty}(-1)^{k}\frac{3}{\sqrt{k+1}}$ **16.** $\sum_{k=1}^{\infty}(-1)^{k}\frac{k+1}{k^3}$

17. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2}{k!}$ **18.** $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k!}{3^k}$

19. $\sum_{k=1}^{\infty}(-1)^{k}\frac{k!}{2k}$ **20.** $\sum_{k=1}^{\infty}(-1)^{k}\frac{4^k}{k!}$

21. $\sum_{k=0}^{\infty}(-1)^{k+1}2e^{-k}$ **22.** $\sum_{k=1}^{\infty}(-1)^{k+1}3e^{1/k}$

23. $\sum_{k=2}^{\infty}(-1)^{k}\ln k$ **24.** $\sum_{k=2}^{\infty}(-1)^{k}\frac{1}{\ln k}$

25. $\sum_{k=0}^{\infty}(-1)^{k+1}\frac{1}{2^k}$ **26.** $\sum_{k=0}^{\infty}(-1)^{k+1}2^k$

T **In exercises 27–34, estimate the sum of each convergent series to within 0.01.**

27. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{4}{k^3}$ **28.** $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2}{k^3}$

29. $\sum_{k=1}^{\infty}(-1)^{k}\frac{k}{2^k}$ **30.** $\sum_{k=1}^{\infty}(-1)^{k}\frac{k^2}{10^k}$

31. $\sum_{k=0}^{\infty}(-1)^{k}\frac{3}{k!}$ **32.** $\sum_{k=0}^{\infty}(-1)^{k+1}\frac{2}{k!}$

33. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{4}{k^4}$ **34.** $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{3}{k^5}$

T **In exercises 35–40, determine how many terms are needed to estimate the sum of the series to within 0.0001.**

35. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2}{k}$ **36.** $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{4}{\sqrt{k}}$

37. $\sum_{k=0}^{\infty}(-1)^{k}\frac{2^k}{k!}$ **38.** $\sum_{k=0}^{\infty}(-1)^{k}\frac{10^k}{k!}$

39. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k!}{k^k}$ **40.** $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{4^k}{k^k}$

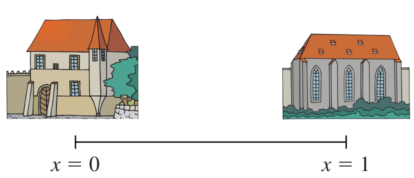
41. In example 4.3, we showed you one way to verify that a sequence is decreasing. As an alternative, explain why if $a_k = f(k)$ and $f'(x) < 0$, for all $x \geq 1$, then the sequence a_k is decreasing. Use this method to prove that $a_k = \frac{k}{k^2+2}$ is decreasing.

42. Use the method of exercise 41 to prove that $a_k = \frac{k}{2^k}$ is decreasing.

43. In this exercise, you will discover why the Alternating Series Test requires that $a_{k+1} \leq a_k$. If $a_k = \begin{cases} 1/k & \text{if } k \text{ is odd} \\ 1/k^2 & \text{if } k \text{ is even} \end{cases}$, argue that $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ diverges to ∞. Thus, an alternating series can diverge even if $\lim_{k\to\infty} a_k = 0$.

44. Verify that the series $\sum_{k=0}^{\infty}(-1)^{k}\frac{1}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ converges. It can be shown that the sum of this series is $\frac{\pi}{4}$. Given this result, we could use this series to obtain an approximation of π. How many terms would be necessary to get eight digits of π correct?

45. A person starts walking from home (at $x = 0$) toward a friend's house (at $x = 1$). Three-fourths of the way there, he changes his mind and starts walking back home. Three-fourths of the way home, he changes his mind again and starts walking back to his friend's house. If he continues this pattern of indecision, always turning around at the three-fourths mark, what will be the eventual outcome? A similar problem appeared in a national magazine and created a minor controversy due to the ambiguous wording of the problem. It is clear that the first turnaround is at $x = \frac{3}{4}$ and the second turnaround is at $\frac{3}{4} - \frac{3}{4}\left(\frac{3}{4}\right) = \frac{3}{16}$. But is the third turnaround three-fourths of the way to $x = 1$ or $x = \frac{3}{4}$? The magazine writer assumed the latter. Show that with this assumption, the person's location forms a geometric series. Find the sum of the series to find where the person ends up.



46. If the problem of exercise 45 is interpreted differently, a more interesting answer results. As before, let $x_1 = \frac{3}{4}$ and $x_2 = \frac{3}{16}$. If the next turnaround is three-fourths of the way from x_2 to 1, then $x_3 = \frac{3}{16} + \frac{3}{4}\left(1 - \frac{3}{16}\right) = \frac{3}{4} + \frac{1}{4}x_2 = \frac{51}{64}$. Three-fourths of the way back to $x = 0$ would put us at $x_4 = x_3 - \frac{3}{4}x_3 = \frac{1}{4}x_3 = \frac{51}{256}$. Show that if n is even, then $x_{n+1} = \frac{3}{4} + \frac{1}{4}x_n$ and $x_{n+2} = \frac{1}{4}x_{n+1}$. Show that the person ends up walking back and forth between two specific locations.

T 47. Use your CAS or calculator to find the sum of the alternating harmonic series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$ accurate to six digits. Compare your approximation to ln 2.

48. Find all values of p such that the series $\sum_{k=1}^{\infty}(-1)^k\frac{1}{k^p}$ converges. Compare your result to the p-series of section 7.3.

EXPLORATORY EXERCISES

1. In this exercise, you will determine whether or not the improper integral $\int_0^1 \sin(1/x)\,dx$ converges. Argue that $\int_{1/\pi}^1 \sin(1/x)\,dx$, $\int_{1/(2\pi)}^{1/\pi} \sin(1/x)\,dx$, $\int_{1/(3\pi)}^{1/(2\pi)} \sin(1/x)\,dx, \cdots$ exist and that (if it exists),

$$\int_0^1 \sin(1/x)\,dx = \int_{1/\pi}^1 \sin(1/x)\,dx + \int_{1/(2\pi)}^{1/\pi} \sin(1/x)\,dx + \int_{1/(3\pi)}^{1/(2\pi)} \sin(1/x)\,dx + \cdots.$$

Verify that the series is an alternating series and show that the hypotheses of the Alternating Series Test are met. Thus, the series and the improper integral both converge.

T 2. Consider the series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$, where x is a constant. Show that the series converges for $x = 1/2$; $x = -1/2$; any x such that $-1 < x \le 1$. Show that the series diverges if $x = -1$, $x < -1$ or $x > 1$. We see in section 7.6 that when the series converges, it converges to $\ln(1 + x)$. Verify this numerically for $x = 1/2$ and $x = -1/2$.

7.5 ABSOLUTE CONVERGENCE AND THE RATIO TEST

You should note that, outside of the Alternating Series Test presented in section 7.4, our other tests for convergence of series (i.e., the Integral Test and the two comparison tests) apply only to series all of whose terms are *positive*. So, what do we do if we're faced with a series that has both positive and negative terms, but that is not an alternating series? For instance, look at the series

$$\sum_{k=1}^{\infty}\frac{\sin k}{k^3} = \sin 1 + \frac{1}{8}\sin 2 + \frac{1}{27}\sin 3 + \frac{1}{64}\sin 4 + \cdots.$$

This has both positive and negative terms, but the terms do not alternate signs. (Calculate the first five or six terms of the series to see this for yourself.) For any such series $\sum_{k=1}^{\infty} a_k$, we can get around this problem by checking if the series of absolute values $\sum_{k=1}^{\infty}|a_k|$ is convergent. When this happens, we say that the original series $\sum_{k=1}^{\infty} a_k$ is **absolutely convergent** (or **converges absolutely**). You should note that to test the convergence of the series of absolute values $\sum_{k=1}^{\infty}|a_k|$ (all of whose terms are positive), we have all of our earlier tests for positive term series available to us.

EXAMPLE 5.1 Testing for Absolute Convergence

Determine if $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2^k}$ is absolutely convergent.

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Solution It is easy to show that this alternating series is convergent. (Try it!) From the graph of the first 20 partial sums in Figure 7.34, it appears that the series converges to approximately 0.35. To determine absolute convergence, we need to determine whether or not the series of absolute values, $\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{2^k} \right|$ is convergent. We have

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{2^k} \right| = \sum_{k=1}^{\infty} \frac{1}{2^k} = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k,$$

which you should recognize as a convergent geometric series ($|r| = \frac{1}{2} < 1$). This says that the original series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k}$ converges absolutely. ■

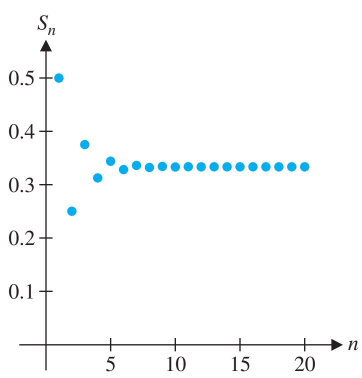

FIGURE 7.34
$S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2^k}$.

You might be wondering about the relationship between convergence and absolute convergence. We'll prove shortly that every absolutely convergent series is also convergent (as in example 5.1). However, the reverse is not true; there are many series that are convergent, but not absolutely convergent. These are called **conditionally convergent** series. Can you think of an example of such a series? If so, it's probably the series in example 5.2.

EXAMPLE 5.2 A Conditionally Convergent Series

Determine if the alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ is absolutely convergent.

Solution In example 4.2, we showed that this series is convergent. To test this for absolute convergence, we consider the series of absolute values,

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k},$$

which is the harmonic series. We showed in section 7.2 (example 2.7) that the harmonic series diverges. This says that $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges, but does not converge absolutely (i.e., it converges conditionally). ■

THEOREM 5.1

If $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

This result says that if a series converges absolutely, then it must also converge. Because of this, when we test series, we first test for absolute convergence. If the series converges absolutely, then we need not test any further to establish convergence.

PROOF

Notice that for any real number, x, we can say that $-|x| \le x \le |x|$. So, for any k, we have

$$-|a_k| \le a_k \le |a_k|.$$

Adding $|a_k|$ to all the terms, we get

$$0 \le a_k + |a_k| \le 2|a_k|. \tag{5.1}$$

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Since $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, we have that $\sum_{k=1}^{\infty} |a_k|$ and hence, also $\sum_{k=1}^{\infty} 2|a_k| = 2\sum_{k=1}^{\infty} |a_k|$ is convergent. Define $b_k = a_k + |a_k|$. From (5.1),

$$0 \le b_k \le 2|a_k|$$

and so, by the Comparison Test, $\sum_{k=1}^{\infty} b_k$ is convergent. Observe that we may write

$$\begin{aligned}\sum_{k=1}^{\infty} a_k &= \sum_{k=1}^{\infty}(a_k + |a_k| - |a_k|) = \sum_{k=1}^{\infty}\underbrace{(a_k + |a_k|)}_{b_k} - \sum_{k=1}^{\infty}|a_k| \\ &= \sum_{k=1}^{\infty} b_k - \sum_{k=1}^{\infty} |a_k|.\end{aligned}$$

Since the two series on the right-hand side are convergent, it follows that $\sum_{k=1}^{\infty} a_k$ must also be convergent. ■

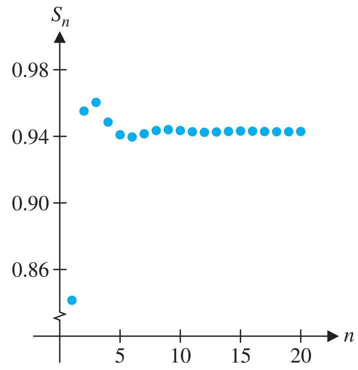

FIGURE 7.35
$S_n = \sum_{k=1}^{n} \frac{\sin k}{k^3}$.

EXAMPLE 5.3 Testing for Absolute Convergence

Determine whether $\sum_{k=1}^{\infty} \frac{\sin k}{k^3}$ is convergent or divergent.

Solution Notice that while this is not a positive-term series, it is also not an alternating series. Because of this, our only choice (given what we know) is to test the series for absolute convergence. From the graph of the first 20 partial sums seen in Figure 7.35, it appears that the series is converging to some value around 0.94. To test for absolute convergence, we consider the series of absolute values, $\sum_{k=1}^{\infty} \left|\frac{\sin k}{k^3}\right|$. Notice that

$$\left|\frac{\sin k}{k^3}\right| = \frac{|\sin k|}{k^3} \le \frac{1}{k^3}, \tag{5.2}$$

since $|\sin k| \le 1$, for all k. Of course, $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series ($p = 3 > 1$). By the Comparison Test and (5.2), $\sum_{k=1}^{\infty} \left|\frac{\sin k}{k^3}\right|$ converges, too. Consequently, the original series $\sum_{k=1}^{\infty} \frac{\sin k}{k^3}$ converges absolutely (and hence, converges). ■

□ The Ratio Test

We now introduce a very powerful tool for testing a series for absolute convergence. This test can be applied to a wide range of series, including the extremely important case of power series that we discuss in section 7.6. As you'll see, this test is also remarkably easy to use.

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THEOREM 5.2 (Ratio Test)

Given $\sum_{k=1}^{\infty} a_k$, with $a_k \neq 0$ for all k, suppose that

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = L.$$

Then,

(i) if $L < 1$, the series converges absolutely,
(ii) if $L > 1$ (or $L = \infty$), the series diverges and
(iii) if $L = 1$, there is no conclusion.

Because the proof of Theorem 5.2 is somewhat involved, we omit it.

EXAMPLE 5.4 Using the Ratio Test

Test $\sum_{k=1}^{\infty} \frac{(-1)^k k}{2^k}$ for convergence.

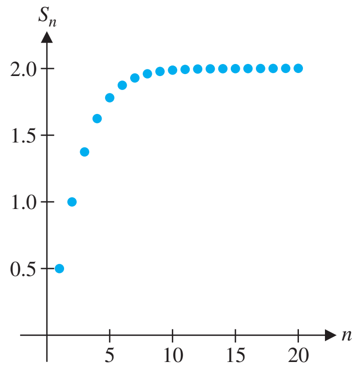


FIGURE 7.36
$S_n = \sum_{k=1}^{n} \frac{k}{2^k}$.

Solution From the graph of the first 20 partial sums of the series of absolute values, $\sum_{k=1}^{\infty} \frac{k}{2^k}$, seen in Figure 7.36, it appears that the series of absolute values converges to about 2. From the Ratio Test, we have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{\frac{k+1}{2^{k+1}}}{\frac{k}{2^k}} = \lim_{k\to\infty}\frac{k+1}{2^{k+1}}\frac{2^k}{k} = \frac{1}{2}\lim_{k\to\infty}\frac{k+1}{k} = \frac{1}{2} < 1$$

Since $2^{k+1} = 2^k \cdot 2^1$.

and so, the series converges absolutely, as expected from Figure 7.36. ■

The Ratio Test is particularly useful when the general term of a series contains an exponential term, as in example 5.4 or a factorial, as in example 5.5.

EXAMPLE 5.5 Using the Ratio Test

Test $\sum_{k=0}^{\infty} \frac{(-1)^k k!}{e^k}$ for convergence.

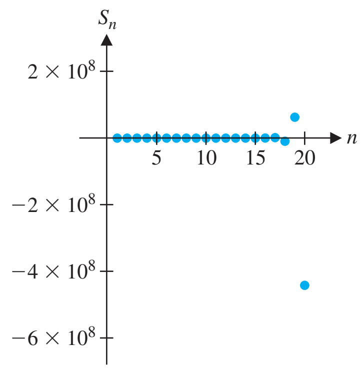


FIGURE 7.37
$S_n = \sum_{k=0}^{n-1} \frac{(-1)^k k!}{e^k}$.

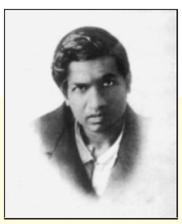
Solution From the graph of the first 20 partial sums of the series seen in Figure 7.37, it appears that the series is diverging. (Look closely at the scale on the y-axis and compute a table of values for yourself.) We can confirm this suspicion with the Ratio Test. We have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{\frac{(k+1)!}{e^{k+1}}}{\frac{k!}{e^k}} = \lim_{k\to\infty}\frac{(k+1)!}{e^{k+1}}\frac{e^k}{k!}$$

$$= \lim_{k\to\infty}\frac{(k+1)\,k!}{ek!} = \frac{1}{e}\lim_{k\to\infty}\frac{k+1}{1} = \infty.$$

Since $(k+1)! = (k+1)\cdot k!$ and $e^{k+1} = e^k \cdot e^1$.

By the Ratio Test, the series diverges, as we suspected. ■



HISTORICAL NOTES

Srinivasa Ramanujan (1887–1920) Indian mathematician whose incredible discoveries about infinite series still mystify mathematicians. Largely self-taught, Ramanujan filled notebooks with conjectures about series, continued fractions and the Riemann-zeta function. Ramanujan rarely gave a proof or even justification of his results. Nevertheless, the famous English mathematician G. H. Hardy said, "They must be true because, if they weren't true, no one would have had the imagination to invent them." (See Exercise 39.)

Recall that in the statement of the Ratio Test (Theorem 5.2), we said that if

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 1,$$

then the Ratio Test yields no conclusion. By this, we mean that in such cases, the series may or may not converge and further testing is required.

EXAMPLE 5.6 A Divergent Series for Which the Ratio Test Fails

Use the Ratio Test for the harmonic series $\sum_{k=1}^{\infty}\frac{1}{k}$.

Solution We have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{\frac{1}{k+1}}{\frac{1}{k}} = \lim_{k\to\infty}\frac{k}{k+1} = 1.$$

In this case, the Ratio Test yields no conclusion, although we already know that the harmonic series diverges. ■

EXAMPLE 5.7 A Convergent Series for Which the Ratio Test Fails

Use the Ratio Test to test the series $\sum_{k=0}^{\infty}\frac{1}{k^2}$.

Solution Here, we have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{1}{(k+1)^2}\frac{k^2}{1}$$

$$= \lim_{k\to\infty}\frac{k^2}{k^2+2k+1} = 1.$$

So again, the Ratio Test yields no conclusion, although we already know that this is a convergent p-series (with $p = 2 > 1$). ■

Look carefully at examples 5.6 and 5.7. You should recognize that the Ratio Test will be inconclusive for any p-series. Fortunately, we don't need the Ratio Test for these series.

We now present one final test for convergence of series.

THEOREM 5.3 (Root Test)

Given $\sum_{k=1}^{\infty} a_k$, suppose that $\lim_{k\to\infty}\sqrt[k]{|a_k|} = L$. Then,

(i) if $L < 1$, the series converges absolutely,
(ii) if $L > 1$ (or $L = \infty$), the series diverges and
(iii) if $L = 1$, there is no conclusion.

Notice how similar the conclusion is to the conclusion of the Ratio Test.

EXAMPLE 5.8 Using the Root Test

Use the Root Test to determine the convergence or divergence of the series $\sum_{k=1}^{\infty}\left(\frac{2k+4}{5k-1}\right)^k$.

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Solution In this case, we consider

$$\lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\sqrt[k]{\left|\frac{2k+4}{5k-1}\right|^k} = \lim_{k\to\infty}\frac{2k+4}{5k-1} = \frac{2}{5} < 1.$$

By the Root Test, the series is absolutely convergent. ■

By this point in your study of series, it may seem as if we have thrown at you a dizzying array of different series and tests for convergence or divergence. Just how are you to keep all of these straight? The only suggestion we have is that you work through *many* problems. We provide a good assortment in the exercise set that follows this section. Some of these require the methods of this section, while others are drawn from the preceding sections (just to keep you thinking about the big picture). For the sake of convenience, we summarize our convergence tests in the table that follows.

Test	*When to use*	*Conclusions*	*Section*
Geometric Series	$\sum_{k=0}^{\infty} ar^k$	Converges to $\frac{a}{1-r}$ if $\lvert r\rvert < 1$; diverges if $\lvert r\rvert \geq 1$.	7.2
***k*th-Term Test**	All series	If $\lim_{k\to\infty} a_k \neq 0$, the series diverges.	7.2
Integral Test	$\sum_{k=1}^{\infty} a_k$ where $f(k) = a_k$ and f is continuous, decreasing and $f(x) \geq 0$	$\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\,dx$ *both* converge or *both* diverge.	7.3
***p*-series**	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	Converges for $p > 1$; diverges for $p \leq 1$.	7.3
Comparison Test	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where $0 \leq a_k \leq b_k$	If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.	7.3
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where $a_k, b_k > 0$ and $\lim_{k\to\infty}\frac{a_k}{b_k} = L > 0$	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ *both* converge or *both* diverge.	7.3
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ where $a_k > 0$ for all k	If $\lim_{k\to\infty} a_k = 0$ and $a_{k+1} \leq a_k$ for all k, then the series converges.	7.4
Absolute Convergence	Series with some positive and some negative terms (including alternating series)	If $\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges, then $\sum_{k=1}^{\infty} a_k$ converges (absolutely).	7.5
Ratio Test	Any series (especially those involving exponentials and/or factorials)	For $\lim_{k\to\infty}\left\lvert\frac{a_{k+1}}{a_k}\right\rvert = L$, if $L < 1$, $\sum_{k=1}^{\infty} a_k$ converges absolutely if $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges, if $L = 1$, no conclusion.	7.5
Root Test	Any series (especially those involving exponentials)	For $\lim_{k\to\infty}\sqrt[k]{\lvert a_k\rvert} = L$, if $L < 1$, $\sum_{k=1}^{\infty} a_k$ converges absolutely if $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges, if $L = 1$, no conclusion.	7.5

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EXERCISES 7.5

WRITING EXERCISES

1. Suppose that two series have identical terms except that in series A all terms are positive and in series B some terms are positive and some terms are negative. Explain why series B is more likely to converge. In light of this, explain why Theorem 5.1 is true.

2. In the Ratio Test, if $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| > 1$, which is bigger, $|a_{k+1}|$ or $|a_k|$? Explain why this implies that the series $\sum_{k=1}^{\infty} a_k$ diverges.

3. In the Ratio Test, if $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = L < 1$, which is bigger, $|a_{k+1}|$ or $|a_k|$? This inequality could also hold if $L = 1$. Compare the relative sizes of $|a_{k+1}|$ and $|a_k|$ if $L = 0.8$ versus $L = 1$. Explain why $L = 0.8$ would be more likely to correspond to a convergent series than $L = 1$.

4. In many series of interest, the terms of the series involve powers of k (e.g., k^2), exponentials (e.g., 2^k) or factorials (e.g., $k!$). For which type(s) of terms is the Ratio Test likely to produce a result (i.e., a limit different than 1)? Briefly explain.

In exercises 1–38, determine if the series is absolutely convergent, conditionally convergent or divergent.

1. $\sum_{k=0}^{\infty} (-1)^k \frac{3}{k!}$

2. $\sum_{k=0}^{\infty} (-1)^k \frac{6}{k!}$

3. $\sum_{k=0}^{\infty} (-1)^k 2^k$

4. $\sum_{k=0}^{\infty} (-1)^k \frac{2}{3^k}$

5. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^2+1}$

6. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2+1}{k}$

7. $\sum_{k=0}^{\infty} (-1)^k \frac{3^k}{k!}$

8. $\sum_{k=0}^{\infty} (-1)^k \frac{10^k}{k!}$

9. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{2k+1}$

10. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{4}{2k+1}$

11. $\sum_{k=1}^{\infty} (-1)^k \frac{k2^k}{3^k}$

12. $\sum_{k=1}^{\infty} (-1)^k \frac{k^2 3^k}{2^k}$

13. $\sum_{k=1}^{\infty} \left(\frac{4k}{5k+1}\right)^k$

14. $\sum_{k=1}^{\infty} \left(\frac{1-3k}{4k}\right)^k$

15. $\sum_{k=1}^{\infty} \frac{-2}{k}$

16. $\sum_{k=1}^{\infty} \frac{4}{k}$

17. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sqrt{k}}{k+1}$

18. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^3+1}$

19. $\sum_{k=1}^{\infty} \frac{k^2}{e^k}$

20. $\sum_{k=1}^{\infty} k^3 e^{-k}$

21. $\sum_{k=2}^{\infty} \frac{e^{3k}}{k^{3k}}$

22. $\sum_{k=1}^{\infty} \left(\frac{e^k}{k^2}\right)^k$

23. $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$

24. $\sum_{k=1}^{\infty} \frac{\cos k}{k^3}$

25. $\sum_{k=1}^{\infty} \frac{\cos k\pi}{k}$

26. $\sum_{k=1}^{\infty} \frac{\sin k\pi}{k}$

27. $\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$

28. $\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln k}$

29. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k\sqrt{k}}$

30. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$

31. $\sum_{k=1}^{\infty} \frac{3}{k^k}$

32. $\sum_{k=0}^{\infty} \frac{2k}{3^k}$

33. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k!}{4^k}$

34. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2 4^k}{k!}$

35. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^{10}}{(2k)!}$

36. $\sum_{k=0}^{\infty} (-1)^k \frac{4^k}{(2k+1)!}$

37. $\sum_{k=0}^{\infty} \frac{(-2)^k(k+1)}{5^k}$

38. $\sum_{k=1}^{\infty} \frac{(-3)^k}{k^2 4^k}$

39. In the 1910s, the Indian mathematician Srinivasa Ramanujan discovered the formula

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26{,}390k)}{(k!)^4 396^{4k}}.$$

Approximate the series with only the $k = 0$ term and show that you get 6 digits of π correct. Approximate the series using the $k = 0$ and $k = 1$ terms and show that you get 14 digits of π correct. In general, each term of this remarkable series increases the accuracy by 8 digits.

40. Prove that Ramanujan's series in exercise 39 converges.

41. To show that $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ converges, use the Ratio Test and the fact that

$$\lim_{k\to\infty}\left(\frac{k+1}{k}\right)^k = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e.$$

42. Determine whether $\sum_{k=1}^{\infty} \frac{k!}{1\cdot 3\cdot 5\cdots(2k-1)}$ converges or diverges.

EXPLORATORY EXERCISES

1. One reason that it is important to distinguish absolute from conditional convergence of a series is the rearrangement of series, to be explored in this exercise. Show that the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k}$

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is absolutely convergent and find its sum S. Find the sum S_+ of the positive terms of the series. Find the sum S_- of the negative terms of the series. Verify that $S = S_+ + S_-$. This may seem obvious, since for the finite sums you are most familiar with, the order of addition never matters. However, you cannot separate the positive and negative terms for conditionally convergent series. For example, show that $\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$ converges (conditionally) but that the series of positive terms and the series of negative terms both diverge. Explain in words why this will always happen for conditionally convergent series. Thus, the order of terms matters for conditionally convergent series. By exploring further, we can uncover a truly remarkable fact: for conditionally convergent series, you can reorder the terms so that the partial sums converge to *any* real number. To illustrate this, suppose we want to reorder the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$ so that the partial sums converge to $\frac{\pi}{2}$. Start by pulling out positive terms $\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\right)$ such that the partial sum is within 0.1 of $\frac{\pi}{2}$. Next, take the first negative term $\left(-\frac{1}{2}\right)$ and positive terms such that the partial sum is within 0.05 of $\frac{\pi}{2}$. Then take the next negative term $\left(-\frac{1}{4}\right)$ and positive terms such that the partial sum is within 0.01 of $\frac{\pi}{2}$. Argue that you could continue in this fashion to reorder the terms so that the partial sums converge to $\frac{\pi}{2}$. (Especially explain why you will never "run out" of positive terms.) Then explain why you cannot do the same with the absolutely convergent series $\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k}$.

7.6 POWER SERIES

We now want to expand our discussion of series to the case where the terms of the series are functions of the variable x. Pay close attention to what we are about to introduce, for this is the culmination of all your hard work in sections 7.1 through 7.5. The primary reason for studying series is that we can use them to represent functions. This opens up all kinds of possibilities for us, from approximating the values of transcendental functions to calculating derivatives and integrals of such functions, to studying differential equations. As well, *defining* functions as convergent series produces an explosion of new functions available to us. In fact, many functions of great significance in applications (for instance, Bessel functions) are defined as a series. We take the first few steps in this section.

As a start, consider the series

$$\sum_{k=0}^{\infty}(x-2)^k = 1 + (x-2) + (x-2)^2 + (x-2)^3 + \cdots.$$

Notice that for each fixed x, this is a geometric series with $r = (x-2)$. Recall that this says that the series will converge whenever $|r| = |x-2| < 1$ and will diverge whenever $|r| = |x-2| \ge 1$. Further, for each x with $|x-2| < 1$ (i.e., $1 < x < 3$), the series converges to

$$\frac{a}{1-r} = \frac{1}{1-(x-2)} = \frac{1}{3-x}.$$

That is, for each x in the interval $(1, 3)$, we have

$$\sum_{k=0}^{\infty}(x-2)^k = \frac{1}{3-x}.$$

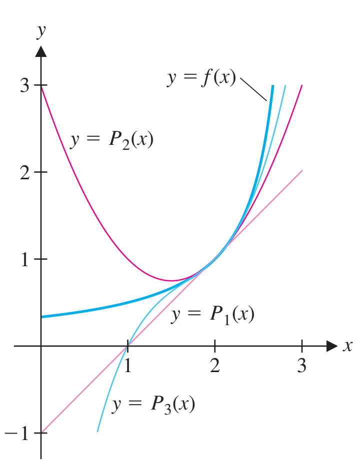

FIGURE 7.38
$y = \frac{1}{3-x}$ and the first three partial sums of $\sum_{k=0}^{\infty}(x-2)^k$.

For all other values of x, the series diverges. In Figure 7.38, we show a graph of $f(x) = \frac{1}{3-x}$, along with the first three partial sums P_n, of this series, where

$$P_n(x) = \sum_{k=0}^{n}(x-2)^k = 1 + (x-2) + (x-2)^2 + \cdots + (x-2)^n,$$

on the interval $[1, 3]$. Notice that as n gets larger, $P_n(x)$ appears to get closer to $f(x)$, for any given x in the interval $(1, 3)$. Further, as n gets larger, $P_n(x)$ tends to stay close to $f(x)$ for a larger range of x-values.

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Make certain that you understand what we've observed here: we have taken a series and noticed that it is equivalent to (i.e., it converges to) a *known* function on a certain interval. You might ask why anyone would care if you could do that. Certainly, $f(x) = \frac{1}{3-x}$ is a perfectly good function and anything you'd want to do with it will most definitely be easier using the algebraic expression $\frac{1}{3-x}$ than using the equivalent series representation, $\sum_{k=0}^{\infty}(x-2)^k$. However, imagine what benefits you might find if you could take a given function (say, one that you don't know a whole lot about) and find an equivalent series representation. This is precisely what we are going to do in section 7.7. For instance, we will be able to show that for all x,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots. \tag{6.1}$$

So, who cares? Well, suppose you wanted to calculate $e^{1.234567}$. How would you do that? Of course, you'd use a calculator. But, haven't you ever wondered how your calculator does it? The problem is that e^x is not an algebraic function. That is, we can't compute its values by using algebraic operations (i.e., addition, subtraction, multiplication, division and nth roots). Over the next few sections, we begin to explore this question. For the moment, let us say this: if we have the series representation (6.1) for e^x, then for any given x, we can compute an approximation to e^x, simply by summing the first few terms of the equivalent series. This is easy to do, since the partial sums of the series are simply polynomials.

In general, any series of the form

Power series

$$\sum_{k=0}^{\infty} b_k(x-c)^k = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + \cdots$$

is called a **power series** in powers of $(x - c)$. We refer to the constants b_k, $k = 0, 1, 2, \ldots$ as the **coefficients** of the series. The first question is: for what values of x does the series converge? Saying this another way, the power series $\sum_{k=0}^{\infty} b_k(x-c)^k$ defines a function of x. Its domain is the set of all x for which the series converges. The primary tool for investigating the convergence or divergence of a power series is the Ratio Test. Notice again that the partial sums of a power series are all polynomials (the simplest functions around).

NOTES

According to the expansion indicated to the right, the first term of the power series is b_0, even when $x = c$, although the summation notation might suggest that the first term is $b_0\,(0)^0$, which is not defined.

EXAMPLE 6.1 Determining Where a Power Series Converges

Determine the values of x for which the power series $\sum_{k=0}^{\infty} \frac{k}{3^{k+1}} x^k$ converges.

Solution Using the Ratio Test, we have convergence if

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)x^{k+1}}{3^{k+2}}\frac{3^{k+1}}{kx^k}\right|$$

$$= \lim_{k\to\infty}\frac{(k+1)|x|}{3k} = \frac{|x|}{3}\lim_{k\to\infty}\frac{k+1}{k}$$

Since $x^{k+1} = x^k \cdot x^1$ and $3^{k+2} = 3^{k+1} \cdot 3^1$.

$$= \frac{|x|}{3} < 1,$$

for $|x| < 3$ or $-3 < x < 3$. So, the series converges absolutely for $-3 < x < 3$ and diverges for $|x| > 3$ (i.e., for $x > 3$ or $x < -3$). Since the Ratio Test gives no conclusion for the endpoints $x = \pm 3$, we must test these separately.

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For $x = 3$, we have the series

$$\sum_{k=0}^{\infty} \frac{k}{3^{k+1}} x^k = \sum_{k=0}^{\infty} \frac{k}{3^{k+1}} 3^k = \sum_{k=0}^{\infty} \frac{k}{3}.$$

Since

$$\lim_{k\to\infty} \frac{k}{3} = \infty \neq 0,$$

the series diverges by the kth-term test for divergence. The series diverges when $x = -3$, for the same reason. Thus, the power series converges for all x in the interval $(-3, 3)$ and diverges for all x outside this interval. ■

Observe that example 6.1 has something in common with the introductory example. In both cases, the series have the form $\sum_{k=0}^{\infty} b_k(x - c)^k$ and there is an interval of the form $(c - r, c + r)$ on which the series converges and outside of which the series diverges. (In the case of example 6.1, notice that $c = 0$.) This interval on which a power series converges is called the **interval of convergence**. The constant r is called the **radius of convergence** (i.e., r is half the length of the interval of convergence). It turns out that there is such an interval for every power series. We have the following result.

THEOREM 6.1

Given any power series, $\sum_{k=0}^{\infty} b_k(x - c)^k$, there are exactly three possibilities:

(i) The series converges for *all* $x \in (-\infty, \infty)$ and the radius of convergence is $r = \infty$;
(ii) The series converges *only* for $x = c$ (and diverges for all other values of x) and the radius of convergence is $r = 0$; or
(iii) The series converges for $x \in (c - r, c + r)$ and diverges for $x < c - r$ and for $x > c + r$, for some number r with $0 < r < \infty$.

The proof of the theorem can be found in Appendix F.

EXAMPLE 6.2 Finding the Interval and Radius of Convergence

Determine the radius and interval of convergence for the power series

$$\sum_{k=0}^{\infty} \frac{10^k}{k!}(x - 1)^k.$$

Solution From the Ratio Test, we have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{10^{k+1}(x-1)^{k+1}}{(k+1)!}\frac{k!}{10^k(x-1)^k}\right|$$

$$= 10|x - 1| \lim_{k\to\infty} \frac{k!}{(k+1)k!}$$

Since $(x-1)^{k+1} = (x-1)^k(x-1)^1$ and $(k+1)! = (k+1)k!$

$$= 10|x - 1| \lim_{k\to\infty} \frac{1}{k+1} = 0 < 1,$$

for *all* x. This says that the series converges absolutely for all x. Thus, the interval of convergence for this series is $(-\infty, \infty)$ and the radius of convergence is $r = \infty$. ■

The interval of convergence for a power series can be a closed interval, an open interval or a half-open interval, as in example 6.3.

EXAMPLE 6.3 **A Half-Open Interval of Convergence**

Determine the radius and interval of convergence for the power series $\sum_{k=1}^{\infty} \frac{x^k}{k4^k}$.

Solution From the Ratio Test, we have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}}{(k+1)4^{k+1}}\frac{k4^k}{x^k}\right|$$
$$= \frac{|x|}{4}\lim_{k\to\infty}\frac{k}{k+1} = \frac{|x|}{4} < 1.$$

So, we are guaranteed absolute convergence for $|x| < 4$ and divergence for $|x| > 4$. It remains only to test the endpoints of the interval: $x = \pm 4$. For $x = 4$, we have

$$\sum_{k=1}^{\infty}\frac{x^k}{k4^k} = \sum_{k=1}^{\infty}\frac{4^k}{k4^k} = \sum_{k=1}^{\infty}\frac{1}{k},$$

which you will recognize as the harmonic series, which diverges. For $x = -4$, we have

$$\sum_{k=1}^{\infty}\frac{x^k}{k4^k} = \sum_{k=1}^{\infty}\frac{(-4)^k}{k4^k} = \sum_{k=1}^{\infty}\frac{(-1)^k}{k},$$

which is the alternating harmonic series, which we know converges (see example 4.2). So, in this case, the interval of convergence is the half-open interval $[-4, 4)$ and the radius of convergence is $r = 4$. ■

Notice that (as stated in Theorem 6.1) every power series, $\sum_{k=0}^{\infty} b_k(x-c)^k$ converges at least for $x = c$, since for $x = c$, we have the trivial case

$$\sum_{k=0}^{\infty} b_k(x-c)^k = b_0 + \sum_{k=1}^{\infty} b_k(c-c)^k = b_0 + \sum_{k=1}^{\infty} b_k 0^k = b_0 + 0 = b_0.$$

EXAMPLE 6.4 **A Power Series That Converges at Only One Point**

Determine the radius of convergence for the power series $\sum_{k=0}^{\infty} k!(x-5)^k$.

Solution From the Ratio Test, we have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)!(x-5)^{k+1}}{k!(x-5)^k}\right|$$
$$= \lim_{k\to\infty}\frac{(k+1)k!|x-5|}{k!}$$
$$= \lim_{k\to\infty}[(k+1)|x-5|]$$
$$= \begin{cases} 0, & \text{if } x = 5 \\ \infty, & \text{if } x \neq 5. \end{cases}$$

Thus, this power series converges only for $x = 5$ and so, its radius of convergence is $r = 0$. ■

Suppose that the power series $\sum_{k=0}^{\infty} b_k(x-c)^k$ has radius of convergence $r > 0$. Then the series converges absolutely for all x in the interval $(c-r, c+r)$ and might converge at one or both of the endpoints, $x = c - r$ and $x = c + r$. Notice that since the series converges for each $x \in (c-r, c+r)$, it defines a function f on the interval $(c-r, c+r)$,

$$f(x) = \sum_{k=0}^{\infty} b_k(x-c)^k = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + \cdots.$$

It turns out that such a function is continuous and differentiable, although the proof is beyond the level of this course. In fact, we differentiate exactly the way you might expect,

Differentiating a power series

$$f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx}[b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + \cdots]$$
$$= b_1 + 2b_2(x-c) + 3b_3(x-c)^2 + \cdots = \sum_{k=1}^{\infty} b_k k(x-c)^{k-1},$$

where the radius of convergence of the resulting series is also r. Since we find the derivative by differentiating each term in the series, we call this **term-by-term** differentiation. Likewise, we can integrate a convergent power series term-by-term:

Integrating a power series

$$\int f(x)\,dx = \int \sum_{k=0}^{\infty} b_k(x-c)^k\,dx = \sum_{k=0}^{\infty} b_k \int (x-c)^k\,dx$$
$$= \sum_{k=0}^{\infty} b_k \frac{(x-c)^{k+1}}{k+1} + K,$$

where the radius of convergence of the resulting series is again r and where K is a constant of integration. The proof of these two results can be found in a text on advanced calculus. It's important to recognize that these two results are *not* obvious. They are not simply an application of the rule that a derivative or integral of a sum is simply the sum of the derivatives or integrals, respectively, since a series is not a sum, but rather, a limit of a sum. (What's the difference, anyway?) Further, these results are true for power series, but are *not* true for series in general.

We summarize the term-by-term differentiation and integration of power series in the following.

Suppose that

$$f(x) = \sum_{k=0}^{\infty} b_k(x-c)^k,$$

where the radius of convergence is $r > 0$. Then,

$$f'(x) = \sum_{k=0}^{\infty} k b_k(x-c)^{k-1}$$

and

$$\int f(x)\,dx = \sum_{k=0}^{\infty} \frac{b_k}{k+1}(x-c)^{k+1} + K,$$

where both of these series also have radius of convergence r.

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EXAMPLE 6.5 A Convergent Series Whose Series of Derivatives Diverges

Find the interval of convergence of the series $\sum_{k=1}^{\infty} \frac{\sin(k^3 x)}{k^2}$ and show that the series of derivatives does not converge for any x.

Solution Notice that

$$\left| \frac{\sin(k^3 x)}{k^2} \right| \le \frac{1}{k^2}, \quad \text{for all } x,$$

since $|\sin(k^3 x)| \le 1$. Further, $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series ($p = 2 > 1$) and so, it follows from the Comparison Test that $\sum_{k=0}^{\infty} \frac{\sin(k^3 x)}{k^2}$ converges absolutely, for all x. On the other hand, the series of derivatives (found by differentiating the series term-by-term) is

$$\sum_{k=1}^{\infty} \frac{d}{dx}\left[\frac{\sin(k^3 x)}{k^2}\right] = \sum_{k=1}^{\infty} \frac{k^3 \cos(k^3 x)}{k^2} = \sum_{k=1}^{\infty} [k \cos(k^3 x)],$$

which *diverges* for all x, by the kth-term test for divergence, since the terms do not tend to zero as $k \to \infty$, for any x. ■

Keep in mind that $\sum_{k=1}^{\infty} \frac{\sin(k^3 x)}{k^2}$ is not a power series. (Why not?) The result of example 6.5 (a convergent series whose series of derivatives diverges) *cannot* occur with any power series with radius of convergence $r > 0$.

In example 6.6, we find that once we have a convergent power series representation for a given function, we can use this to obtain power series representations for any number of other functions, by differentiating and integrating the series term by term.

EXAMPLE 6.6 Differentiating and Integrating a Power Series

Use the power series $\sum_{k=0}^{\infty} (-1)^k x^k$ to find power series representations of $\frac{1}{(1+x)^2}$, $\frac{1}{1+x^2}$ and $\tan^{-1} x$.

Solution Notice that $\sum_{k=0}^{\infty} (-1)^k x^k = \sum_{k=0}^{\infty} (-x)^k$ is a geometric series with ratio $r = -x$. This series converges, then, whenever $|r| = |-x| = |x| < 1$, to

$$\frac{a}{1-r} = \frac{1}{1-(-x)} = \frac{1}{1+x}.$$

That is, for $-1 < x < 1$,

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k. \tag{6.2}$$

Differentiating both sides of (6.2), we get

$$\frac{-1}{(1+x)^2} = \sum_{k=0}^{\infty} (-1)^k k x^{k-1}, \quad \text{for } -1 < x < 1.$$

Multiplying both sides by -1 gives us a new power series representation:

$$\frac{1}{(1+x)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} k x^{k-1},$$

valid for $-1 < x < 1$. Notice that we can also obtain a new power series from (6.2) by substitution. For instance, if we replace x with x^2, we get

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k (x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}, \tag{6.3}$$

valid for $-1 < x^2 < 1$ (which is equivalent to having $x^2 < 1$ or $-1 < x < 1$).

Integrating both sides of (6.3) gives us

$$\int \frac{1}{1+x^2}\,dx = \sum_{k=0}^{\infty} (-1)^k \int x^{2k}\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} + c. \tag{6.4}$$

You should recognize the integral on the left-hand side of (6.4) as $\tan^{-1} x$. That is,

$$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} + c, \text{ for } -1 < x < 1. \tag{6.5}$$

Taking $x = 0$ gives us

$$\tan^{-1} 0 = \sum_{k=0}^{\infty} \frac{(-1)^k 0^{2k+1}}{2k+1} + c = c,$$

so that $c = \tan^{-1} 0 = 0$. Equation (6.5) now gives us a power series representation for $\tan^{-1} x$, namely:

$$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots, \text{ for } -1 < x < 1. \blacksquare$$

Notice that working as in example 6.6, we can produce power series representations of any number of functions. In section 7.7, we present a systematic method for producing power series representations for a wide range of functions.

EXERCISES 7.6

WRITING EXERCISES

1. Power series have the form $\sum_{k=0}^{\infty} a_k (x-c)^k$. Explain why the farther x is from c, the larger the terms of the series are and the less likely the series is to converge. Describe how this general trend relates to the radius of convergence.

2. Applying the Ratio Test to $\sum_{k=0}^{\infty} a_k (x-c)^k$ requires you to evaluate $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k}(x-c) \right|$. For $x = c$, this limit equals 0 and the series converges. As x increases or decreases, $|x-c|$ increases. If the series has a finite radius of convergence $r > 0$, what is the value of the limit when $|x-c| = r$? Explain how the limit changes when $|x-c| < r$ and $|x-c| > r$ and how this determines the convergence or divergence of the series.

3. As shown in example 6.2, $\sum_{k=0}^{\infty} \frac{10^k}{k!}(x-1)^k$ converges for all x. If $x = 1001$, the value of $(x-1)^k = 1000^k$ gets very large very fast, as k increases. Explain why, for the series to converge, the value of $k!$ must get large faster than 1000^k. To illustrate how fast the factorial grows, compute 50!, 100! and 200! (if your calculator can handle these).

4. In a power series representation of $\sqrt{x+1}$ about $c = 0$, explain why the radius of convergence cannot be greater than 1. (Think about the domain of $\sqrt{x+1}$.)

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In exercises 1–10, find a power series representation of $f(x)$ about $c = 0$ (refer to example 6.6). Also, determine the radius and interval of convergence and graph $f(x)$ together with the partial sums $\sum_{k=0}^{3} a_k x^k$ and $\sum_{k=0}^{6} a_k x^k$.

1. $f(x) = \dfrac{2}{1-x}$
2. $f(x) = \dfrac{3}{x-1}$
3. $f(x) = \dfrac{3}{1+x^2}$
4. $f(x) = \dfrac{2}{1-x^2}$
5. $f(x) = \dfrac{2x}{1-x^3}$
6. $f(x) = \dfrac{3x}{1+x^2}$
7. $f(x) = \dfrac{4}{1+4x}$
8. $f(x) = \dfrac{3}{1-4x}$
9. $f(x) = \dfrac{2}{4+x}$
10. $f(x) = \dfrac{3}{6-x}$

In exercises 11–16, determine the interval of convergence and the function to which the given power series converges.

11. $\sum_{k=0}^{\infty} (x+2)^k$
12. $\sum_{k=0}^{\infty} (x-3)^k$
13. $\sum_{k=0}^{\infty} (2x-1)^k$
14. $\sum_{k=0}^{\infty} (3x+1)^k$
15. $\sum_{k=0}^{\infty} (-1)^k \left(\dfrac{x}{2}\right)^k$
16. $\sum_{k=0}^{\infty} 3\left(\dfrac{x}{4}\right)^k$

In exercises 17–34, determine the radius and interval of convergence.

17. $\sum_{k=0}^{\infty} \dfrac{2^k}{k!}(x-2)^k$
18. $\sum_{k=0}^{\infty} \dfrac{3^k}{k!}x^k$
19. $\sum_{k=0}^{\infty} \dfrac{k}{4^k}x^k$
20. $\sum_{k=0}^{\infty} \dfrac{k}{2^k}x^k$
21. $\sum_{k=1}^{\infty} \dfrac{(-1)^k}{k3^k}(x-1)^k$
22. $\sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k4^k}(x+2)^k$
23. $\sum_{k=0}^{\infty} k!(x+1)^k$
24. $\sum_{k=0}^{\infty} k!(x-2)^k$
25. $\sum_{k=1}^{\infty} \dfrac{1}{k}(x-1)^k$
26. $\sum_{k=0}^{\infty} k(x-2)^k$
27. $\sum_{k=0}^{\infty} k^2(x-3)^k$
28. $\sum_{k=1}^{\infty} \dfrac{1}{k^2}(x+2)^k$
29. $\sum_{k=0}^{\infty} \dfrac{k!}{(2k)!}x^k$
30. $\sum_{k=0}^{\infty} \dfrac{(k!)^2}{(2k)!}x^k$
31. $\sum_{k=1}^{\infty} \dfrac{2^k}{k^2}(x+2)^k$
32. $\sum_{k=0}^{\infty} \dfrac{k^2}{k!}(x+1)^k$
33. $\sum_{k=1}^{\infty} \dfrac{4^k}{\sqrt{k}}x^k$
34. $\sum_{k=1}^{\infty} \dfrac{(-1)^k x^k}{\sqrt{k}}$

In exercises 35–42, find a power series representation and radius of convergence by integrating or differentiating one of the series from exercises 1–10.

35. $f(x) = 3\tan^{-1} x$
36. $f(x) = 2\ln(1-x)$
37. $f(x) = \dfrac{2x}{(1-x^2)^2}$
38. $f(x) = \dfrac{3}{(x-1)^2}$
39. $f(x) = \ln(1+x^2)$
40. $f(x) = \ln(1+4x)$
41. $f(x) = \dfrac{1}{(1+4x)^2}$
42. $f(x) = \dfrac{2}{(4+x)^2}$

In exercises 43–46, find the interval of convergence of the (non-power) series and the corresponding series of derivatives.

43. $\sum_{k=1}^{\infty} \dfrac{\cos(k^3 x)}{k^2}$
44. $\sum_{k=1}^{\infty} \dfrac{\cos(x/k)}{k}$
45. $\sum_{k=0}^{\infty} e^{kx}$
46. $\sum_{k=0}^{\infty} e^{-2kx}$

47. For any constants a and $b > 0$, determine the interval and radius of convergence of $\sum_{k=0}^{\infty} \dfrac{(x-a)^k}{b^k}$.

48. Prove that if $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence r, with $0 < r < \infty$, then $\sum_{k=0}^{\infty} a_k x^{2k}$ has radius of convergence $\sqrt{r}$.

49. If $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence r, with $0 < r < \infty$, determine the radius of convergence of $\sum_{k=0}^{\infty} a_k (x-c)^k$ for any constant c.

50. If $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence r, with $0 < r < \infty$, determine the radius of convergence of $\sum_{k=0}^{\infty} a_k \left(\dfrac{x}{b}\right)^k$ for any constant $b \neq 0$.

51. Show that $f(x) = \dfrac{x+1}{(1-x)^2} = \dfrac{\frac{2x}{1-x}+1}{1-x}$ has the power series representation $f(x) = 1 + 3x + 5x^2 + 7x^3 + 9x^4 + \cdots$. Find the radius of convergence. Set $x = \dfrac{1}{1000}$ and discuss the interesting decimal representation of $\dfrac{1{,}001{,}000}{998{,}001}$.

52. Use long division to show that $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$.

53. Even great mathematicians can make mistakes. Leonhard Euler started with the equation $\dfrac{x}{x-1} + \dfrac{x}{1-x} = 0$, rewrote it as $\dfrac{1}{1-1/x} + \dfrac{x}{1-x} = 0$, found power series representations for each function and concluded that $\cdots + \dfrac{1}{x^2} + \dfrac{1}{x} + 1 + x +$

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$x^2 + \cdots = 0$. Substitute $x = 1$ to show that the conclusion is false, then find the mistake in Euler's derivation.

T 54. If your CAS or calculator has a command named "Taylor," use it to verify your answers to exercises 35–42.

55. An **electric dipole** consists of a charge q located at $x = 1$ and a charge $-q$ located at $x = -1$. The electric field at any $x > 1$ is given by $E(x) = \frac{kq}{(x-1)^2} - \frac{kq}{(x+1)^2}$ for some constant k. Find a power series representation for $E(x)$.

T 56. Show that a power series representation of $f(x) = \ln(1 + x^2)$ is given by $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+2}}{k+1}$. For the partial sums $P_n(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+2}}{k+1}$, compute $|f(0.9) - P_n(0.9)|$ for each of $n = 2, 4, 6$. Discuss the pattern. Then compute $|f(1.1) - P_n(1.1)|$ for each of $n = 2, 4, 6$. Discuss the pattern. Discuss the relevance of the radius of convergence to these calculations.

EXPLORATORY EXERCISES

1. Note that the radius of convergence in each of exercises 1–5 is 1. Given that the functions in exercises 1, 2, 4 and 5 are undefined at $x = 1$, explain why the radius of convergence can't be larger than 1. The restricted radius in exercise 3 can be understood using complex numbers. Show that $1 + x^2 = 0$ for $x = \pm i$, where $i = \sqrt{-1}$. In general, a complex number $a + bi$ is associated with the point (a, b). Find the "distance" between the complex numbers 0 and i by finding the distance between the associated points (0, 0) and (0, 1). Discuss how this compares to the radius of convergence. Then use the ideas in this exercise to quickly conjecture the radius of convergence of power series with center $c = 0$ for the functions $f(x) = \frac{4}{1+4x}$, $f(x) = \frac{2}{4+x}$ and $f(x) = \frac{2}{4+x^2}$.

2. For each series $f(x)$, compare the intervals of convergence of $f(x)$ and $\int f(x)\,dx$, where the antiderivative is taken term by term. (a) $f(x) = \sum_{k=0}^{\infty} (-1)^k x^k$; (b) $f(x) = \sum_{k=0}^{\infty} \sqrt{k} x^k$; (c) $f(x) = \sum_{k=0}^{\infty} \frac{1}{k} x^k$. As stated in the text, the radius of convergence remains the same after integration (or differentiation). Based on the examples in this exercise, does integration make it more or less likely that the series will converge at the endpoints? Conversely, will differentiation make it more or less likely that the series will converge at the endpoints?

7.7 TAYLOR SERIES

You may still be wondering about the reason why we have developed series. Each time we have developed a new concept, we have worked hard to build a case for why we want to do what we're doing. For example, in developing the derivative, we set out to find the slope of a tangent line and to find instantaneous velocity, only to find that they were essentially the same thing. When we developed the definite integral, we did so in the course of trying to find area under the curve. But, we have not yet completely revealed why we're pursuing series, even though we've been developing them for more than five sections now. Well, the punchline is close at hand. In this section, we develop a compelling reason for considering series. They are not merely another mathematical curiosity, but rather, are an essential means for exploring and computing with transcendental functions (e.g., $\sin x$, $\cos x$, $\ln x$, e^x, etc.).

Suppose that the power series $\sum_{k=0}^{\infty} b_k (x-c)^k$ has radius of convergence $r > 0$. As we've observed, this means that the series converges absolutely to some function f on the interval $(c - r, c + r)$. We have

$$f(x) = \sum_{k=0}^{\infty} b_k (x-c)^k = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + b_4(x-c)^4 + \cdots,$$

for each $x \in (c - r, c + r)$. Differentiating term by term, we get that

$$f'(x) = \sum_{k=0}^{\infty} b_k k (x-c)^{k-1} = b_1 + 2b_2(x-c) + 3b_3(x-c)^2 + 4b_4(x-c)^3 + \cdots,$$

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again, for each $x \in (c - r, c + r)$. Likewise, we get

$$f''(x) = \sum_{k=0}^{\infty} b_k k(k-1)(x-c)^{k-2} = 2b_2 + 3 \cdot 2b_3(x-c) + 4 \cdot 3b_4(x-c)^2 + \cdots$$

and

$$f'''(x) = \sum_{k=0}^{\infty} b_k k(k-1)(k-2)(x-c)^{k-3} = 3 \cdot 2b_3 + 4 \cdot 3 \cdot 2b_4(x-c) + \cdots$$

and so on (all valid for $c - r < x < c + r$). Notice that if we substitute $x = c$ in each of the above derivatives, all the terms of the series drop out, except one. We get

$$f(c) = b_0,$$
$$f'(c) = b_1,$$
$$f''(c) = 2b_2,$$
$$f'''(c) = 3!\, b_3$$

and so on. Observe, that in general, we have

$$f^{(k)}(c) = k!\, b_k. \tag{7.1}$$

Solving (7.1) for b_k, we have

$$b_k = \frac{f^{(k)}(c)}{k!}, \text{ for } k = 0, 1, 2, \ldots.$$

To summarize, we found that if $\sum_{k=0}^{\infty} b_k(x-c)^k$ is a convergent power series with radius of convergence $r > 0$, then the series converges to some function f that we can write as

$$f(x) = \sum_{k=0}^{\infty} b_k(x-c)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k, \text{ for } x \in (c-r, c+r).$$

Now, think about this problem from another angle. Instead of starting with a series, suppose that you start with an infinitely differentiable function, f (i.e., f can be differentiated infinitely often). Then, we can construct the series

Taylor series expansion of $f(x)$ about $x = c$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k,$$

called a **Taylor series** expansion for f. (See the historical note on Brook Taylor in section 4.6.) There are two important questions we need to answer.

- Does a series constructed in this way converge? If so, what is its radius of convergence?
- If the series converges, it converges to a function. What is that function? (For instance, is it f?)

We can answer the first of these questions on a case-by-case basis, usually by applying the Ratio Test. The second question will require further insight.

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EXAMPLE 7.1 Constructing a Taylor Series Expansion

Construct the Taylor series expansion for $f(x) = e^x$, about $x = 0$ (i.e., take $c = 0$).

Solution Here, we have the extremely simple case where

$$f'(x) = e^x,\ f''(x) = e^x \text{ and so on, } f^{(k)}(x) = e^x, \text{ for } k = 0, 1, 2, \ldots.$$

This gives us the Taylor series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k = \sum_{k=0}^{\infty} \frac{e^0}{k!}x^k = \sum_{k=0}^{\infty} \frac{1}{k!}x^k.$$

From the Ratio Test, we have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{|x|^{k+1}}{(k+1)!}\frac{k!}{|x|^k} = |x|\lim_{k\to\infty}\frac{k!}{(k+1)\,k!}$$

$$= |x|\lim_{k\to\infty}\frac{1}{k+1} = |x|\,(0) = 0 < 1, \text{ for all } x.$$

So, the Taylor series $\sum_{k=0}^{\infty}\frac{1}{k!}x^k$ converges for all real numbers x. At this point, though, we do not know the function to which the series converges. (Could it be e^x?) ■

REMARK 7.1

The special case of a Taylor series expansion about $x = 0$ is often called a Maclaurin series. (See the historical note about Colin Maclaurin in section 7.3.) That is, the series $\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$ is the Maclaurin series expansion for f.

Before we present any further examples of Taylor series, let's see if we can determine the function to which a given Taylor series converges. First, notice that the partial sums of a Taylor series (like any power series) are simply polynomials. We define

$$P_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k$$

$$= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

Observe that $P_n(x)$ is a polynomial of degree n, as $\frac{f^{(k)}(c)}{k!}$ is a constant for each k. We refer to P_n as the **Taylor polynomial of degree n** for f expanded about $x = c$.

EXAMPLE 7.2 Constructing and Graphing Taylor Polynomials

For $f(x) = e^x$, find the Taylor polynomial of degree n expanded about $x = 0$.

Solution As in example 7.1, we have that $f^{(k)}(x) = e^x$, for all k. So, we have the nth degree Taylor polynomial is

$$P_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}(x-0)^k = \sum_{k=0}^{n}\frac{e^0}{k!}x^k$$

$$= \sum_{k=0}^{n}\frac{1}{k!}x^k = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

Since we established in example 7.1 that the Taylor series for $f(x) = e^x$ about $x = 0$ converges for all x, this says that the sequence of partial sums (i.e., the sequence of Taylor polynomials) converges for all x. In an effort to determine the function to which the Taylor polynomials are converging, we have plotted $P_1(x)$, $P_2(x)$, $P_3(x)$ and $P_4(x)$, together with the graph of $f(x) = e^x$ in Figures 7.39a–d, respectively.

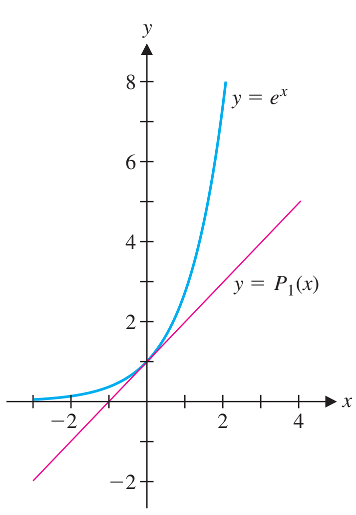

FIGURE 7.39a
$y = e^x$ and $y = P_1(x)$.

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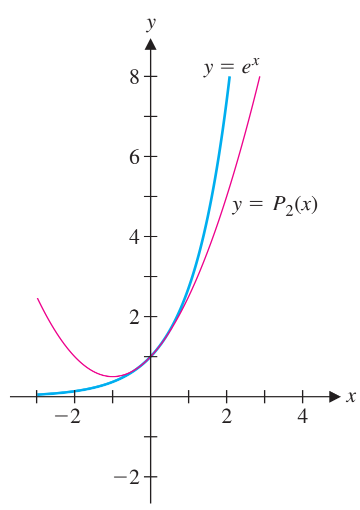

FIGURE 7.39b
$y = e^x$ and $y = P_2(x)$.

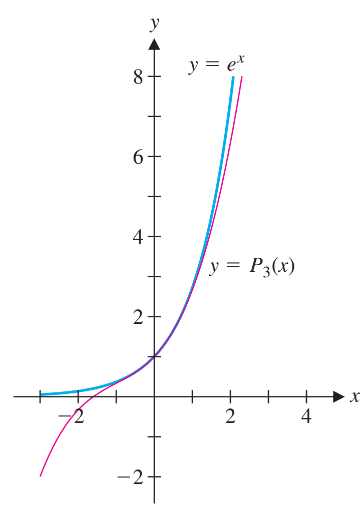

FIGURE 7.39c
$y = e^x$ and $y = P_3(x)$.

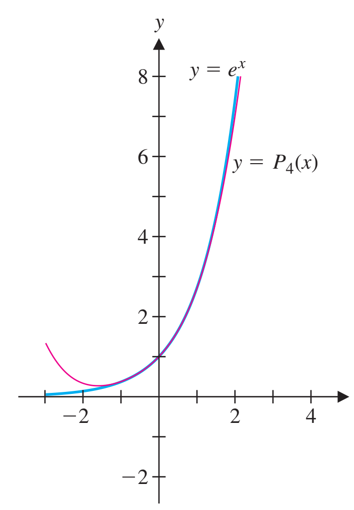

FIGURE 7.39d
$y = e^x$ and $y = P_4(x)$.

Notice that as n gets larger, the graphs of $P_n(x)$ appear (at least on the interval displayed) to be approaching the graph of $f(x) = e^x$. Since we know that the Taylor series converges and the graphical evidence suggests that the partial sums of the series are approaching $f(x) = e^x$, it is reasonable to conjecture that the series converges to e^x. This is, in fact, exactly what is happening, as we can prove using Theorems 7.1 and 7.2. ■

THEOREM 7.1 (Taylor's Theorem)

Suppose that f has $(n + 1)$ derivatives on the interval $(c - r, c + r)$, for some $r > 0$. Then, for $x \in (c - r, c + r)$, $f(x) \approx P_n(x)$ and the error in using $P_n(x)$ to approximate $f(x)$ is

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x - c)^{n+1}, \tag{7.2}$$

for some number z between x and c.

The error term $R_n(x)$ in (7.2) is often called the **remainder term.** Note that this term looks very much like the first neglected term of the Taylor series, except that $f^{(n+1)}$ is evaluated at some (unknown) number z between x and c, instead of at c. This remainder term serves two purposes: it enables us to obtain an estimate of the error in using a Taylor polynomial to approximate a given function and as we'll see in the next theorem, it gives us the means to prove that a Taylor series for a given function f converges to f.

The proof of Taylor's Theorem is somewhat technical and so we leave it for a more advanced text.

Note: If we could show that

$$\lim_{n\to\infty} R_n(x) = 0, \text{ for } all\ x \text{ in } (c - r, c + r),$$

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then we would have that

$$0 = \lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} [f(x) - P_n(x)] = f(x) - \lim_{n\to\infty} P_n(x)$$

or

$$\lim_{n\to\infty} P_n(x) = f(x), \text{ for } \textit{all } x \in (c-r, c+r).$$

That is, the sequence of partial sums of the Taylor series (i.e., the sequence of Taylor polynomials) converges to $f(x)$ for each $x \in (c-r, c+r)$. We summarize this in Theorem 7.2.

REMARK 7.2

Observe that for $n = 0$, Taylor's Theorem simplifies to a very familiar result. We have

$$R_0(x) = f(x) - P_0(x) = \frac{f'(z)}{(0+1)!}(x-c)^{0+1}.$$

Since $P_0(x) = f(c)$, we have simply

$$f(x) - f(c) = f'(z)(x-c).$$

Dividing by $(x-c)$, gives us

$$\frac{f(x)-f(c)}{x-c} = f'(z),$$

which is the conclusion of the Mean Value Theorem. In this way, observe that Taylor's Theorem is a generalization of the Mean Value Theorem.

THEOREM 7.2

Suppose that f has derivatives of all orders in the interval $(c-r, c+r)$, for some $r > 0$ and that $\lim_{n\to\infty} R_n(x) = 0$, for all x in $(c-r, c+r)$. Then, the Taylor series for f expanded about $x = c$ converges to $f(x)$, that is,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k,$$

for all x in $(c-r, c+r)$.

We now return to the Taylor series expansion of $f(x) = e^x$ about $x = 0$, constructed in example 7.1 and investigated further in example 7.2 and prove that it converges to e^x, as we had suspected.

EXAMPLE 7.3 Proving That a Taylor Series Converges to the Desired Function

Show that the Taylor series for $f(x) = e^x$ expanded about $x = 0$ converges to e^x.

Solution We already found the indicated Taylor series, $\sum_{k=0}^{\infty} \frac{1}{k!}x^k$ in example 7.1. Here, we have $f^{(k)}(x) = e^x$, for all $k = 0, 1, 2, \ldots$. This gives us the remainder term

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-0)^{n+1} = \frac{e^z}{(n+1)!}x^{n+1}, \tag{7.3}$$

where z is somewhere between x and 0 (and depends also on the value of n). We first find a bound on the size of e^z. Notice that if $x > 0$, then $0 < z < x$ and so,

$$e^z < e^x.$$

If $x \le 0$, then $x \le z \le 0$, so that

$$e^z \le e^0 = 1.$$

We define M to be the larger of these two bounds on e^z. That is, we let

$$M = \max\{e^x, 1\}.$$

Then, for any x and any n, we have

$$e^z \le M.$$

Together with (7.3), this gives us the error estimate

$$|R_n(x)| = \frac{e^z}{(n+1)!}|x|^{n+1} \le M\frac{|x|^{n+1}}{(n+1)!}. \tag{7.4}$$

To prove that the Taylor series converges to e^x, we want to use (7.4) to show that $\lim_{n\to\infty} R_n(x) = 0$, for all x. However, for any given x, how can we compute $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!}$? While we cannot do so directly, we can use the following indirect approach. We consider the *series* $\sum_{n=0}^{\infty} \frac{|x|^{n+1}}{(n+1)!}$, as follows. Applying the Ratio Test, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|^{n+2}}{(n+2)!}\frac{(n+1)!}{|x|^{n+1}} = |x|\lim_{k\to\infty}\frac{1}{n+2} = 0,$$

for all x. This then says that the series $\sum_{n=0}^{\infty} \frac{|x|^{n+1}}{(n+1)!}$ converges absolutely for all x. By the kth-term test for divergence, since this last series converges, its general term must tend to 0 as $n \to \infty$. That is,

$$\lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!} = 0$$

and so, from (7.4), $\lim_{n\to\infty} R_n(x) = 0$, for all x. From Theorem 7.2, we now have that the Taylor series converges to e^x for all x. That is,

$$e^x = \sum_{k=0}^{\infty}\frac{1}{k!}x^k = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots. \tag{7.5}$$

The trick, if there is one, in finding a Taylor series expansion is in *accurately* calculating enough derivatives for you to recognize the general form of the nth derivative. So, take your time and *BE CAREFUL!* Once this is done, you need to show that $R_n(x) \to 0$, as $n \to \infty$, for all x, to ensure that the series converges to the function you are expanding.

One of the reasons for calculating Taylor series is that we can use their partial sums to compute approximate values of a function.

EXAMPLE 7.4 Using a Taylor Series to Obtain an Approximation of *e*

Use the Taylor series for e^x in (7.5) to obtain an approximation to the number e.

Solution We have

$$e = e^1 = \sum_{k=0}^{\infty}\frac{1}{k!}1^k = \sum_{k=0}^{\infty}\frac{1}{k!}.$$

We list some partial sums of this series in the accompanying table. From this we get the very accurate approximation

$$e \approx 2.718281828.$$

M	$\sum_{k=0}^{M}\frac{1}{k!}$
5	2.716666667
10	2.718281801
15	2.718281828
20	2.718281828

EXAMPLE 7.5 A Taylor Series Expansion of sin *x*

Find the Taylor series for $f(x) = \sin x$, expanded about $x = \frac{\pi}{2}$ and prove that the series converges to $\sin x$ for all x.

Solution In this case, the Taylor series is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}\left(\frac{\pi}{2}\right)}{k!}\left(x - \frac{\pi}{2}\right)^k.$$

First, we compute some derivatives and their value at $x = \frac{\pi}{2}$. We have

$$\begin{aligned} f(x) &= \sin x & f\left(\tfrac{\pi}{2}\right) &= 1, \\ f'(x) &= \cos x & f'\left(\tfrac{\pi}{2}\right) &= 0, \\ f''(x) &= -\sin x & f''\left(\tfrac{\pi}{2}\right) &= -1, \\ f'''(x) &= -\cos x & f'''\left(\tfrac{\pi}{2}\right) &= 0, \\ f^{(4)}(x) &= \sin x & f^{(4)}\left(\tfrac{\pi}{2}\right) &= 1 \end{aligned}$$

and so on. Recognizing that every other term is zero and every other term is ± 1, we see that the Taylor series is

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{f^{(k)}\left(\frac{\pi}{2}\right)}{k!}\left(x - \frac{\pi}{2}\right)^k &= 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x - \frac{\pi}{2}\right)^6 + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(x - \frac{\pi}{2}\right)^{2k}. \end{aligned}$$

In order to test the series for convergence, we consider the remainder term

$$|R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!}\left(x - \frac{\pi}{2}\right)^{n+1} \right|, \qquad (7.6)$$

for some z between x and $\frac{\pi}{2}$. From our derivative calculations, note that

$$f^{(n+1)}(z) = \begin{cases} \pm\cos z, & \text{if } n \text{ is even} \\ \pm\sin z, & \text{if } n \text{ is odd} \end{cases}.$$

From this, observe that

$$\left| f^{(n+1)}(z) \right| \le 1,$$

for every n. (Notice that this is true whether n is even or odd.) From (7.6), we now have

$$|R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} \right| \left| x - \frac{\pi}{2} \right|^{n+1} \le \frac{1}{(n+1)!}\left| x - \frac{\pi}{2} \right|^{n+1} \to 0,$$

as $n \to \infty$, for every x, as in example 7.3. This says that

$$\boxed{\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(x - \frac{\pi}{2}\right)^{2k} = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \cdots,}$$

for all x. In Figures 7.40a–d, we show graphs of $f(x) = \sin x$ together with the Taylor polynomials $P_2(x)$, $P_4(x)$, $P_6(x)$ and $P_8(x)$ (the first few partial sums of the series). Notice that the higher the degree of the Taylor polynomial is, the larger the interval is over which the polynomial provides a close approximation to $f(x) = \sin x$.

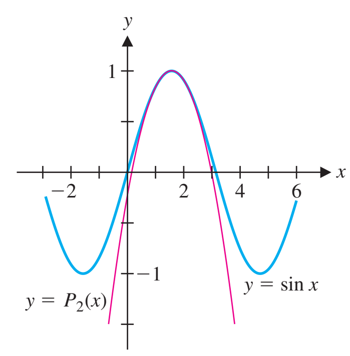

FIGURE 7.40a
$y = \sin x$ and $y = P_2(x)$.

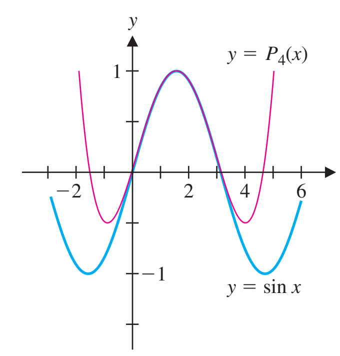

FIGURE 7.40b
$y = \sin x$ and $y = P_4(x)$.

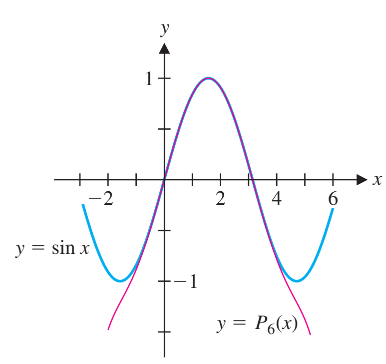

FIGURE 7.40c
$y = \sin x$ and $y = P_6(x)$.

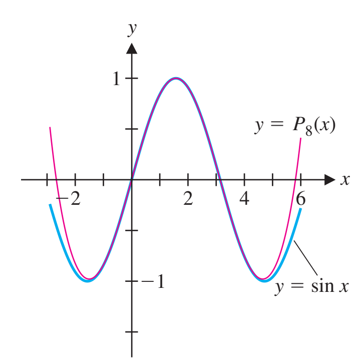

FIGURE 7.40d
$y = \sin x$ and $y = P_8(x)$.

In example 7.6, we illustrate how to use Taylor's Theorem to estimate the error in using a Taylor polynomial to approximate the value of a function.

EXAMPLE 7.6 Estimating the Error in a Taylor Polynomial Approximation

Use a Taylor polynomial to approximate the value of $\ln(1.1)$ and estimate the error in this approximation.

Solution First, note that since ln 1 is known exactly and 1 is close to 1.1 (Why would this matter?), we expand $f(x) = \ln x$ in a Taylor series about $x = 1$. We compute an adequate number of derivatives so that the pattern becomes clear. We have

$$\begin{aligned}
f(x) &= \ln x & f(1) &= 0 \\
f'(x) &= x^{-1} & f'(1) &= 1 \\
f''(x) &= -x^{-2} & f''(1) &= -1 \\
f'''(x) &= 2x^{-3} & f'''(1) &= 2 \\
f^{(4)}(x) &= -3 \cdot 2x^{-4} & f^{(4)}(1) &= -3! \\
f^{(5)}(x) &= 4!\,x^{-5} & f^{(5)}(1) &= 4! \\
&\vdots & &\vdots \\
f^{(k)}(x) &= (-1)^{k+1}(k-1)!\,x^{-k} & f^{(k)}(1) &= (-1)^{k+1}(k-1)! \quad (k \geq 1).
\end{aligned}$$

We get the Taylor series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k$$
$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \cdots + (-1)^{k+1}\frac{(k-1)!}{k!}(x-1)^k + \cdots$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k.$$

We could use the remainder term to show that the series converges to $f(x) = \ln x$, for $0 < x < 2$, but this is not the original question here. (This is left as an exercise.) As an illustration, we construct the Taylor polynomial, $P_4(x)$,

$$P_4(x) = \sum_{k=1}^{4} \frac{(-1)^{k+1}}{k}(x-1)^k \quad \text{from the preceding}$$
$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4.$$

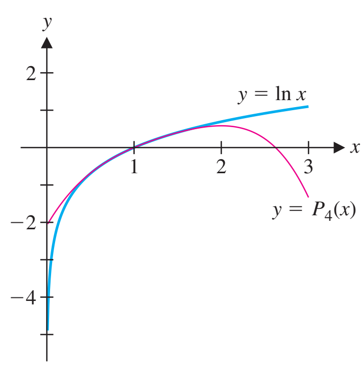

FIGURE 7.41
$y = \ln x$ and $y = P_4(x)$.

We show a graph of $y = \ln x$ and $y = P_4(x)$ in Figure 7.41. Taking $x = 1.1$ gives us the approximation

$$\ln(1.1) \approx P_4(1.1) = 0.1 - \frac{1}{2}(0.1)^2 + \frac{1}{3}(0.1)^3 - \frac{1}{4}(0.1)^4 \approx 0.095308333.$$

We can use the remainder term to estimate the error in this approximation. We have

$$|Error| = |\ln(1.1) - P_4(1.1)| = |R_4(1.1)|$$
$$= \left|\frac{f^{(4+1)}(z)}{(4+1)!}(1.1-1)^{4+1}\right| = \frac{4!|z|^{-5}}{5!}(0.1)^5,$$

where z is between 1 and 1.1. This gives us the following bound on the error:

$$|Error| = \frac{(0.1)^5}{5z^5} < \frac{(0.1)^5}{5(1^5)} = 0.000002,$$

since $1 < z < 1.1$ implies that $\frac{1}{z} < \frac{1}{1} = 1$. Another way to think of this is that our approximation of $\ln(1.1) \approx 0.095308333$ is off by no more than ± 0.000002. ■

A more significant question related to example 7.6 is to determine how many terms of the Taylor series are needed in order to guarantee a given accuracy. We use the remainder term to accomplish this in example 7.7.

EXAMPLE 7.7 Finding the Number of Terms Needed for a Given Accuracy

Find the number of terms in the Taylor series for $f(x) = \ln x$ expanded about $x = 1$ that will guarantee an accuracy of at least 1×10^{-10} in the approximation of $\ln(1.1)$ and $\ln(1.5)$.

Solution From our calculations in example 7.6 and (7.2), we have that for some number z between 1 and 1.1,

$$|R_n(1.1)| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}(1.1-1)^{n+1}\right|$$
$$= \frac{n!|z|^{-n-1}}{(n+1)!}(0.1)^{n+1} = \frac{(0.1)^{n+1}}{(n+1)\,z^{n+1}} < \frac{(0.1)^{n+1}}{n+1},$$

since $1 < z < 1.1$ implies that $\frac{1}{z} < \frac{1}{1} = 1$. Further, since we want the error to be less than 1×10^{-10}, we require that

$$|R_n(1.1)| < \frac{(0.1)^{n+1}}{n+1} < 1 \times 10^{-10}.$$

You can solve this inequality for n by trial and error to find that $n = 9$ will guarantee the required accuracy. Notice that larger values of n will also guarantee this accuracy, since $\frac{(0.1)^{n+1}}{n+1}$ is a decreasing function of n. We then have the approximation

$$\ln(1.1) \approx P_9(1.1) = \sum_{k=0}^{9} \frac{(-1)^{k+1}}{k}(1.1-1)^k$$
$$= (0.1) - \frac{1}{2}(0.1)^2 + \frac{1}{3}(0.1)^3 - \frac{1}{4}(0.1)^4 + \frac{1}{5}(0.1)^5$$
$$- \frac{1}{6}(0.1)^6 + \frac{1}{7}(0.1)^7 - \frac{1}{8}(0.1)^8 + \frac{1}{9}(0.1)^9$$
$$\approx 0.095310179813,$$

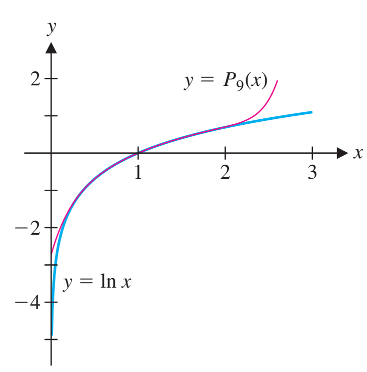

FIGURE 7.42
$y = \ln x$ and $y = P_9(x)$.

which from our error estimate we know is correct to within 1×10^{-10}. We show a graph of $y = \ln x$ and $y = P_9(x)$ in Figure 7.42. In comparing Figure 7.42 with Figure 7.41, observe that while $P_9(x)$ provides an improved approximation to $P_4(x)$ over the interval of convergence (0, 2), it does not provide a better approximation outside of this interval.

Similarly, notice that for some number z between 1 and 1.5,

$$|R_n(1.5)| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}(1.5-1)^{n+1}\right| = \frac{n!|z|^{-n-1}}{(n+1)!}(0.5)^{n+1}$$
$$= \frac{(0.5)^{n+1}}{(n+1)\,z^{n+1}} < \frac{(0.5)^{n+1}}{n+1},$$

since $1 < z < 1.5$ implies that $\frac{1}{z} < \frac{1}{1} = 1$. So, here we require that

$$|R_n(1.5)| < \frac{(0.5)^{n+1}}{n+1} < 1 \times 10^{-10}.$$

Solving this by trial and error shows that $n = 28$ will guarantee the required accuracy. Observe that this says that many more terms are needed to approximate $f(1.5)$ than for $f(1.1)$, to obtain the same accuracy. This further illustrates the general principle that the

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farther away x is from the point about which we expand, the slower the convergence of the Taylor series will be. ■

For your convenience, we have compiled a list of common Taylor series in the following table.

Taylor Series		*Interval of Convergence*	*Where to find*
$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$	$= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$	$(-\infty, \infty)$	examples 7.1 and 7.3
$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$	$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$	$(-\infty, \infty)$	exercise 2
$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$	$= 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$	$(-\infty, \infty)$	exercise 1
$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(x - \frac{\pi}{2}\right)^{2k}$	$= 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \cdots$	$(-\infty, \infty)$	example 7.5
$\ln x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (x-1)^k$	$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots$	$(0, 2]$	examples 7.6, 7.7
$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$	$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$	$(-1, 1)$	example 6.6

Notice that once you have found a Taylor series expansion for a given function, you can find any number of other Taylor series simply by making a substitution.

EXAMPLE 7.8 Finding New Taylor Series from Old Ones

Find Taylor series in powers of x for e^{2x}, e^{x^2} and e^{-2x}.

Solution Rather than compute the Taylor series for these functions from scratch, recall that we had established in example 7.3 that

$$e^t = \sum_{k=0}^{\infty} \frac{1}{k!} t^k = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \cdots, \tag{7.7}$$

for all $t \in (-\infty, \infty)$. We use the variable t here instead of x, so that we can more easily make substitutions. Taking $t = 2x$ in (7.7), we get the new Taylor series

$$e^{2x} = \sum_{k=0}^{\infty} \frac{1}{k!}(2x)^k = \sum_{k=0}^{\infty} \frac{2^k}{k!} x^k = 1 + 2x + \frac{2^2}{2!}x^2 + \frac{2^3}{3!}x^3 + \cdots.$$

Similarly, letting $t = x^2$ in (7.7), we get the Taylor series

$$e^{x^2} = \sum_{k=0}^{\infty} \frac{1}{k!}(x^2)^k = \sum_{k=0}^{\infty} \frac{1}{k!} x^{2k} = 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \cdots.$$

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Finally, taking $t = -2x$ in (7.7), we get

$$e^{-2x} = \sum_{k=0}^{\infty} \frac{1}{k!}(-2x)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} 2^k x^k = 1 - 2x + \frac{2^2}{2!}x^2 - \frac{2^3}{3!}x^3 + \cdots.$$

Notice that all of these last three series converge for all $x \in (-\infty, \infty)$. (Why is that?) ∎

EXERCISES 7.7

WRITING EXERCISES

1. Describe how the Taylor polynomial with $n = 1$ compares to the linear approximation (see section 3.1). Give an analogous interpretation of the Taylor polynomial with $n = 2$. That is, how do various graphical properties (position, slope, concavity) of the Taylor polynomial compare with those of the function $f(x)$ at $x = c$?
2. Briefly discuss how a computer might use Taylor polynomials to compute $\sin(1.2)$. In particular, how would the computer know how many terms to compute? How would the number of terms necessary to compute $\sin(1.2)$ compare to the number needed to compute $\sin(100)$? Describe a trick that would make it much easier for the computer to compute $\sin(100)$. (Hint: The sine function is periodic.)
3. Taylor polynomials are built up from a knowledge of $f(c)$, $f'(c)$, $f''(c)$ and so on. Explain in graphical terms why information at one point (e.g., position, slope, concavity, etc.) can be used to construct the graph of the function on the entire interval of convergence.
4. If $f(c)$ is the position of an object at time $t = c$, then $f'(c)$ is the object's velocity and $f''(c)$ is the object's acceleration at time c. Explain in physical terms how knowledge of these values at one time (plus $f'''(c)$, etc.) can be used to predict the position of the object on the interval of convergence.
5. Our table of common Taylor series lists two different series for $\sin x$. Explain how the same function could have two different Taylor series representations. For a given problem (e.g., approximate $\sin 2$), explain how you would choose which Taylor series to use.
6. Explain why the Taylor series with center $c = 0$ of $f(x) = x^2 - 1$ is simply $x^2 - 1$.

In exercises 1–8, find the Maclaurin series (i.e., Taylor series about $c = 0$) and its interval of convergence.

1. $f(x) = \cos x$
2. $f(x) = \sin x$
3. $f(x) = \dfrac{3}{x-2}$
4. $f(x) = \cos 2x$
5. $f(x) = \ln(1 + x)$
6. $f(x) = e^{-x}$
7. $f(x) = 1/(1 + x)^2$
8. $f(x) = 1/(1 - x)$

In exercises 9–14, find the Taylor series about the indicated center and determine the interval of convergence.

9. $f(x) = e^{x-1}$, $c = 1$
10. $f(x) = \cos x$, $c = -\pi/2$
11. $f(x) = \ln x$, $c = e$
12. $f(x) = e^x$, $c = 2$
13. $f(x) = 1/x$, $c = 1$
14. $f(x) = 1/x$, $c = -1$

T In exercises 15–22, graph $f(x)$ and the Taylor polynomials for the indicated center c and degree n.

15. $f(x) = \cos x$, $c = 0$, $n = 5$; $n = 9$
16. $f(x) = \ln x$, $c = 1$, $n = 4$; $n = 8$
17. $f(x) = \sqrt{x}$, $c = 1$, $n = 3$; $n = 6$
18. $f(x) = \dfrac{1}{1+x}$, $c = 0$, $n = 4$; $n = 8$
19. $f(x) = e^x$, $c = 2$, $n = 3$; $n = 6$
20. $f(x) = \sin^{-1} x$, $c = 0$, $n = 4$; $n = 8$
21. $f(x) = \dfrac{1}{\sqrt{x}}$, $c = 4$, $n = 2$; $n = 4$
22. $f(x) = \sqrt{1 + x^2}$, $c = 0$, $n = 2$; $n = 4$

In exercises 23–26, prove that the Taylor series converges to $f(x)$ by showing that $R_n(x) \to 0$ as $n \to \infty$.

23. $\sin x = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$
24. $\cos x = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$
25. $\ln x = \displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k}$, $1 \le x \le 2$
26. $e^{-x} = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$

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T In exercises 27–32, (a) use a Taylor polynomial of degree 4 to approximate the given number, (b) estimate the error in the approximation, (c) estimate the number of terms needed in a Taylor polynomial to guarantee an accuracy of 10^{-10}.

27. $\ln(1.05)$ **28.** $\ln(0.9)$ **29.** $\sqrt{1.1}$

30. $\sqrt{1.2}$ **31.** $e^{0.1}$ **32.** $e^{-0.1}$

In exercises 33–38, use a Taylor series to verify the given formula.

33. $\sum_{k=0}^{\infty} \frac{2^k}{k!} = e^2$ **34.** $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1}$

35. $\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1}}{(2k+1)!} = 0$ **36.** $\sum_{k=0}^{\infty} \frac{(-1)^k (\pi/2)^{2k+1}}{(2k+1)!} = 1$

37. $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$ **38.** $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln 2$

In exercises 39–46, use a known Taylor series to find the Taylor series about $c = 0$ for the given function and find its radius of convergence.

39. $f(x) = e^{-2x}$ **40.** $f(x) = e^{3x}$

41. $f(x) = xe^{-x^2}$ **42.** $f(x) = \frac{e^x - 1}{x}$

43. $f(x) = \sin x^2$ **44.** $f(x) = x \sin 2x$

45. $f(x) = \cos 3x$ **46.** $f(x) = \cos x^3$

47. You may have wondered why it is necessary to show that $\lim_{n\to\infty} R_n(x) = 0$ to conclude that a Taylor series converges to $f(x)$. Consider $f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$. Show that $f'(0) = f''(0) = 0$. (Hint: Use the fact that $\lim_{n\to 0} \frac{e^{-1/n^2}}{n^n} = 0$ for any positive integer n.) It turns out that $f^{(n)}(0) = 0$ for all n. Thus, the Taylor series of $f(x)$ about $c = 0$ equals 0, a convergent series which does not converge to $f(x)$.

48. Find the Taylor series expansion of $f(x) = |x|$ with center $x = 1$. Argue that the radius of convergence is ∞. However, show that the Taylor series for $f(x)$ does not converge to $f(x)$ for all x.

49. The Environmental Protection Agency publishes an overall fuel economy rating R for each car. It combines the car's miles per gallon rating in the city (c) with the car's miles per gallon rating on the highway (h) using the formula $R = \frac{1}{0.55/c + 0.45/h}$. Treating h as a constant, find the first three terms in the Maclaurin series of $R(c)$. If a car's city rating improves, discuss the effect on its overall rating.

50. For the fuel rating equation of exercise 49, treat c as a constant and find the first three terms in the Maclaurin series of $R(h)$. If a car's highway rating improves, discuss the effect on its overall rating. Based on your results here and in exercise 49, which rating (c or h) does the EPA consider to be more important?

T 51. We have seen that $\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} + \cdots$. Determine how many terms are needed to approximate $\sin 1$ to within 10^{-5}. Show that $\sin 1 = \int_0^1 \cos x\, dx$. Determine how many terms are needed for Simpson's Rule to approximate this integral to within 10^{-5}. Compare the efficiency of using Maclaurin series and Simpson's Rule for this problem.

T 52. As in exercise 51, compare the efficiency of using Maclaurin series and Simpson's Rule in estimating e to within 10^{-5}.

53. Find the Maclaurin series of $f(x) = \sqrt{a^2 + x^2} - \sqrt{a^2 - x^2}$ for some nonzero constant a.

T 54. In many applications the **error function** $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$ is important. Compute and graph the fourth order Taylor polynomial for erf(x) about $c = 0$.

55. Suppose that a plane is at location $f(0) = 10$ miles with velocity $f'(0) = 10$ miles/min, acceleration $f''(0) = 2$ miles/min^2 and $f'''(0) = -1$ miles/min^3. Predict the location of the plane at time $t = 2$ min.

56. Suppose that an astronaut is at (0, 0) and the moon is represented by a circle of radius 1 centered at (10, 5). The astronaut's capsule follows a path $y = f(x)$ with current position $f(0) = 0$, slope $f'(0) = 1/5$, concavity $f''(0) = -1/10$, $f'''(0) = 1/25$, $f^{(4)}(0) = 1/25$ and $f^{(5)}(0) = -1/50$. Graph a Taylor polynomial approximation of $f(x)$. Based on your current information, do you advise the astronaut to change paths? How confident are you in the accuracy of your approximation?

57. Find the Taylor series for e^x about a general center c.

58. Find the Taylor series for $\sqrt{x}$ about a general center $c = a^2$.

Exercises 59–62 involve the binomial expansion.

59. Show that the Maclaurin series for $(1 + x)^r$ is $1 + \sum_{k=1}^{\infty} \frac{r(r-1)\cdots(r-k+1)}{k!} x^k$ for any constant r.

60. Simplify the series in exercise 59 for $r = 2$; $r = 3$; r is a positive integer.

61. Use the result of exercise 59 to write out the Maclaurin series for $f(x) = \sqrt{1+x}$.

62. Use the result of exercise 59 to write out the Maclaurin series for $f(x) = (1 + x)^{3/2}$.

63. Find the Maclaurin series of $f(x) = \cosh x$ and $f(x) = \sinh x$. Compare to the Maclaurin series of $\cos x$ and $\sin x$.

64. Use the Maclaurin series for $\tan x$ and the result of exercise 63 to conjecture the Maclaurin series for $\tanh x$.

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EXPLORATORY EXERCISES

1. Almost all of our series results apply to series of complex numbers. Defining $i = \sqrt{-1}$, show that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and so on. Replacing x with ix in the Maclaurin series for e^x, separate terms containing i from those that don't contain i (after the simplifications indicated above) and derive **Euler's formula:** $e^{ix} = \cos x + i \sin x$.

2. Using the technique of exercise 1, show that $\cos(ix) = \cosh x$ and $\sin(ix) = i \sinh x$. That is, the trig functions and their hyperbolic counterparts are closely related as functions of complex variables.

7.8 APPLICATIONS OF TAYLOR SERIES

In section 7.7, we developed the concept of a Taylor series expansion and gave many illustrations of how to compute Taylor series expansions. We also gave a few hints as to how these expansions might be used. In this section, we expand on our earlier presentation, by giving a few examples of how Taylor series are used in applications. You probably recognize that the work in the preceding section was challenging. The good news is that your hard work has significant payoffs, as we illustrate with the following problems. It's worth noting that many of the early practitioners of the calculus (including Newton and Leibniz) worked actively with series.

In this section, we will use series to approximate the values of transcendental functions, evaluate limits and integrals and define important new functions. As you continue your studies in mathematics and related fields, you are likely to see far more applications of Taylor series than we can include here.

You may have wondered how calculators and computers calculate values of transcendental functions, like $\sin(1.234567)$. We can now use Taylor series to do so, using only basic arithmetic operations.

EXAMPLE 8.1 Using Taylor Polynomials to Approximate a Sine Value

Use a Taylor series to compute $\sin(1.234567)$ accurate to within 10^{-11}.

Solution It's not hard to find the Taylor series expansion for $f(x) = \sin x$ about $x = 0$. (We left this as an exercise in section 7.7.) We have

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots,$$

where the interval of convergence is $(-\infty, \infty)$. Notice that if we take $x = 1.234567$, the series representation of $\sin 1.234567$ is

$$\sin 1.234567 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (1.234567)^{2k+1},$$

which is an alternating series. We can use a partial sum of this series to approximate the desired value, but just how accurate will a given partial sum be? Recall that for alternating series, the error in a partial sum is bounded by the absolute value of the first neglected term. (Note that you could also use the remainder term from Taylor's Theorem to bound the error.) To ensure that the error is less than 10^{-11}, we must find an integer k such that

$\dfrac{1.234567^{2k+1}}{(2k+1)!} < 10^{-11}$. By trial and error, we find that

$$\frac{1.234567^{17}}{17!} \approx 1.010836 \times 10^{-13} < 10^{-11},$$

so that $k = 8$ will do. Observe that this says that the first neglected term corresponds to $k = 8$ and so, we compute the partial sum

$$\begin{aligned}\sin 1.234567 &\approx \sum_{k=0}^{7} \frac{(-1)^k}{(2k+1)!}(1.234567)^{2k+1} \\ &= 1.234567 - \frac{1.234567^3}{3!} + \frac{1.234567^5}{5!} - \frac{1.234567^7}{7!} + \cdots - \frac{1.234567^{15}}{15!} \\ &\approx 0.94400543137.\end{aligned}$$

Check your calculator or computer to verify that this matches your calculator's estimate. ■

If you look carefully at example 8.1, you might discover that we were a bit hasty. Certainly, we answered the question and produced an approximation with the desired accuracy, but was this the easiest way in which to do this? The answer is no, as we simply grabbed the most handy Taylor series expansion of $f(x) = \sin x$. You should try to resist the impulse to automatically use the Taylor series expansion about $x = 0$ (i.e., the Maclaurin series), rather than making a more efficient choice. We illustrate this in example 8.2.

EXAMPLE 8.2 Choosing a More Appropriate Taylor Series Expansion

Repeat example 8.1, but this time, make a more appropriate choice of the Taylor series.

Solution Recall from our discussion in section 7.7 that Taylor series converge much faster close to the point about which you expand, than they do far away. So, if we need to compute sin 1.234567, is there a handy Taylor series expansion of $f(x) = \sin x$ about some point closer to $x = 1.234567$? Keeping in mind that we only know the value of $\sin x$ exactly at a few points, you should quickly recognize that a series expanded about $x = \frac{\pi}{2} \approx 1.57$ is a better choice than one expanded about $x = 0$. (Another reasonable choice is the Taylor series expansion about $x = \frac{\pi}{3}$.) In example 7.5, recall that we had found that

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(x - \frac{\pi}{2}\right)^{2k} = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \cdots,$$

where the interval of convergence is $(-\infty, \infty)$. Taking $x = 1.234567$, gives us

$$\begin{aligned}\sin 1.234567 &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(1.234567 - \frac{\pi}{2}\right)^{2k} \\ &= 1 - \frac{1}{2}\left(1.234567 - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(1.234567 - \frac{\pi}{2}\right)^4 - \cdots,\end{aligned}$$

which is again an alternating series. Using the remainder term from Taylor's Theorem to bound the error, we have that

$$\begin{aligned}|R_n(1.234567)| &= \left|\frac{f^{(2n+2)}(z)}{(2n+2)!}\right|\left|1.234567 - \frac{\pi}{2}\right|^{2n+2} \\ &\le \frac{|1.234567 - \frac{\pi}{2}|^{2n+2}}{(2n+2)!}.\end{aligned}$$

(Note that we might also have chosen to use Theorem 4.2.) By trial and error, you can find that

$$\frac{|1.234567 - \frac{\pi}{2}|^{2n+2}}{(2n+2)!} < 10^{-11}$$

for $n = 4$, so that an approximation with the required degree of accuracy is

$$\begin{aligned}\sin 1.234567 &\approx \sum_{k=0}^{4} \frac{(-1)^k}{(2k)!}\left(1.234567 - \frac{\pi}{2}\right)^{2k} \\ &= 1 - \frac{1}{2}\left(1.234567 - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(1.234567 - \frac{\pi}{2}\right)^4 \\ &\quad - \frac{1}{6!}\left(1.234567 - \frac{\pi}{2}\right)^6 + \frac{1}{8!}\left(1.234567 - \frac{\pi}{2}\right)^8 \\ &\approx 0.94400543137.\end{aligned}$$

Compare this result to example 8.1, where we needed to compute many more terms of the Taylor series to obtain the same degree of accuracy. ■

We can also use Taylor series to quickly conjecture the value of a difficult limit. Be careful, though: the theory of when these conjectures are guaranteed to be correct is beyond the level of this text. However, we can certainly obtain helpful hints about certain limits.

EXAMPLE 8.3 Using Taylor Polynomials to Conjecture the Value of a Limit

Use Taylor series to conjecture $\lim_{x \to 0} \frac{\sin(x^3) - x^3}{x^9}$.

Solution Again recall that the Maclaurin series for $\sin x$ is

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots,$$

where the interval of convergence is $(-\infty, \infty)$. Substituting x^3 for x gives us

$$\sin\left(x^3\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (x^3)^{2k+1} = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots.$$

This gives us

$$\frac{\sin\left(x^3\right) - x^3}{x^9} = \frac{\left(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots\right) - x^3}{x^9} = -\frac{1}{3!} + \frac{x^6}{5!} + \cdots$$

and so, we conjecture that

$$\lim_{x \to 0} \frac{\sin\left(x^3\right) - x^3}{x^9} = -\frac{1}{3!} = -\frac{1}{6}.$$

You can verify that this limit is correct using l'Hôpital's Rule (three times, simplifying each time). ■

So, for what else can Taylor series be used? There are many answers to this question, but this next one is quite useful. Since Taylor polynomials are used to approximate functions on

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a given interval and since there's nothing easier to integrate than a polynomial, we consider using a Taylor polynomial approximation to produce an approximation of a definite integral. It turns out that such an approximation is often better than that obtained from the numerical methods developed in section 4.9. We illustrate this in example 8.4.

EXAMPLE 8.4 Using Taylor Series to Approximate a Definite Integral

Use a Taylor polynomial with $n = 8$ to approximate $\int_{-1}^{1} \cos(x^2)\,dx$.

Solution Note that you do not know an antiderivative of $\cos(x^2)$ and so, have no choice but to rely on a numerical approximation of the value of the integral. Since you are integrating on the interval $(-1, 1)$, a Maclaurin series expansion (i.e., a Taylor series expansion about $x = 0$) is a good choice. It's a simple matter to show that

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots,$$

which converges on all of $(-\infty, \infty)$. Replacing x by x^2 gives us the Taylor series expansion

$$\cos(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{4k} = 1 - \frac{1}{2}x^4 + \frac{1}{4!}x^8 - \frac{1}{6!}x^{12} + \cdots,$$

so that

$$\cos(x^2) \approx 1 - \frac{1}{2}x^4 + \frac{1}{4!}x^8.$$

This leads us to the approximation

$$\int_{-1}^{1} \cos(x^2)\,dx \approx \int_{-1}^{1} \left(1 - \frac{1}{2}x^4 + \frac{1}{4!}x^8\right) dx$$

$$= \left(x - \frac{x^5}{10} + \frac{x^9}{216}\right)\Big|_{x=-1}^{x=1}$$

$$= \frac{977}{540} \approx 1.809259.$$

Our CAS gives us $\int_{-1}^{1} \cos(x^2)\,dx \approx 1.809048$, so our approximation appears to be very accurate. ■

You might reasonably argue that we don't need Taylor series to obtain approximations like those in example 8.4, as you could always use other, simpler numerical methods like Simpson's Rule to do the job. That's often true, but just try to use Simpson's Rule on the integral in example 8.5.

EXAMPLE 8.5 Using Taylor Series to Approximate the Value of an Integral

Use a Taylor polynomial with $n = 5$ to approximate $\displaystyle\int_{-1}^{1} \frac{\sin x}{x}\,dx$.

Solution Note that you do not know an antiderivative of $\dfrac{\sin x}{x}$. Further, observe that the integrand is discontinuous at $x = 0$. However, this does *not* need to be treated as an improper integral, since $\lim_{x\to 0}\frac{\sin x}{x} = 1$. (This says that the integrand has a removable discontinuity at $x = 0$.) From the first few terms of the Maclaurin series for $f(x) = \sin x$, we have the Taylor polynomial approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!},$$

so that

$$\frac{\sin x}{x} \approx 1 - \frac{x^2}{3!} + \frac{x^4}{5!}.$$

Notice that since this is a polynomial, it is simple to integrate. Consequently,

$$\begin{aligned}\int_{-1}^{1}\frac{\sin x}{x}\,dx &\approx \int_{-1}^{1}\left(1 - \frac{x^2}{6} + \frac{x^4}{120}\right)dx \\ &= \left(x - \frac{x^3}{18} + \frac{x^5}{600}\right)\Bigg|_{x=-1}^{x=1} \\ &= \left(1 - \frac{1}{18} + \frac{1}{600}\right) - \left(-1 + \frac{1}{18} - \frac{1}{600}\right) \\ &= \frac{1703}{900} \approx 1.8922.\end{aligned}$$

Our CAS gives us $\int_{-1}^{1}\frac{\sin x}{x}\,dx \approx 1.89216$, so our approximation is quite good. On the other hand, if you try to apply Simpson's Rule or Trapezoidal Rule, the algorithm will not work, as they will attempt to evaluate $\frac{\sin x}{x}$ at $x = 0$. (Most graphing calculators and some computer algebra systems also fail to give an answer here, due to the division by zero at $x = 0$.) ■

While you have now calculated Taylor series expansions of many familiar functions, many other functions are actually *defined* by a power series. These include many functions in the very important class of **special functions** that frequently arise in physics and engineering applications. These functions cannot be written in terms of elementary functions (the algebraic, trigonometric, exponential and logarithmic functions with which you are familiar) and are only known from their series definitions. Among the more important special functions are the Bessel functions, which are used in the study of fluid motion, acoustics, wave propagation and other areas of applied mathematics. The **Bessel function of order *p*** is defined by the power series

$$J_p(x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+p}}{2^{2k+p}k!(k+p)!}, \tag{8.1}$$

for nonnegative integers p. You might find it surprising that we *define* a function by a power series expansion, but in fact, this is very common. In particular, in the process of solving differential equations, we often derive the solution as a series. As it turns out, most of these series solutions are not elementary functions. Specifically, Bessel functions arise in the solution of the differential equation $x^2y'' + xy' + (x^2 - p^2)y = 0$. In examples 8.6 and 8.7, we explore several interesting properties of Bessel functions.

EXAMPLE 8.6 The Radius of Convergence of a Bessel Function

Find the radius of convergence for the series defining the Bessel function $J_0(x)$.

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Solution From equation (8.1) with $p = 0$, we have $J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}$. Observe that the Ratio Test gives us

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{2k+2}}{2^{2k+2}[(k+1)!]^2}\frac{2^{2k}(k!)^2}{x^{2k}}\right| = \lim_{k\to\infty}\left|\frac{x^2}{4(k+1)^2}\right| = 0 < 1,$$

for all x. The series then converges absolutely for all x and so, the radius of convergence is ∞. ■

In example 8.7, we explore an interesting relationship between the zeros of two Bessel functions.

EXAMPLE 8.7 The Zeros of Bessel Functions

Verify graphically that on the interval [0, 10], the zeros of $J_0(x)$ and $J_1(x)$ alternate.

Solution Unless you have a CAS with these Bessel functions available as built-in functions, you will need to graph partial sums of the defining series:

$$J_0(x) \approx \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2} \quad \text{and} \quad J_1(x) \approx \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}.$$

Before graphing these, you must first determine how large n should be in order to produce a reasonable graph. Notice that for each fixed $x > 0$, both of the defining series are alternating series. Consequently, the error in using a partial sum to approximate the function is bounded by the first neglected term. That is,

$$\left|J_0(x) - \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}\right| \le \frac{x^{2n+2}}{2^{2n+2}[(n+1)!]^2}$$

and

$$\left|J_1(x) - \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}\right| \le \frac{x^{2n+3}}{2^{2n+3}(n+1)!(n+2)!},$$

with the largest error in each occurring at $x = 10$. Notice that for $n = 12$, we have that

$$\left|J_0(x) - \sum_{k=0}^{12} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}\right| \le \frac{x^{2(12)+2}}{2^{2(12)+2}[(12+1)!]^2} \le \frac{10^{26}}{2^{26}(13!)^2} < 0.04$$

and

$$\left|J_1(x) - \sum_{k=0}^{12} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}\right| \le \frac{x^{2(12)+3}}{2^{2(12)+3}(12+1)!(12+2)!} \le \frac{10^{27}}{2^{27}(13!)(14!)} < 0.04.$$

Consequently, using a partial sum with $n = 12$ will result in approximations that are within 0.04 of the correct value for each x in the interval [0, 10]. This is plenty of accuracy for our present purposes. Figure 7.43 shows graphs of partial sums with $n = 12$ for $J_0(x)$ and $J_1(x)$.

Notice that $J_1(0) = 0$ and in the figure, you can clearly see that $J_0(x) = 0$ at about $x = 2.4$, $J_1(x) = 0$ at about $x = 3.9$, $J_0(x) = 0$ at about $x = 5.6$, $J_1(x) = 0$ at about $x = 7.0$ and $J_0(x) = 0$ at about $x = 8.8$. From this, it is now apparent that the zeros of $J_0(x)$ and $J_1(x)$ do indeed alternate on the interval [0, 10]. ■

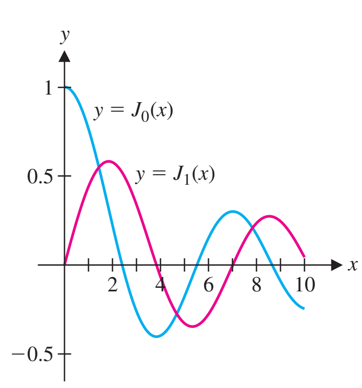

FIGURE 7.43
$y = J_0(x)$ and $y = J_1(x)$.

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It turns out that the result of example 8.7 generalizes to any interval of positive numbers and any two Bessel functions of consecutive order. That is, between consecutive zeros of $J_p(x)$ is a zero of $J_{p+1}(x)$ and between consecutive zeros of $J_{p+1}(x)$ is a zero of $J_p(x)$. We explore this further in the exercises.

EXERCISES 7.8

WRITING EXERCISES

1. In example 8.2, we showed that an expansion about $x = \frac{\pi}{2}$ is more accurate for approximating $\sin(1.234567)$ than an expansion about $x = 0$ with the same number of terms. Explain why an expansion about $x = 1.2$ would be even more efficient, but is not practical.
2. Assuming that you don't need to rederive the Maclaurin series of $\cos x$, compare the amount of work done in example 8.4 to the work needed to compute a Simpson's Rule approximation with $n = 16$.
3. In equation (8.1), we defined the Bessel functions as series. This may seem like a convoluted way of defining a function, but compare the levels of difficulty doing the following with a Bessel function versus $\sin x$: computing $f(0)$, computing $f(1.2)$, evaluating $f(2x)$, computing $f'(x)$, computing $\int f(x)\,dx$ and computing $\int_0^1 f(x)\,dx$.
4. Discuss how you might estimate the error in the approximation of example 8.4.

In exercises 1–6, use an appropriate Taylor series to approximate the given value, accurate to within 10^{-11}.

1. $\sin 1.61$
2. $\sin 6.32$
3. $\cos 0.34$
4. $\cos 3.04$
5. $e^{-0.2}$
6. $e^{0.4}$

In exercises 7–12, use a known Taylor series to conjecture the value of the limit.

7. $\lim\limits_{x\to 0} \dfrac{\cos x^2 - 1}{x^4}$
8. $\lim\limits_{x\to 0} \dfrac{\sin x^2 - x^2}{x^6}$
9. $\lim\limits_{x\to 1} \dfrac{\ln x - (x-1)}{(x-1)^2}$
10. $\lim\limits_{x\to 0} \dfrac{\tan^{-1} x - x}{x^3}$
11. $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x}$
12. $\lim\limits_{x\to 0} \dfrac{e^{-2x} - 1}{x}$

In exercises 13–18, use a known Taylor polynomial with n nonzero terms to estimate the value of the integral.

13. $\displaystyle\int_{-1}^{1} \frac{\sin x}{x}\,dx,\ n = 3$
14. $\displaystyle\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \cos x^2\,dx,\ n = 4$
15. $\displaystyle\int_{-1}^{1} e^{-x^2}\,dx,\ n = 5$
16. $\displaystyle\int_{0}^{1} \tan^{-1} x\,dx,\ n = 5$
17. $\displaystyle\int_{1}^{2} \ln x\,dx,\ n = 5$
18. $\displaystyle\int_{0}^{1} e^{\sqrt{x}}\,dx,\ n = 4$
19. Find the radius of convergence of $J_1(x)$.
20. Find the radius of convergence of $J_2(x)$.
21. Find the number of terms needed to approximate $J_2(x)$ within 0.04 for x in the interval [0, 10].
22. Show graphically that the zeros of $J_1(x)$ and $J_2(x)$ alternate on the interval (0, 10).
23. Einstein's theory of relativity states that the mass of an object traveling at velocity v is $m(v) = m_0/\sqrt{1 - v^2/c^2}$, where m_0 is the rest mass of the object and c is the speed of light. Show that $m \approx m_0 + \left(\dfrac{m_0}{2c^2}\right) v^2$. Use this approximation to estimate how large v would need to be to increase the mass by 10%.
24. Find the fourth-degree Taylor polynomial for $m(v)$ in exercise 23.
25. The weight (force due to gravity) of an object of mass m and altitude x miles above the surface of the earth is $w(x) = \dfrac{mgR^2}{(R+x)^2}$, where R is the radius of the earth and g is the acceleration due to gravity. Show that $w(x) \approx mg(1 - 2x/R)$. Estimate how large x would need to be to reduce the weight by 10%.
26. Find the second-degree Taylor polynomial for $w(x)$ in exercise 25. Use it to estimate how large x needs to be to reduce the weight by 10%.
27. Based on your answers to exercises 25 and 26, is weight significantly different at a high-altitude location (e.g., 7500 ft) compared to sea level?
28. The radius of the earth is up to 300 miles larger at the equator than it is at the poles. Which would have a larger effect on weight, altitude or latitude?

In exercises 29–32, use the Maclaurin series expansion $\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^{15} - \cdots$.

29. The tangential component of the space shuttle's velocity during re-entry is approximately $v(t) = v_c \tanh\left(\dfrac{g}{v_c}t + \tanh^{-1}\dfrac{v_0}{v_c}\right)$,

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where v_0 is the velocity at time 0 and v_c is the terminal velocity (see Long and Weiss, *The American Mathematical Monthly,* February, 1999). If $\tanh^{-1}\frac{v_0}{v_c} = \frac{1}{2}$, show that $v(t) \approx gt + \frac{1}{2}v_c$. Is this estimate of $v(t)$ too large or too small?

30. Show that in exercise 29, $v(t) \to v_c$ as $t \to \infty$. Use the approximation in exercise 29 to estimate the time needed to reach 90% of the terminal velocity.

31. The downward velocity of a skydiver of mass m is $v(t) = \sqrt{40mg}\tanh\left(\sqrt{\frac{g}{40m}}t\right)$. Show that $v(t) \approx gt - \frac{g^2}{120m}t^3$.

32. The velocity of a water wave of length L in water of depth h satisfies the equation $v^2 = \frac{gL}{2\pi}\tanh\frac{2\pi h}{L}$. Show that $v \approx \sqrt{gh}$.

33. The power of a reflecting telescope is proportional to the surface area S of the parabolic reflector, where $S = \frac{8\pi}{3}c^2\left[\left(\frac{d^2}{16c^2}+1\right)^{3/2} - 1\right]$. Here, d is the diameter of the parabolic reflector, which has depth k with $c = \frac{d^2}{4k}$. Expand the term $\left(\frac{d^2}{16c^2}+1\right)^{3/2}$ and show that if $\frac{d^2}{16c^2}$ is small, then $S \approx \frac{\pi d^2}{4}$.

T 34. The energy density of electromagnetic radiation at wavelength λ from a blackbody at temperature T degrees (Kelvin) is given by **Planck's law** of blackbody radiation: $f(\lambda) = \frac{8\pi hc}{\lambda^5(e^{hc/\lambda kT} - 1)}$, where h is Planck's constant, c is the speed of light and k is Boltzmann's constant. To find the wavelength of peak emission, maximize $f(\lambda)$ by minimizing $g(\lambda) = \lambda^5(e^{hc/\lambda kT} - 1)$. Use the Maclaurin series for e^x to expand the expression in parentheses and find λ to minimize the resulting function. (Hint: Use $\frac{hc}{k} \approx 0.014$.) Compare this to **Wien's law:** $\lambda_{\max} = \frac{0.002898}{T}$. Wien's law is accurate for small λ. Discuss the flaw in our use of Maclaurin series.

35. Use the Maclaurin series for e^x to expand the denominator in Planck's law of exercise 34 and show that $f(\lambda) \approx \frac{8\pi kt}{\lambda^4}$. State whether this approximation is better for small or large wavelengths λ. This is known in physics as the **Rayleigh-Jeans law.**

36. A disk of radius a has a charge of constant density σ. Point P lies at a distance r directly above the disk. The **electrical potential** at point P is given by $V = 2\pi\sigma(\sqrt{r^2+a^2} - r)$. Show that for large r, $V \approx \frac{\pi a^2 \sigma}{r}$.

EXPLORATORY EXERCISES

1. The Bessel functions and **Legendre polynomials** are examples of the so-called special functions. For nonnegative integers n, the Legendre polynomials are defined by

$$P_n(x) = 2^{-n}\sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!}{(n-k)!k!(n-2k)!}x^{n-2k}.$$

Here, $[n/2]$ is the greatest integer less than or equal to $n/2$ (for example, $[1/2] = 0$ and $[2/2] = 1$). Show that $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$. Show that for these three functions,

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0, \text{ for } m \neq n.$$

This fact, which is true for all Legendre polynomials, is called the **orthogonality condition.** Orthogonal functions are commonly used to provide simple representations of complicated functions.

7.9 FOURIER SERIES

Over the last several sections, we have seen how we can represent a function by a power series (i.e., a Taylor series). This is an extraordinary development, in particular because we can use the partial sums of such a series expansion (i.e., the Taylor polynomials) to compute approximate values of the function for values of x close to the point c about which you expanded. For the first time, this gave us the means of calculating approximate values of transcendental functions like e^x, $\ln x$ and $\sin x$, which we otherwise could not compute. Although this is a good reason for studying series expansions of functions, it is not the only reason we need them.

Observe that many phenomena we encounter in the world around us are periodic in nature. That is, they repeat themselves over and over again. For instance, light, sound,

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radio waves and x-rays (to mention only a few) are all periodic. For such phenomena, Taylor polynomial approximations have some obvious shortcomings. Look back at any of the graphs you (or we) constructed of the Taylor polynomials of a periodic function and you'll notice that as x gets farther away from c (the point about which you expanded), the difference between the function and a given Taylor polynomial grows. Such behavior, as we illustrate in Figure 7.44a for the case of $f(x) = \sin x$ expanded about $x = \frac{\pi}{2}$, is typical of the convergence of Taylor series.

Because the Taylor polynomials provide an accurate approximation only in the vicinity of c, we say that they are accurate *locally*. In general, no matter how large you make n, the approximation is still only valid locally. In many situations, notably in communications, we need to find an approximation to a given periodic function that is valid *globally* (i.e., for all x). Consider the Taylor polynomial graphed in Figure 7.44b and convince yourself that Taylor polynomials will not satisfy this need. For this reason, we construct a different type of series expansion for periodic functions, one where each of the terms in the expansion is periodic.

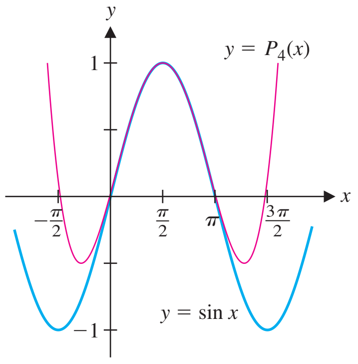

FIGURE 7.44a
$y = \sin x$ and $y = P_4(x)$.

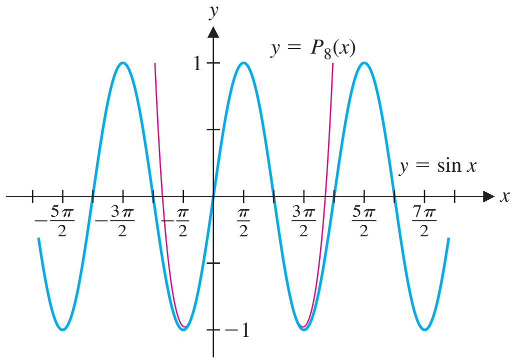

FIGURE 7.44b
$y = \sin x$ and $y = P_8(x)$.

Recall that we say that a function f is **periodic** of **period** $T > 0$ if $f(x + T) = f(x)$, for all x in the domain of f. Can you think of any periodic functions? Surely, $\sin x$ and $\cos x$ come to mind. These are both periodic of period 2π. Further, $\sin(2x)$, $\cos(2x)$, $\sin(3x)$, $\cos(3x)$ and so on are all periodic of period 2π. In fact, the simplest periodic functions you can think of (aside from constant functions) are the functions

$$\sin(kx) \text{ and } \cos(kx), \text{ for } k = 1, 2, 3, \ldots.$$

Note that each of these is periodic of period 2π, as follows. For any integer k, let $f(x) = \sin(kx)$. We then have

$$f(x + 2\pi) = \sin[k(x + 2\pi)] = \sin(kx + 2k\pi) = \sin(kx) = f(x).$$

Likewise, you can show that $\cos(kx)$ has period 2π.

So, if you wanted to expand a periodic function in a series, the simplest periodic functions to use in the terms of the series are just these functions. Consequently, we consider a series of the form

Fourier series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k \cos(kx) + b_k \sin(kx)],$$

called a **Fourier series.** Notice that if the series converges, it will converge to a periodic function whose period is 2π, since every term in the series has period 2π. The coefficients of the series, $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are called the **Fourier coefficients.** You may have noticed the unusual way in which we wrote the leading term of the series $\left(\frac{a_0}{2}\right)$. We did this in order to simplify the formulas for computing these coefficients, as we'll see later.

There are a number of important questions we must address.

- What functions can be expanded in a Fourier series?
- How do we compute the Fourier coefficients?
- Does the Fourier series converge? If so, to what function does the series converge?

We begin our investigation much as we did with power series. Suppose that a given Fourier series converges on the interval $[-\pi, \pi]$. It then represents a function f on that interval,

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k \cos(kx) + b_k \sin(kx)], \tag{9.1}$$

where f must be periodic outside of $[-\pi, \pi]$. Although some of the details of the proof are beyond the level of this course, we want to give you some idea of how the Fourier coefficients are computed. If we integrate both sides of equation (9.1) with respect to x on the interval $[-\pi, \pi]$, we get

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\,dx &= \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx + \int_{-\pi}^{\pi} \sum_{k=1}^{\infty}[a_k \cos(kx) + b_k \sin(kx)]\,dx \\ &= \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx + \sum_{k=1}^{\infty}\left[a_k \int_{-\pi}^{\pi} \cos(kx)\,dx + b_k \int_{-\pi}^{\pi} \sin(kx)\,dx\right],\end{aligned} \tag{9.2}$$

HISTORICAL NOTES

Jean Baptiste Joseph Fourier (1768–1830) French mathematician who invented Fourier series. Fourier was heavily involved in French politics, becoming a member of the Revolutionary Committee, serving as scientific advisor to Napoleon and establishing educational facilities in Egypt. Fourier held numerous offices, including secretary of the Cairo Institute and Prefect of Grenoble. Fourier introduced his trigonometric series as an essential technique for developing his highly original and revolutionary theory of heat.

assuming we can interchange the order of integration and summation. In general, the order may *not* be interchanged (this is beyond the level of this course), but for many Fourier series, doing so is permissible. Observe that for every $k = 1, 2, 3, \ldots$, we have

$$\int_{-\pi}^{\pi} \cos(kx)\,dx = \frac{1}{k}\sin(kx)\Big|_{-\pi}^{\pi} = \frac{1}{k}[\sin(k\pi) - \sin(-k\pi)] = 0$$

and

$$\int_{-\pi}^{\pi} \sin(kx)\,dx = -\frac{1}{k}\cos(kx)\Big|_{-\pi}^{\pi} = -\frac{1}{k}[\cos(k\pi) - \cos(-k\pi)] = 0.$$

This reduces equation (9.2) to simply

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx = a_0\pi.$$

Solving this for a_0, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx. \tag{9.3}$$

If we multiply both sides of equation (9.1) by $\cos(nx)$ (where n is an integer, $n \geq 1$), and then integrate with respect to x on the interval $[-\pi, \pi]$, observe that we get

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx &= \int_{-\pi}^{\pi} \frac{a_0}{2}\cos(nx)\,dx \\ &\quad + \int_{-\pi}^{\pi}\sum_{k=1}^{\infty}[a_k\cos(kx)\cos(nx) + b_k\sin(kx)\cos(nx)]\,dx \\ &= \frac{a_0}{2}\int_{-\pi}^{\pi}\cos(nx)\,dx \\ &\quad + \sum_{k=1}^{\infty}\left[a_k\int_{-\pi}^{\pi}\cos(kx)\cos(nx)\,dx + b_k\int_{-\pi}^{\pi}\sin(kx)\cos(nx)\,dx\right], \end{aligned} \tag{9.4}$$

again assuming we can interchange the order of integration and summation. Next, recall that

$$\int_{-\pi}^{\pi}\cos(nx)\,dx = 0, \text{ for all } n = 1, 2, \ldots.$$

It's an easy exercise to show that

$$\int_{-\pi}^{\pi}\sin(kx)\cos(nx)\,dx = 0, \text{ for all } n = 1, 2, \ldots \text{ and for all } k = 1, 2, \ldots$$

and that

$$\int_{-\pi}^{\pi}\cos(kx)\cos(nx)\,dx = \begin{cases} 0, & \text{if } n \neq k \\ \pi, & \text{if } n = k \end{cases}.$$

Notice that this says that every term in the series in equation (9.4) except one (the term corresponding to $k = n$) is zero and equation (9.4) reduces to simply

$$\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = a_n\pi.$$

This gives us (after substituting k for n)

Fourier coefficients

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx, \text{ for } k = 1, 2, 3, \ldots. \tag{9.5}$$

Likewise, multiplying both sides of equation (9.1) by $\sin(nx)$ and integrating from $-\pi$ to π gives us

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx, \text{ for } k = 1, 2, 3, \ldots. \tag{9.6}$$

Equations (9.3), (9.5) and (9.6) are called the **Euler-Fourier formulas.** Notice that equation (9.3) is the same as (9.5) with $k = 0$. (This was the reason we chose the leading term of the series to be $\frac{a_0}{2}$, instead of simply a_0.)

Let's summarize what we've done so far. We observed that if a Fourier series converges on some interval, then it converges to a function f where the Fourier coefficients satisfy the Euler-Fourier formulas (9.3), (9.5) and (9.6).

Just as we did with power series, given any integrable function f, we can compute the coefficients in (9.3), (9.5) and (9.6) and write down a Fourier series. But, will the series converge and if it does, to what function will it converge? We'll answer these questions shortly. For the moment, let's try to compute the terms of a Fourier series and see what we can observe.

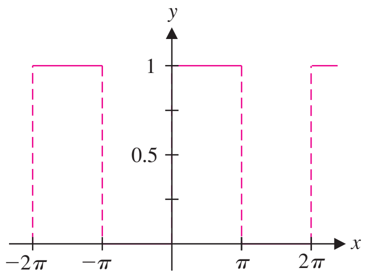

FIGURE 7.45
Square-wave function.

EXAMPLE 9.1 Finding a Fourier Series Expansion

Find the Fourier series corresponding to the **square-wave** function

$$f(x) = \begin{cases} 0, & \text{if } -\pi < x \le 0 \\ 1, & \text{if } 0 < x \le \pi \end{cases},$$

where f is assumed to be periodic outside of the interval $[-\pi, \pi]$ (see the graph in Figure 7.45).

Solution Notice that even though a_0 satisfies the same formula as a_k, for $k \ge 1$, we must always compute a_0 separately from the rest of the a_k's. From equation (9.3), we get

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\,dx = 0 + \frac{\pi}{\pi} = 1.$$

From (9.5), we also have that for $k \ge 1$,

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (0)\cos(kx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\cos(kx)\,dx \\ &= \frac{1}{\pi k}\sin(kx)\bigg|_0^{\pi} = \frac{1}{\pi k}[\sin(k\pi) - \sin(0)] = 0. \end{aligned}$$

Finally, from (9.6), we have

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (0)\sin(kx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} (1)\sin(kx)\,dx \\ &= -\frac{1}{\pi k}\cos(kx)\bigg|_0^{\pi} = -\frac{1}{\pi k}[\cos(k\pi) - \cos(0)] = -\frac{1}{\pi k}[(-1)^k - 1] \\ &= \begin{cases} 0, & \text{if } k \text{ is even} \\ \dfrac{2}{\pi k}, & \text{if } k \text{ is odd} \end{cases}. \end{aligned}$$

Notice that we can write the even- and odd-indexed coefficients separately as $b_{2k} = 0$, for $k = 1, 2, \ldots$ and $b_{2k-1} = \dfrac{2}{(2k-1)\pi}$, for $k = 1, 2, \ldots$. We then have the Fourier series

$$\begin{aligned} \frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k\cos(kx) + b_k\sin(kx)] &= \frac{1}{2} + \sum_{k=1}^{\infty} b_k\sin(kx) = \frac{1}{2} + \sum_{k=1}^{\infty} b_{2k-1}\sin(2k-1)x \\ &= \frac{1}{2} + \sum_{k=1}^{\infty}\frac{2}{(2k-1)\pi}\sin[(2k-1)x] \\ &= \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin(3x) + \frac{2}{5\pi}\sin(5x) + \cdots. \end{aligned}$$

Notice that none of our existing convergence tests are appropriate for Fourier series. Since we can't test this, we consider the graphs of the first few partial sums of the series defined by

$$F_n(x) = \frac{1}{2} + \sum_{k=1}^{n}\frac{2}{(2k-1)\pi}\sin[(2k-1)x].$$

In Figures 7.46a–d, we graph a number of these partial sums.

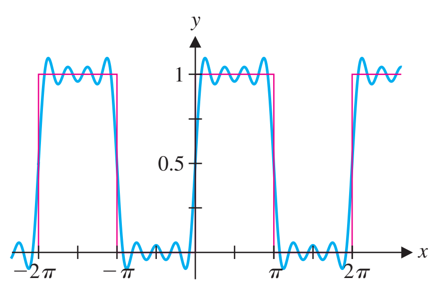

FIGURE 7.46a
$y = F_4(x)$ and $y = f(x)$.

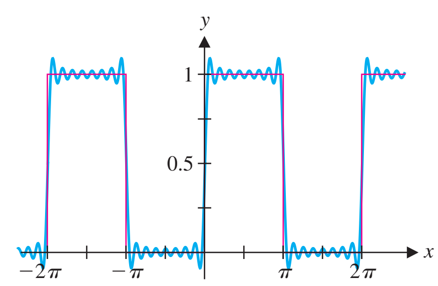

FIGURE 7.46b
$y = F_8(x)$ and $y = f(x)$.

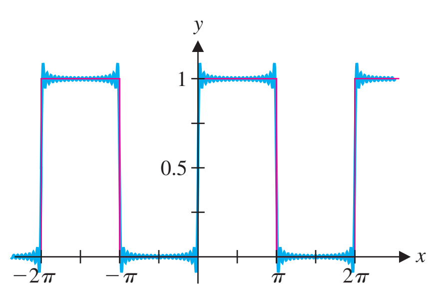

FIGURE 7.46c
$y = F_{20}(x)$ and $y = f(x)$.

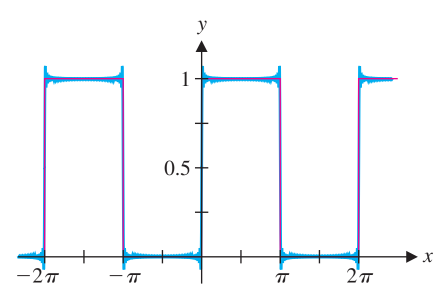

FIGURE 7.46d
$y = F_{50}(x)$ and $y = f(x)$.

Notice that as n gets larger and larger, the graph of $F_n(x)$ appears to be approaching the graph of the square-wave function $f(x)$ shown in red and seen in Figure 7.45. Based on this, we might conjecture that the Fourier series converges to the function $f(x)$. As it turns out, this is not quite correct. We'll soon see that the series converges to $f(x)$, everywhere, *except* at points of discontinuity. ■

Next, we give an example of constructing a Fourier series for another common waveform.

EXAMPLE 9.2 A Fourier Series Expansion for the Triangular Wave Function

Find the Fourier series expansion of $f(x) = |x|$, for $-\pi \le x \le \pi$, where f is assumed to be periodic outside of the interval $[-\pi, \pi]$, of period 2π.

Solution In this case, f is the **triangular wave** function graphed in Figure 7.47. From the Euler-Fourier formulas, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -x\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx$$

$$= -\frac{1}{\pi}\frac{x^2}{2}\bigg|_{-\pi}^{0} + \frac{1}{\pi}\frac{x^2}{2}\bigg|_{0}^{\pi} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Similarly, for each $k \ge 1$, we get

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(kx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (-x)\cos(kx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(kx)\,dx.$$

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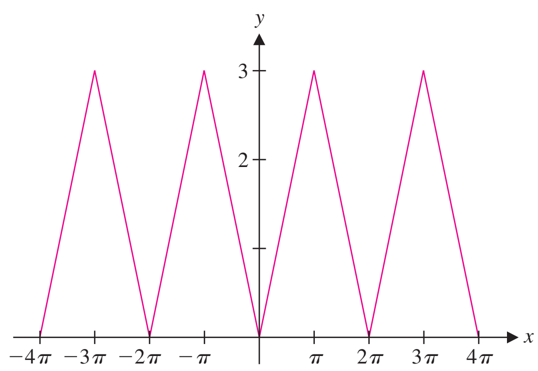

FIGURE 7.47
Triangular wave.

Both integrals require the same integration by parts. We let

$$u = x \qquad dv = \cos(kx)\,dx$$
$$du = dx \qquad v = \frac{1}{k}\sin(kx)$$

so that

$$\begin{aligned}
a_k &= -\frac{1}{\pi}\int_{-\pi}^{0} x\cos(kx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(kx)\,dx \\
&= -\frac{1}{\pi}\left[\frac{x}{k}\sin(kx)\right]_{-\pi}^{0} + \frac{1}{\pi k}\int_{-\pi}^{0}\sin(kx)\,dx + \frac{1}{\pi}\left[\frac{x}{k}\sin(kx)\right]_{0}^{\pi} - \frac{1}{\pi k}\int_{0}^{\pi}\sin(kx)\,dx \\
&= -\frac{1}{\pi}\left[0 + \frac{\pi}{k}\sin(-\pi k)\right] - \frac{1}{\pi k^2}\cos(kx)\Big|_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{\pi}{k}\sin(\pi k) - 0\right] + \frac{1}{\pi k^2}\cos(kx)\Big|_{0}^{\pi} \\
&= 0 - \frac{1}{\pi k^2}[\cos 0 - \cos(-k\pi)] + 0 + \frac{1}{\pi k^2}[\cos(k\pi) - \cos 0] \\
&= \frac{2}{\pi k^2}[\cos(k\pi) - 1] = \begin{cases} 0, & \text{if } k \text{ is even} \\ \dfrac{-4}{\pi k^2}, & \text{if } k \text{ is odd} \end{cases}.
\end{aligned}$$

Since $\sin \pi k = 0$ and $\sin(-\pi k) = 0$.

Since $\cos(k\pi) = 1$, when k is even and $\cos(k\pi) = -1$, when k is odd.

Writing the even- and odd-indexed coefficients separately, we have $a_{2k} = 0$, for $k = 1, 2, \ldots$ and $a_{2k-1} = \dfrac{-4}{\pi(2k-1)^2}$, for $k = 1, 2, \ldots$. We leave it as an exercise to show that

$$b_k = 0, \text{ for all } k.$$

This gives us the Fourier series

$$\begin{aligned}
\frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k\cos(kx) + b_k\sin(kx)] &= \frac{\pi}{2} + \sum_{k=1}^{\infty} a_k\cos(kx) = \frac{\pi}{2} + \sum_{k=1}^{\infty} a_{2k-1}\cos(2k-1)x \\
&= \frac{\pi}{2} - \sum_{k=1}^{\infty}\frac{4}{\pi(2k-1)^2}\cos(2k-1)x \\
&= \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos(3x) - \frac{4}{25\pi}\cos(5x) - \cdots.
\end{aligned}$$

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You can show that this series converges absolutely for all x, by using the Comparison Test, since

$$|a_k| = \left| \frac{4}{\pi(2k-1)^2} \cos(2k-1)x \right| \le \frac{4}{\pi(2k-1)^2}$$

and the series $\sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)^2}$ converges. (Hint: Compare this last series to the convergent p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$, using the Limit Comparison Test.) To get an idea of the function to which the series is converging, we plot several of the partial sums of the series,

$$F_n(x) = \frac{\pi}{2} - \sum_{k=1}^{n} \frac{4}{\pi(2k-1)^2} \cos(2k-1)x.$$

See if you can conjecture the sum of the series by looking at Figures 7.48a and b. Notice how quickly the partial sums of the series appear to converge to the triangular wave function f (shown in red; also see Figure 7.47). As it turns out, the graph of the partial sums will not change appreciably if you plot $F_n(x)$ for much larger values of n. (Try this!) We'll see later how to be sure that the Fourier series converges to $f(x)$ for all x. There's something further to note here: the accuracy of the approximation is fairly uniform. That is, the difference between a given partial sum and f is roughly the same for each x. Take care to distinguish this behavior from that of Taylor polynomial approximations, where the farther you get away from the point about which you've expanded, the worse the approximation tends to get.

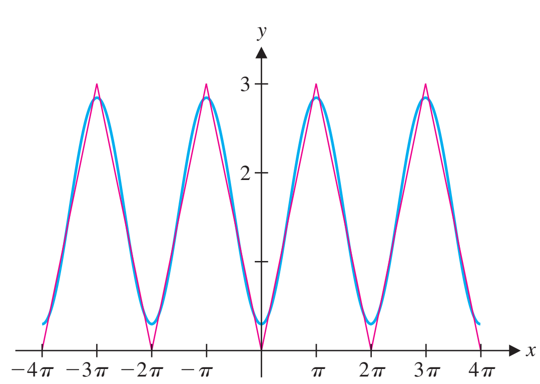

FIGURE 7.48a
$y = F_1(x)$ and $y = f(x)$.

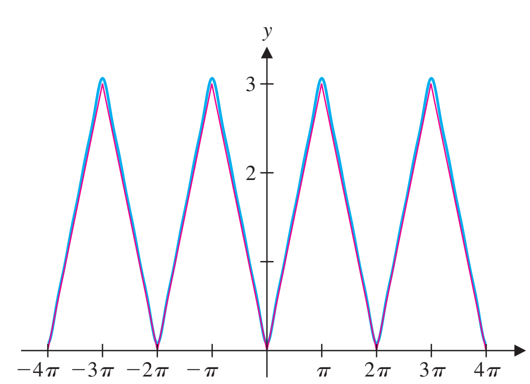

FIGURE 7.48b
$y = F_4(x)$ and $y = f(x)$.

□ Functions of Period Other Than 2π

Now, suppose you have a function f that is periodic of period T, but $T \ne 2\pi$. In this case, we want to expand f in a series of simple functions of period T. First, define $l = \frac{T}{2}$ and notice that

$$\cos\left(\frac{k\pi x}{l}\right) \quad \text{and} \quad \sin\left(\frac{k\pi x}{l}\right)$$

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are periodic of period $T = 2l$, for each $k = 1, 2, \ldots$ [Hint: To prove this, let $f(x) = \cos\left(\frac{k\pi x}{l}\right)$ and show that $f(x + 2l) = f(x)$, for all x and for each $k = 1, 2, \ldots$. Likewise, for $g(x) = \sin\left(\frac{k\pi x}{l}\right)$, show that $g(x + 2l) = g(x)$ for all x and for $k = 1, 2, \ldots$.] The Fourier series expansion of f of period $2l$ is then

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{k\pi x}{l}\right) + b_k \sin\left(\frac{k\pi x}{l}\right) \right].$$

We leave it as an exercise to show that the Fourier coefficients in this case are given by the Euler-Fourier formulas:

$$a_k = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{k\pi x}{l}\right) dx, \text{ for } k = 0, 1, 2, \ldots \tag{9.7}$$

and

$$b_k = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{k\pi x}{l}\right) dx, \text{ for } k = 1, 2, 3, \ldots. \tag{9.8}$$

Notice that (9.3), (9.5) and (9.6) are equivalent to (9.7) and (9.8) with $l = \pi$.

EXAMPLE 9.3 A Fourier Series Expansion for a Square Wave Function

Find a Fourier series expansion for the function

$$f(x) = \begin{cases} -2, & \text{if } -1 < x \le 0 \\ 2, & \text{if } 0 < x \le 1 \end{cases},$$

where f is defined so that it is periodic of period 2 outside of the interval $[-1, 1]$.

Solution The graph of f is the square-wave seen in Figure 7.49. From the Euler- Fourier formulas (9.7) and (9.8) with $l = 1$, we have

$$a_0 = \frac{1}{1} \int_{-1}^{1} f(x)\, dx = \int_{-1}^{0} (-2)\, dx + \int_{0}^{1} 2\, dx = 0.$$

Likewise, we get

$$a_k = \frac{1}{1} \int_{-1}^{1} f(x) \cos\left(\frac{k\pi x}{1}\right) dx = 0, \text{ for } k = 1, 2, 3, \ldots.$$

(This is left as an exercise.)

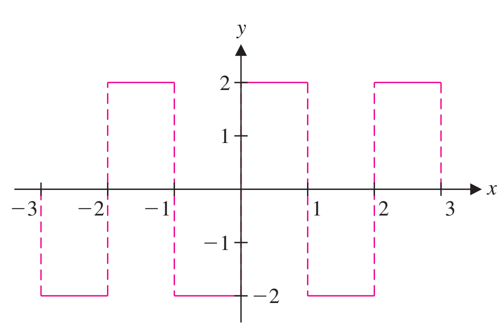

FIGURE 7.49
Square wave.

Finally, we have

$$b_k = \frac{1}{1}\int_{-1}^{1} f(x)\sin\left(\frac{k\pi x}{1}\right)dx = \int_{-1}^{0}(-2)\sin(k\pi x)\,dx + \int_{0}^{1} 2\sin(k\pi x)\,dx$$

$$= \frac{2}{k\pi}\cos(k\pi x)\Big|_{-1}^{0} - \frac{2}{k\pi}\cos(k\pi x)\Big|_{0}^{1} = \frac{4}{k\pi}[\cos 0 - \cos(k\pi)]$$

$$= \frac{4}{k\pi}[1-\cos(k\pi)] = \begin{cases} 0, & \text{if } k \text{ is even} \\ \dfrac{8}{k\pi}, & \text{if } k \text{ is odd} \end{cases}.$$

Since $\cos(k\pi) = 1$, when k is even and $\cos(k\pi) = -1$, when k is odd.

This gives us the Fourier series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k\cos(k\pi x) + b_k\sin(k\pi x)] = \sum_{k=1}^{\infty} b_k\sin(k\pi x) = \sum_{k=1}^{\infty} b_{2k-1}\sin[(2k-1)\pi x]$$

$$= \sum_{k=1}^{\infty}\frac{8}{(2k-1)\pi}\sin[(2k-1)\pi x].$$

Since $b_{2k} = 0$ and $b_{2k-1} = \frac{8}{(2k-1)\pi}$.

Although we as yet have no tools for determining the convergence or divergence of this series, we graph a few of the partial sums of the series,

$$F_n(x) = \sum_{k=1}^{n}\frac{8}{(2k-1)\pi}\sin[(2k-1)\pi x]$$

in Figures 7.50a–d. From the graphs, it appears that the series is converging to the square wave function f, except at the points of discontinuity, $x = 0, \pm 1, \pm 2, \pm 3, \ldots$. At those

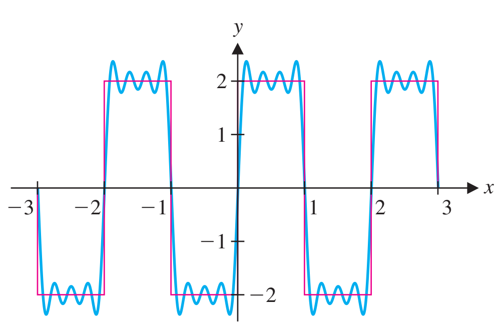

FIGURE 7.50a
$y = F_4(x)$ and $y = f(x)$.

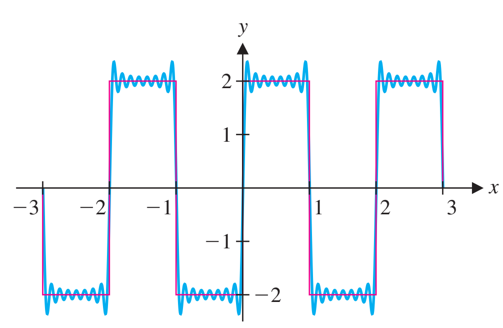

FIGURE 7.50b
$y = F_8(x)$ and $y = f(x)$.

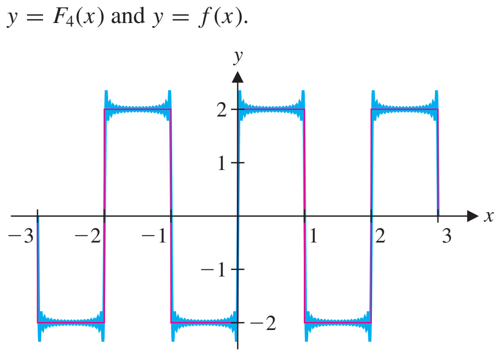

FIGURE 7.50c
$y = F_{20}(x)$ and $y = f(x)$.

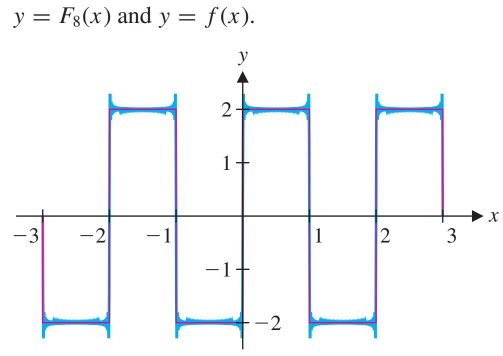

FIGURE 7.50d
$y = F_{50}(x)$ and $y = f(x)$.

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points, the series appears to converge to 0. You can easily verify this by observing that the terms of the series are

$$\frac{8}{(2k-1)\pi}\sin[(2k-1)\pi x] = 0, \text{ for integer values of } x.$$

Since each term in the series is zero, the series converges to 0 at all integer values of x. You might think of this as follows: at the points where f is discontinuous, the series is converging to the average of the two function values on either side of the discontinuity. As we will see, this is typical of the convergence of Fourier series. ■

We now state the major result on the convergence of Fourier series in Theorem 9.1.

THEOREM 9.1 **(Fourier Convergence Theorem)**

Suppose that f is periodic of period $2l$ and that f and f' are continuous on the interval $[-l, l]$, except for at most a finite number of jump discontinuities. Then, f has a convergent Fourier series expansion. Further, the series converges to $f(x)$, when f is continuous at x and to

$$\frac{1}{2}\left[\lim_{t\to x^+} f(t) + \lim_{t\to x^-} f(t)\right]$$

at any points x where f has a jump discontinuity.

PROOF

The proof of the theorem is beyond the level of this text and can be found in texts on advanced calculus or Fourier analysis. ■

REMARK 9.1

The Fourier Convergence Theorem says that a Fourier series may converge to a discontinuous function, even though every term in the series is continuous (and differentiable) for all x.

EXAMPLE 9.4 **Proving Convergence of a Fourier Series**

Use the Fourier Convergence Theorem to prove that the Fourier series expansion of period 2π,

$$\frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi}\cos(2k-1)x,$$

derived in example 9.2, for $f(x) = |x|$, for $-\pi \le x \le \pi$ and periodic outside of $[-\pi, \pi]$, converges to $f(x)$ everywhere.

Solution First, note that f is continuous everywhere (see Figure 7.47). We also have that since

$$f(x) = |x| = \begin{cases} -x, & \text{if } -\pi \le x < 0 \\ x, & \text{if } 0 \le x < \pi \end{cases}$$

and is periodic outside $[-\pi, \pi]$, then

$$f'(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}.$$

So, f' is also continuous on $[-\pi, \pi]$, except for jump discontinuities at $x = 0$ and $x = \pm\pi$. From the Fourier Convergence Theorem, we now have that the Fourier series

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converges to f everywhere (since f is continuous everywhere). Because of this, we can write

$$f(x) = \frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi} \cos(2k-1)x,$$

for all x. ■

As you can see from the Fourier Convergence Theorem, Fourier series do not always converge to the function you are expanding.

EXAMPLE 9.5 Investigating Convergence of a Fourier Series

Use the Fourier Convergence Theorem to investigate the convergence of the Fourier series

$$\sum_{k=1}^{\infty} \frac{8}{(2k-1)\pi} \sin[(2k-1)\pi x],$$

derived as an expansion of the square-wave function

$$f(x) = \begin{cases} -2, & \text{if } -1 < x \le 0 \\ 2, & \text{if } 0 < x \le 1 \end{cases},$$

where f is taken to be periodic outside of $[-1, 1]$ (see example 9.3).

Solution First, note that f is continuous, except for jump discontinuities at $x = 0, \pm 1, \pm 2, \ldots$. Further,

$$f'(x) = \begin{cases} 0, & \text{if } -1 < x < 0 \\ 0, & \text{if } 0 < x < 1 \end{cases}$$

and is periodic outside of $[-1, 1]$. Thus, f' is also continuous everywhere, except for jump discontinuities at integer values of x. From the Fourier Convergence Theorem, the Fourier series will converge to $f(x)$ everywhere, except at the discontinuities, $x = 0, \pm 1, \pm 2, \ldots$, where the series converges to the average of the one-sided limits, that is, 0. (Why 0?) Since the series does not converge to f everywhere, we cannot say that the function and the series are *equal*. In this case, we usually write

$$f(x) \sim \sum_{k=1}^{\infty} \frac{8}{(2k-1)\pi} \sin[(2k-1)\pi x],$$

to indicate that the series *corresponds* to f (but is not necessarily equal to f). In the case of Fourier series, this says that the series converges to $f(x)$ at every x where f is continuous and to the average of the one-sided limits at any jump discontinuities. Notice that this is the behavior seen in the graphs of the partial sums of the series seen in Figures 7.50a–d. ■

☐ Fourier Series and Music Synthesizers

You may be wondering why we have taken the trouble to investigate Fourier series. There must be some significant payoff for all of this work. Most people find it surprising that series of sines and cosines can converge to form line segments, parabolas and other shapes on finite intervals. As interesting as this is, this is not why we have pursued this study. As it turns out, Fourier series are widely used in engineering, physics, chemistry and so on. We give you a sense of how Fourier series are used with the following brief discussion of music synthesizers and through a variety of exercises.

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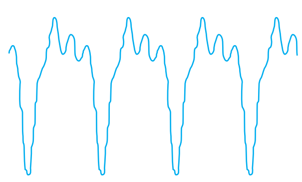

FIGURE 7.51
Saxophone waveform.

Suppose that you had a music machine that could generate pure tones at various pitches and volumes. What types of sounds could you synthesize by combining several pure tones together? To answer this question, we first translate the problem into mathematics. A pure tone can be modeled by $A \sin \omega t$, where the amplitude A determines the volume and the frequency ω determines the pitch. For example, to mimic a saxophone, you must match the characteristic waveform of a saxophone (see Figure 7.51). The shape of the waveform affects the **timbre** of the tone, a quality most humans readily discern (a saxophone *sounds* different than a trumpet, doesn't it?).

Consider the following music synthesizer problem. Given a waveform such as the one shown in Figure 7.51, can you add together several pure tones of the form $A \sin \omega t$ to approximate the waveform? Note that if the pure tones are of the form $b_1 \sin t$, $b_2 \sin 2t$, $b_3 \sin 3t$ and so on, this is essentially a Fourier series problem. That is, we want to approximate a given wave function $f(t)$ by a sum of these pure tones, as follows:

$$f(t) \approx b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \cdots + b_n \sin nt.$$

Although the cosine terms are all missing, notice that this is the partial sum of a Fourier series. (Such series are called **Fourier sine series** and are explored in the exercises.) For music synthesizers, the Fourier coefficients are simply the amplitudes of the various harmonics in a given waveform. In this context, you can think of the bass and treble knobs on a stereo as manipulating the amplitudes of different terms in a Fourier series. Cranking up the bass emphasizes low frequency terms (i.e., increases the coefficients of the first few terms of the Fourier series), while turning up the treble emphasizes the high frequency terms. An equalizer (see Figure 7.52) gives you more direct control of individual frequencies.

FIGURE 7.52
A graphic equalizer.

In general, the idea of analyzing a wave phenomenon by breaking the wave down into its component frequencies is essential to much of modern science and engineering. This type of **spectral analysis** is used in numerous scientific disciplines.

EXERCISES 7.9

WRITING EXERCISES

1. Explain why the Fourier series of $f(x) = 1 + 3\cos x - \sin 2x$ on the interval $[-\pi, \pi]$ is simply $1 + 3\cos x - \sin 2x$. (Hint: Explain what the goal of a Fourier series representation is and note that in this case no work needs to be done.) Would this change if the interval were $[-1, 1]$ instead?

2. Polynomials are built up from the basic operations of arithmetic. We often use Taylor series to rewrite an awkward function (e.g., $\sin x$) into arithmetic form. Many natural phenomena are waves, which are well modeled by sines and cosines. Discuss the extent to which the following statement is true: Fourier series allow us to rewrite algebraic functions (e.g., x^2) into a natural (wave) form.

3. Theorem 9.1 states that a Fourier series may converge to a function with jump discontinuities. In examples 9.1 and 9.3, identify the locations of the jump discontinuities and the values to which the Fourier series converges at these points. In what way are these values reasonable compromises?

4. Carefully examine Figures 7.46 and 7.50. For which x's does the Fourier series seem to converge rapidly? slowly? Note that for every n the partial sum $F_n(x)$ passes *exactly* through the limiting point for jump discontinuities. Describe the behavior of the partial sums *near* the jump discontinuities. This overshoot/undershoot behavior is referred to as the **Gibbs phenomenon** (see exercises 49 and 53).

In exercises 1–8, find the Fourier series of the function on the interval $[-\pi, \pi]$. Graph the function and the partial sums $F_4(x)$ and $F_8(x)$ on the interval $[-2\pi, 2\pi]$.

1. $f(x) = x$
2. $f(x) = x^2$
3. $f(x) = 2|x|$
4. $f(x) = 3x$
5. $f(x) = \begin{cases} 1, & \text{if } -\pi < x < 0 \\ -1, & \text{if } 0 < x < \pi \end{cases}$

6. $f(x) = \begin{cases} 1, & \text{if } -\pi < x < 0 \\ 0, & \text{if } 0 < x < \pi \end{cases}$

7. $f(x) = 3\sin 2x$

8. $f(x) = 2\sin 3x$

In exercises 9–14, find the Fourier series of the function on the given interval.

9. $f(x) = -x$, $[-1, 1]$

10. $f(x) = |x|$, $[-1, 1]$

11. $f(x) = x^2$, $[-1, 1]$

12. $f(x) = 3x$, $[-2, 2]$

13. $f(x) = \begin{cases} 0, & \text{if } -1 < x < 0 \\ x, & \text{if } 0 < x < 1 \end{cases}$

14. $f(x) = \begin{cases} 0, & \text{if } -1 < x < 0 \\ 1 - x, & \text{if } 0 < x < 1 \end{cases}$

In exercises 15–20, do not compute the Fourier series, but graph the function to which the Fourier series converges, showing at least three full periods of the limit function.

15. $f(x) = x$, $[-2, 2]$

16. $f(x) = x^2$, $[-3, 3]$

17. $f(x) = \begin{cases} -x, & \text{if } -1 < x < 0 \\ 0, & \text{if } 0 < x < 1 \end{cases}$

18. $f(x) = \begin{cases} 1, & \text{if } -2 < x < 0 \\ 3, & \text{if } 0 < x < 2 \end{cases}$

19. $f(x) = \begin{cases} -1, & \text{if } -2 < x < -1 \\ 0, & \text{if } -1 < x < 1 \\ 1, & \text{if } 1 < x < 2 \end{cases}$

20. $f(x) = \begin{cases} 2, & \text{if } -2 < x < -1 \\ -2, & \text{if } -1 < x < 1 \\ 0, & \text{if } 1 < x < 2 \end{cases}$

In exercises 21–24, use the Fourier Convergence Theorem to investigate the convergence of the Fourier series in the given exercise.

21. exercise 1

22. exercise 3

23. exercise 5

24. exercise 13

25. Substitute $x = 1$ into the Fourier series formula of exercise 11 to prove that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

26. Use the Fourier series of example 9.1 to prove that $\sum_{k=1}^{\infty} \frac{\sin(2k-1)}{(2k-1)} = \frac{\pi}{4}$.

27. Use the Fourier series of example 9.2 to prove that $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$.

28. Combine the results of exercises 25 and 27 to find $\sum_{k=1}^{\infty} \frac{1}{(2k)^2}$.

Exercises 29–34 involve even and odd functions.

29. You have undoubtedly noticed that many Fourier series consist of only cosine or only sine terms. This can be easily understood in terms of even and odd functions. A function $f(x)$ is **even** if $f(-x) = f(x)$ for all x. A function is **odd** if $f(-x) = -f(x)$ for all x. Show that $\cos x$ is even, $\sin x$ is odd and $\cos x + \sin x$ is neither.

30. If $f(x)$ is even, show that $g(x) = f(x)\cos x$ is even and $h(x) = f(x)\sin x$ is odd.

31. If $f(x)$ is odd, show that $g(x) = f(x)\cos x$ is odd and $h(x) = f(x)\sin x$ is even.

32. If f and g are even, what can you say about fg?

33. If f is even and g is odd, what can you say about fg?

34. If f and g are odd, what can you say about fg?

35. Prove the general Euler-Fourier formulas (9.7) and (9.8).

36. If $g(x)$ is an odd function (see exercise 29), show that $\int_{-l}^{l} g(x)\,dx = 0$ for any (positive) constant l. (Hint: Compare $\int_{-l}^{0} g(x)\,dx$ and $\int_{0}^{l} g(x)\,dx$. You will need to make the change of variable: $t = -x$ in one of the integrals.) Using the results of exercise 30, show that if $f(x)$ is even, then $b_k = 0$ for all k and the Fourier series of $f(x)$ consists only of a constant and cosine terms. If $f(x)$ is odd, show that $a_k = 0$ for all k and the Fourier series of $f(x)$ consists only of sine terms.

In exercises 37–42, use the even/odd properties of $f(x)$ to predict (don't compute) whether the Fourier series will contain only cosine terms, only sine terms or both.

37. $f(x) = x^3$

38. $f(x) = x^4$

39. $f(x) = e^x$

40. $f(x) = |x|$

41. $f(x) = \begin{cases} 0, & \text{if } -1 < x < 0 \\ x, & \text{if } 0 < x < 1 \end{cases}$

42. $f(x) = \begin{cases} -1, & \text{if } -2 < x < 0 \\ 3, & \text{if } 0 < x < 2 \end{cases}$

43. The function $f(x) = \begin{cases} -1, & \text{if } -2 < x < 0 \\ 3, & \text{if } 0 < x < 2 \end{cases}$ is neither even nor odd, but can be written as $f(x) = g(x) + 1$ where $g(x) = \begin{cases} -2, & \text{if } -2 < x < 0 \\ 2, & \text{if } 0 < x < 2 \end{cases}$. Explain why the Fourier series of $f(x)$ will contain sine terms and the constant 1, but no cosine terms.

44. Suppose that you want to find the Fourier series of $f(x) = x + x^2$. Explain why to find b_k you would only need to integrate $x \sin\left(\frac{k\pi x}{l}\right)$ and to find a_k you would only need to integrate $x^2 \cos\left(\frac{k\pi x}{l}\right)$.

Exercises 45–48 are adapted from the owner's manual of a high-end music synthesizer.

45. A fundamental choice to be made when generating a new tone on a music synthesizer is the waveform. The options are sawtooth, square and pulse. You worked with the sawtooth wave in exercise 9. Graph the limiting function for the function in exercise 9 on the interval $[-4, 4]$. Explain why "sawtooth"

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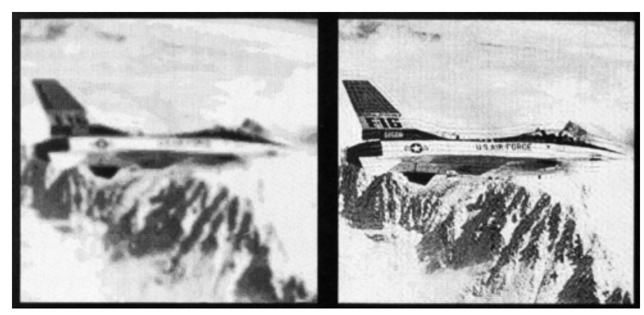
is a good name. A square wave is shown in Figure 7.49. A pulse wave of period 2 with width $1/n$ is generated by $f(x) = \begin{cases} -2, & \text{if } 1/n < |x| < 1 \\ 2, & \text{if } |x| \le 1/n \end{cases}$. Graph pulse waves of width $1/3$ and $1/4$ on the interval $[-4, 4]$.

46. The **harmonic content** of a wave equals the ratio of integral harmonic waves to the fundamental wave. To understand what this means, write the Fourier series of exercise 9 as $\frac{2}{\pi}\left(\sin \pi x + \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{4}\sin 4\pi x + \cdots\right)$. The harmonic content of the sawtooth wave is $\frac{1}{n}$. Explain how this relates to the relative sizes of the Fourier coefficients. The harmonic content of the square wave is $\frac{1}{n}$ with even-numbered harmonics missing. Compare this description to the Fourier series of example 9.3. The harmonic content of the pulse wave of width $\frac{1}{3}$ is $\frac{1}{n}$ with every third harmonic missing. Without computing the Fourier coefficients, write out the general form of the Fourier series of $f(x) = \begin{cases} -2, & \text{if } 1/3 < |x| < 1 \\ 2, & \text{if } |x| \le 1/3 \end{cases}$.

47. The cutoff frequency setting on a music synthesizer has a dramatic effect on the timbre of the tone produced. In terms of harmonic content (see exercise 46), when the cutoff frequency is set at $n > 0$, all harmonics beyond the nth harmonic are set equal to 0. In Fourier series terms, explain how this corresponds to the partial sum $F_n(x)$. For the sawtooth and square waves, graph the waveforms with the cutoff frequency set at 4. Compare these to the waveforms with the cutoff frequency set at 2. As the setting is lowered, you hear more of a "pure" tone. Briefly explain why.

48. The resonance setting on a music synthesizer also changes timbre significantly. Set at 1, you get the basic waveform (e.g., sawtooth or square). Set at 2, the harmonic content of the first four harmonics are divided by 2, the fifth harmonic is multiplied by $\frac{3}{4}$, the sixth harmonic is left the same, the seventh harmonic is divided by 2 and the remaining harmonics are set to 0. Graph the sawtooth and square waves with resonance set to 2. Which one is starting to resemble the saxophone waveform of Figure 7.51?

49. The Gibbs phenomenon is the tendency of approximating partial sums of Fourier series to badly undershoot and overshoot the limit function near jump discontinuities (places where the limit function changes rapidly). A black-and-white photograph can be digitized by partitioning the photograph into small rectangles and assigning each rectangle a number. For example, a pure white rectangle might be a 1 and a pure black rectangle a 10 with grey rectangles assigned values between 1 and 10. The digitized photograph can then be approximated by a Fourier series. A sharp edge in a photograph would have a rapid change from black to white, similar to a jump discontinuity. The rapid change in values from black to white requires high frequency components in the Fourier series (i.e., the terms of the series for large values of k). The photograph on the left is out of focus, to the point that we can't read the markings on the plane (you can imagine this would be important in many tense situations). By increasing the coefficients in the high frequencies in the Fourier series representation of the photograph, you can get the sharper photograph on the right. This photograph also has a distinct "halo effect" around the plane. Explain how the halo could be related to the Gibbs phenomenon (photos reprinted by permission from *Visualization* by Friedhoff and Benzon).



50. Piano tuning is relatively simple due to the phenomenon studied in this exercise. Compare the graphs of $\sin 8t + \sin 8.2t$ and $2\sin 8t$. Note especially that the amplitude of $\sin 8t + \sin 8.2t$ appears to slowly rise and fall. In the trig identity $\sin 8t + \sin 8.2t = [2\cos(0.2t)]\sin(8.1t)$, think of $2\cos(0.2t)$ as the amplitude of $\sin(8.1t)$ and explain why the amplitude varies slowly. Piano tuners often start by striking a tuning fork of a certain pitch (e.g., $\sin 8t$) and then striking the corresponding piano note. If the piano is slightly out-of-tune (e.g., $\sin 8.2t$), the tuning fork plus piano produces a combined tone that noticeably increases and decreases in volume. Use your graph to explain why this occurs.

51. The function $\sin 8\pi t$ represents a 4-Hz signal (1 Hz equals 1 cycle per second) if t is measured in seconds. If you received this signal, your task might be to take your measurements of the signal and try to reconstruct the function. For example, if you measured three samples per second, you would have the data $f(0) = 0$, $f(1/3) = \sqrt{3}/2$, $f(2/3) = -\sqrt{3}/2$ and $f(1) = 0$. Knowing the signal is of the form $A\sin Bt$, you would use the data to try to solve for A and B. In this case, you don't have enough information to guarantee getting the right values for A and B. Prove this by finding several values of A and B with $B \ne 8\pi$ that match the data. A famous result of H. Nyquist from 1928 states that to reconstruct a signal of frequency f you need at least $2f$ samples.

52. The energy of a signal $f(x)$ on the interval $[-\pi, \pi]$ is defined by $E = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx$. If $f(x)$ has a Fourier series $f(x) = \frac{a_0}{2} + \sum_{k=0}^{\infty}(a_k \cos kx + b_k \sin kx)$, show that $E = A_0^2 + A_1^2 + A_2^2 + \cdots$, where $A_k = \sqrt{a_k^2 + b_k^2}$. The sequence $\{A_k\}$ is called the **energy spectrum** of $f(x)$.

53. Carefully examine the graphs in Figure 7.46. You can see the Gibbs phenomenon at $x = 0$. Does it appear that the size of the Gibbs overshoot changes as the number of terms increases?

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We examine this question here. For the partial sum $F_{2n-1}(x)$, as defined in example 9.1, it can be shown that the absolute maximum occurs at $\frac{\pi}{2n}$. Evaluate $F_{2n-1}\left(\frac{\pi}{2n}\right)$ for $n = 4$, $n = 6$ and $n = 8$. Show that for large n, the size of the bump is $\left|F_{2n-1}\left(\frac{\pi}{2n}\right) - f\left(\frac{\pi}{2n}\right)\right| \approx 0.09$. Gibbs showed that in general the size of the bump at a jump discontinuity is about 0.09 times the size of the jump.

54. Some fixes have been devised to reduce the Gibbs phenomenon. Define the σ-factors by $\sigma_k = \frac{\sin(k\pi/n)}{(k\pi/n)}$ for $k = 1, 2, \ldots, n$ and consider the modified Fourier sum $\frac{a_0}{2} + \sum_{k=0}^{n}(a_k\sigma_k \cos kx + b_k\sigma_k \sin kx)$. For example 9.1, plot the modified sums for $n = 4$ and $n = 8$ and compare to Figure 7.46.

EXPLORATORY EXERCISES

1. Suppose that you wanted to approximate a waveform with sine functions (no cosines), as in the music synthesizer problem. Such a *Fourier sine series* will be derived in this exercise. You essentially use Fourier series with a trick to guarantee sine terms only. Start with your waveform as a function defined on the interval $[0, l]$, for some length l. Then define a function $g(x)$ that equals $f(x)$ on $[0, l]$ *and* that is an odd function. Show that $g(x) = \begin{cases} f(x) & \text{if } 0 \le x \le l \\ -f(-x) & \text{if } -l < x < 0 \end{cases}$ works. Explain why the Fourier series expansion of $g(x)$ on $[-l, l]$ would contain sine terms only. This series is the sine series expansion of $f(x)$. Show the following helpful shortcut: the sine series coefficients are $b_k = \frac{1}{l}\int_{-l}^{l} g(x)\sin\left(\frac{k\pi}{l}\right)dx = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{k\pi}{l}\right)dx$. Then compute the sine series expansion of $f(x) = x^2$ on $[0, 1]$ and graph the limit function on $[-3, 3]$.

Similarly, develop a **Fourier cosine series** and find the cosine series expansion of $f(x) = x$ on $[0, 1]$.

2. Fourier series is a part of the field of **Fourier analysis,** which is central to many engineering applications. Fourier analysis includes the Fourier transforms (and the FFT or fast Fourier transform) and inverse Fourier transforms, to which you will get a brief introduction in this exercise. Given measurements of a signal (waveform), the goal is to construct the Fourier series of a function. To start with a simple version of the problem, suppose the signal has the form $f(x) = \frac{a_0}{2} + a_1\cos\pi x + a_2\cos 2\pi x + b_1\sin\pi x + b_2\sin 2\pi x$ and you have the measurements $f(-1) = 0$, $f\left(-\frac{1}{2}\right) = 1$, $f(0) = 2$, $f\left(\frac{1}{2}\right) = 1$ and $f(1) = 0$. Substituting into the general equation for $f(x)$, show that

$$f(-1) = \frac{a_0}{2} - a_1 + a_2 = 0$$

$$f\left(-\frac{1}{2}\right) = \frac{a_0}{2} - a_2 - b_1 = 1$$

$$f(0) = \frac{a_0}{2} + a_1 + a_2 = 2$$

$$f\left(\frac{1}{2}\right) = \frac{a_0}{2} - a_2 + b_1 = 1$$

$$\text{and} \quad f(1) = \frac{a_0}{2} - a_1 + a_2 = 0.$$

Note that the first and last equations are identical and that b_2 never appears in an equation. Thus, you have four equations and four unknowns. Solve the equations. (Hint: Start by comparing the second and fourth equations, then the third and fifth equations.) You should conclude that $f(x) = 1 + \cos\pi x$, with no information about b_2. To determine b_2, we would need another function value. In general, the number of measurements determines how many terms you can find in the Fourier series (see exercise 51). Fortunately, there is an easier way of determining the Fourier coefficients. Recall that $a_k = \int_{-1}^{1} f(x)\cos n\pi x\,dx$ and $b_k = \int_{-1}^{1} f(x)\sin n\pi x\,dx$. You can estimate the integral using function values at $x = -1/2$, $x = 0$, $x = 1/2$ and $x = 1$. Find a version of a Riemann sum approximation that gives $a_0 = 2$, $a_1 = 1$, $a_2 = 0$ and $b_1 = 0$. What value is given for b_2? Use this Riemann sum rule to find the appropriate coefficients for the data $f\left(-\frac{3}{4}\right) = \frac{3}{4}$, $f\left(-\frac{1}{2}\right) = \frac{1}{2}$, $f\left(-\frac{1}{4}\right) = \frac{1}{4}$, $f(0) = 0$, $f\left(\frac{1}{4}\right) = -\frac{1}{4}$, $f\left(\frac{1}{2}\right) = -\frac{1}{2}$, $f\left(\frac{3}{4}\right) = -\frac{3}{4}$ and $f(1) = -1$. Compare to the Fourier series of exercise 13.

7.10 USING SERIES TO SOLVE DIFFERENTIAL EQUATIONS

In chapter 6, we saw how to solve several different types of differential equations. Among second order equations, we saw how to solve only those with constant coefficients, such as

$$y'' - 6y' + 9y = 0.$$

In cases such as this, we looked for a solution of the form of $y = e^{rx}$. So, what if the coefficients aren't constant? For instance, suppose you wanted to solve the equation

$$y'' + 2xy' + 2y = 0.$$

We leave it as an exercise to show that substituting $y = e^{rx}$ in this case does not lead to a solution. However, it turns out that in many cases such as this, we can find a solution by assuming that the solution can be written as a power series, such as

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

The idea is to substitute this series into the differential equation and then use the resulting equation to determine the coefficients, $a_0, a_1, a_2, \ldots, a_n$. Before we see how to do this in general, we illustrate this for a simple equation, whose solution is already known, to demonstrate that we arrive at the same solution either way.

EXAMPLE 10.1 Power Series Solution of a Differential Equation

Use a power series to determine the general solution of

$$y'' + y = 0.$$

Solution First, observe that since this equation has constant coefficients, we already know how to find a solution. We leave it as an exercise to show that the general solution is

$$y = c_1 \sin x + c_2 \cos x,$$

where c_1 and c_2 are constants.

We now look for a solution of the equation in the form of the power series

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n.$$

To substitute this into the equation, we first need to obtain representations for y' and y''. Assuming that the power series is convergent and has a positive radius of convergence, recall that we can differentiate it term-by-term to obtain the derivatives

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$y'' = 2a_2 + 6a_3 x + \cdots = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting these power series into the differential equation, we get

$$0 = y'' + y = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n. \tag{10.1}$$

The immediate objective here is to combine the two series in (10.1) into one power series. Since the powers in the one series are of the form x^{n-2} and in the other series are of the form x^n, we will first need to rewrite one of the two series. Notice that we have that

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Substituting this into equation (10.1) gives us

$$0 = y'' + y = \sum_{n=0}^{\infty}(n+2)(n+1)\,a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^n$$
$$= \sum_{n=2}^{\infty}[(n+2)(n+1)\,a_{n+2} + a_n]x^n. \tag{10.2}$$

Read equation (10.2) carefully; it says that the power series on the far right converges to the constant function $f(x) = 0$. Another way to think of this is as the Taylor series expansion of the zero function. In view of this, all of the coefficients must be zero. That is,

$$0 = (n+2)(n+1)\,a_{n+2} + a_n,$$

for $n = 0, 1, 2, \ldots$. We solve this for the coefficient with the largest index, to obtain

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}, \tag{10.3}$$

for $n = 0, 1, 2, \ldots$. Equation (10.3) is called the **recurrence relation.** From here, we'd like to use (10.3) to determine all of the coefficients of the series solution. This may seem like a tall order, but it's not as difficult as it sounds. The general idea is to write out (10.3) for a number of specific values of n and then try to recognize a pattern that the coefficients follow. We begin by recognizing that (10.3) relates a_{n+2} to a_n, for each n. In other words, a_2 is related to a_0; a_4 is related to a_2, which in turn is related to a_0 and so on. So, all of the coefficients with even indices ($a_2, a_4, a_6, \ldots$) are all related to a_0. Likewise, you should be able to see that all of the coefficients with odd indices are related to a_1. To recognize the pattern, we simply write out a few terms, as follows. From (10.3), we have for the even-indexed coefficients that

$$a_2 = \frac{-a_0}{2 \cdot 1} = \frac{-1}{2!}a_0,$$
$$a_4 = \frac{-a_2}{4 \cdot 3} = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1}\,a_0 = \frac{1}{4!}a_0,$$
$$a_6 = \frac{-a_4}{6 \cdot 5} = \frac{-1}{6!}a_0,$$
$$a_8 = \frac{-a_6}{8 \cdot 7} = \frac{1}{8!}a_0$$

and so on. (Try to write down a_{10} by recognizing the pattern, without referring to the recurrence relation.) Since we can write each even-indexed coeffiecient as a_{2n}, for some n, note that we can now write down a simple formula that works for any of these coefficients. We have

$$a_{2n} = \frac{(-1)^n}{(2n)!}\,a_0, \tag{10.4}$$

for $n = 0, 1, 2, \ldots$. Similarly, using (10.3), we have that the odd-indexed coefficients are

$$a_3 = \frac{-a_1}{3 \cdot 2} = \frac{-1}{3!}a_1,$$
$$a_5 = \frac{-a_3}{5 \cdot 4} = \frac{1}{5!}a_1,$$
$$a_7 = \frac{-a_5}{7 \cdot 6} = \frac{-1}{7!}a_1,$$
$$a_9 = \frac{-a_7}{9 \cdot 8} = \frac{1}{9!}a_1$$

and so on. Since we can write each odd-indexed coefficient as a_{2n+1} (or alternatively as a_{2n-1}), for some n, note that we have the following simple formula for the odd-indexed coefficients:

$$a_{2n+1} = \frac{(-1)^n}{(2n+1)!}a_1.$$

Since we have now written every coefficient in terms of either a_0 or a_1, we can rewrite the solution by separating the a_0 terms from the a_1 terms. We have

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\ &= a_0\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) + a_1\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right) \\ &= a_0 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}}_{y_1(x)} + a_1 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}}_{y_2(x)} \\ &= a_0 y_1(x) + a_1 y_2(x), \end{aligned} \tag{10.5}$$

where $y_1(x)$ and $y_2(x)$ are two solutions of the differential equation (assuming the series converge). At this point, you should be able to easily check that both of the indicated power series converge absolutely for all x, by using the Ratio Test. Beyond this, you might also recognize that the series solutions $y_1(x)$ and $y_2(x)$ that we obtained are in fact, the Maclaurin series expansions of $\cos x$ and $\sin x$, respectively. In light of this, (10.5) is an equivalent solution to that found by using the methods of Chapter 6. ■

The method used to solve the differential equation in example 10.1 is certainly far more complicated than the methods we used in Chapter 6 for solving the same equation. It is not our intention here to provide you with a new and even more complicated method for solving the same old equations. Rather, this new method can be used to solve a wider range of differential equations than those solvable using our earlier methods. Now that we have verified that our new power series method gives the same solution as our earlier method, we turn our attention to equations that cannot be solved using our earlier, simpler methods. We begin by returning to an equation mentioned in the introduction to this section.

EXAMPLE 10.2 Solving a Differential Equation with Variable Coefficients

Find the general solution of the differential equation

$$y'' + 2xy' + 2y = 0.$$

Solution First, observe that since the coefficient of y' is not constant, we have little choice but to look for a series solution of the equation. As in example 10.1, we begin by assuming that we may write the solution as a power series,

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

As before, we have

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Substituting these three power series into the equation, we get

$$\begin{aligned} 0 = y'' + 2xy' + 2y &= \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} + 2x\sum_{n=1}^{\infty} na_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} + \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n, \end{aligned} \tag{10.6}$$

where in the middle term, we moved the x into the series and combined powers of x. In order to combine the three series, we must only rewrite the first series so that its general term is a multiple of x^n, instead of x^{n-2}. As we did in example 10.1, we have

$$\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)\,a_{n+2} x^n,$$

and so, from (10.6), we have

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} + \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)\,a_{n+2} x^n + \sum_{n=0}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n \\ &= \sum_{n=0}^{\infty} [(n+2)(n+1)\,a_{n+2} + 2na_n + 2a_n]x^n \\ &= \sum_{n=0}^{\infty} [(n+2)(n+1)\,a_{n+2} + 2\,(n+1)\,a_n]x^n. \end{aligned} \tag{10.7}$$

To get this, we used the fact that $\sum_{n=1}^{\infty} 2na_n x^n = \sum_{n=0}^{\infty} 2na_n x^n$. (Notice that the first term in the series on the right is zero!) Reading equation (10.7) carefully, note that we again have a power series converging to the zero function, from which it follows that all of the coefficients must be zero:

$$0 = (n+2)(n+1)\,a_{n+2} + 2(n+1)a_n,$$

for $n = 0, 1, 2, \ldots$. Again solving for the coefficient with the largest index, we get the recurrence relation

$$a_{n+2} = -\frac{2(n+1)a_n}{(n+2)(n+1)}$$

or

$$a_{n+2} = -\frac{2a_n}{n+2}.$$

Much like we saw in example 10.1, the recurrence relation tells us that every second coefficient is related, so that all of the even-indexed coefficients are related to a_0 and all of the odd-indexed coefficients are related to a_1. In order to try to recognize the pattern, we write out a number of terms, using the recurrence relation. We have

$$\begin{aligned} a_2 &= -\frac{2}{2}a_0 = -a_0, \\ a_4 &= -\frac{2}{4}a_2 = \frac{1}{2}a_0, \\ a_6 &= -\frac{2}{6}a_4 = -\frac{1}{3!}a_0, \\ a_8 &= -\frac{2}{8}a_6 = \frac{1}{4!}a_0 \end{aligned}$$

and so on. At this point, you should recognize the pattern for these coefficients. (If not, write out a few more terms.) Note that we can write the even-indexed coefficients as

$$a_{2n} = \frac{(-1)^n}{n!}a_0,$$

for $n = 0, 1, 2, \ldots$. Be sure to match this formula against those coefficients calculated above to see that they match. Continuing with the odd-indexed coefficients, we have from the recurrence relation that

$$\begin{aligned} a_3 &= -\frac{2}{3}a_1, \\ a_5 &= -\frac{2}{5}a_3 = \frac{2^2}{5 \cdot 3}a_1, \\ a_7 &= -\frac{2}{7}a_5 = -\frac{2^3}{7 \cdot 5 \cdot 3}a_1, \\ a_9 &= -\frac{2}{9}a_7 = \frac{2^4}{9 \cdot 7 \cdot 5 \cdot 3}a_1 \end{aligned}$$

and so on. While you might recognize the pattern here, unlike the case for the even-indexed coefficients, it's a bit harder to write down this pattern succinctly. Observe that the products in the denominators are not quite factorials. Rather, they are the products of the first so many odd numbers. The solution to this is to write this as a factorial, but then cancel out all of the even integers in the product. In particular, note that

$$\frac{1}{9 \cdot 7 \cdot 5 \cdot 3} = \frac{\overbrace{8}^{2\cdot 4} \cdot \overbrace{6}^{2\cdot 3} \cdot \overbrace{4}^{2\cdot 2} \cdot \overbrace{2}^{2\cdot 4}}{9!} = \frac{2^4 \cdot 4!}{9!},$$

so that a_9 becomes

$$a_9 = \frac{2^4}{9 \cdot 7 \cdot 5 \cdot 3}\, a_1 = \frac{2^4 \cdot 2^4 \cdot 4!}{9!}\, a_1 = \frac{2^{2\cdot 4} \cdot 4!}{9!}\, a_1.$$

More generally, we now have

$$a_{2n+1} = \frac{(-1)^n\, 2^{2n} n!}{(2n+1)!}\, a_1,$$

for $n = 0, 1, 2, \ldots$.

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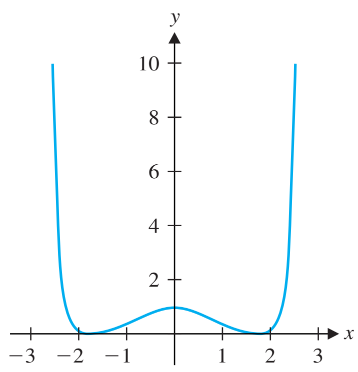

FIGURE 7.53a
$y = y_1(x)$.

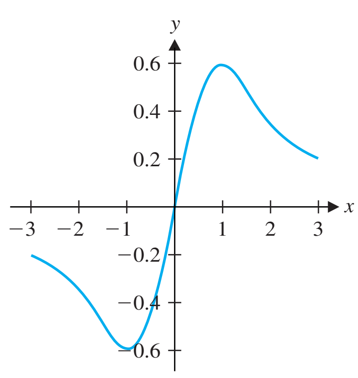

FIGURE 7.53b
$y = y_2(x)$.

Now that we have expressions for all of the coefficients, we can write the solution of the differential equation as

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left(a_{2n}x^{2n} + a_{2n+1}x^{2n+1}\right) \\ &= a_0 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}}_{y_1(x)} + a_1 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} n!}{(2n+1)!} x^{2n+1}}_{y_2(x)} \\ &= a_0 y_1(x) + a_1 y_2(x), \end{aligned}$$

where y_1 and y_2 are two power series solutions of the differential equation. We leave it as an exercise to use the Ratio Test to show that both of these series converge absolutely for all x. You are unlikely to recognize these two power series as Taylor series of familiar functions as we did in example 10.1, but even so, these are perfectly good solutions. (Actually, you might recognize the power series for $y_1(x)$ as e^{-x^2}, but in practice recognizing series solutions as power series of familiar functions is rather unlikely.) To give you an idea of the behavior of these functions, we draw a graph of $y_1(x)$ in Figure 7.53a and of $y_2(x)$ in Figure 7.53b. We obtained these graphs by plotting the partial sums of these series. In particular, it's worth noting that neither of these solutions is in the form $y = e^{rx}$, for any value of r. So, looking for a solution in this form, as we did for the case of a differential equation with constant coefficients cannot work here. ■

From examples 10.1 and 10.2, you might get the idea that if you look for a series solution, you can always recognize the pattern of the coefficients and write the pattern down succinctly. Unfortunately, this is not at all true. Most often, the pattern is difficult to see and even more difficult to write down compactly. Still, series solutions are a valuable means of solving a differential equation. In the worst case, you can always compute a number of the coefficients of the series from the recurrence relation and then use the first so many terms of the series as an approximation to the actual solution.

In the final example, we illustrate the more common case where the coefficients are a bit more challenging to find.

EXAMPLE 10.3 A Series Solution Where the Coefficients Are Harder to Find

Use a power series to find the general solution of **Airy's equation**

$$y'' - xy = 0.$$

Solution As in both our previous examples, we begin by assuming that we may write the solution as a power series

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Again, we have

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Subsituting these power series into the equation, we get

$$\begin{aligned} 0 = y'' - xy &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1}. \end{aligned}$$

In order to combine the two preceding series, we must rewrite one or both series so that they both have the same power of x. For simplicity, we rewrite the first series only. We have

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= \sum_{n=-1}^{\infty} (n+3)(n+2)\, a_{n+3} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= (2)(1)\, a_2 + \sum_{n=0}^{\infty} (n+3)(n+2)\, a_{n+3} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= 2a_2 + \sum_{n=0}^{\infty} [(n+3)(n+2)\, a_{n+3} - a_n] x^{n+1}, \end{aligned}$$

where we wrote out the first term of the first series and then combined the two series, once both had an index that started with $n = 0$. Again, this is a power series expansion of the zero function and so, all of the coefficients must be zero. That is,

$$0 = 2a_2 \tag{10.8}$$

and

$$0 = (n+3)(n+2)\, a_{n+3} - a_n, \tag{10.9}$$

for $n = 0, 1, 2, \ldots$. Notice that equation (10.8) says that $a_2 = 0$. As we have seen before, (10.9) gives us the recurrence relation

$$a_{n+3} = \frac{1}{(n+3)(n+2)} a_n, \tag{10.10}$$

for $n = 0, 1, 2, \ldots$. Notice that here, instead of having all of the even-indexed coefficients related to a_0 and all of the odd-indexed coefficients related to a_1, we have a slightly different situation. In this case, (10.10) tells us that every *third* coefficient is related. In particular, notice that since $a_2 = 0$, (10.10) now says that

$$\begin{aligned} a_5 &= \frac{1}{5 \cdot 4} a_2 = 0, \\ a_8 &= \frac{1}{8 \cdot 7} a_5 = 0 \end{aligned}$$

and so on. So, every third coefficient starting with a_2 is zero. But, how do we concisely write down something like this? Think about the notation a_{2n} and a_{2n+1} that we have used previously. You can view a_{2n} as a representation of every second coefficient stating with a_0. Likewise, a_{2n+1} represents every second coefficient starting with a_1. In the present case, if we want to write down every third coefficient starting with a_2, we write a_{3n+2}. We can now observe that

$$a_{3n+2} = 0,$$

for $n = 0, 1, 2, \ldots$. Continuing on with the remaining coefficients, we have from (10.10) that

$$\begin{aligned}
a_3 &= \frac{1}{3 \cdot 2} a_0, \\
a_6 &= \frac{1}{6 \cdot 5} a_3 = \frac{1}{6 \cdot 5 \cdot 3 \cdot 2} a_0, \\
a_9 &= \frac{1}{9 \cdot 8} a_6 = \frac{1}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2} a_0
\end{aligned}$$

and so on. Hopefully, you see the pattern that's developing for these coefficients. The trouble here is that it's not as easy to write down this pattern as it was in the first two examples. Notice that the denominator in the expression for a_9 is almost 9!, but with every third factor in the product deleted. Since we don't have a way of succinctly writing this down, we write the coefficients by indicating the pattern, as follows:

$$a_{3n} = \frac{(3n-2)(3n-5)\cdots 7 \cdot 4 \cdot 1}{(3n)!} a_0,$$

where this is not intended as a literal formula, as explicit substitution of $n = 0$ or $n = 1$ would result in negative values. Rather, this is an indication of the general pattern. Similarly, the recurrence relation gives us

$$\begin{aligned}
a_4 &= \frac{1}{4 \cdot 3} a_1, \\
a_7 &= \frac{1}{7 \cdot 6} a_4 = \frac{1}{7 \cdot 6 \cdot 4 \cdot 3} a_1, \\
a_{10} &= \frac{1}{10 \cdot 9} a_7 = \frac{1}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3} a_1
\end{aligned}$$

and so on. More generally, we can establish the pattern:

$$a_{3n+1} = \frac{(3n-1)(3n-4)\cdots 8 \cdot 5 \cdot 2}{(3n+1)!} a_1,$$

where again, this is not intended as a literal formula.

Now that we have found all of the coefficients, we can write the solution, by separately writing out every third term of the series, as follows:

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left(a_{3n} x^{3n} + a_{3n+1} x^{3n+1} + a_{3n+2} x^{3n+2} \right) \\
&= a_0 \underbrace{\sum_{n=0}^{\infty} \frac{(3n-2)(3n-5)\cdots 7 \cdot 4 \cdot 1}{(3n)!} x^{3n}}_{y_1(x)} + a_1 \underbrace{\sum_{n=0}^{\infty} \frac{(3n-1)(3n-4)\cdots 8 \cdot 5 \cdot 2}{(3n+1)!} x^{3n+1}}_{y_2(x)} \\
&= a_0 y_1(x) + a_1 y_2(x).
\end{aligned}$$

We leave it as an exercise to use the Ratio Test to show that the power series defining y_1 and y_2 are absolutely convergent for all x. ■

You may have noticed that in all three of our examples, we assumed that there was a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots,$$

only to arrive at the general solution

$$y = a_0 y_1(x) + a_1 y_2(x),$$

where y_1 and y_2 were power series solutions of the equation. This is in fact not coincidental. One can show that (at least for certain equations) this is always the case. One clue as to why this might be so lies in the following.

Suppose that we want to solve the initial value problem consisting of a second order differential equation and the initial conditions $y(0) = A$ and $y'(0) = B$. Taking $y(x) = \sum_{n=0}^{\infty} a_n x^n$ gives us

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots.$$

So, imposing the initial conditions, we have

$$A = y(0) = a_0 + a_1(0) + a_2(0)^2 + \cdots = a_0$$

and

$$B = y'(0) = a_1 + 2a_2(0) + 3a_3(0)^2 + \cdots = a_1.$$

So, irrespective of the particular equation we're solving, we always have $y(0) = a_0$ and $y'(0) = a_1$.

You might ask what you'd do if the initial conditions were imposed at some point other than at $x = 0$, say at $x = x_0$. In this case, we look for a power series solution of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

It's easy to show that in this case, we still have $y(x_0) = a_0$ and $y'(x_0) = a_1$.

In the exercises, we explore finding series solutions about a variety of different points.

EXERCISES 7.10

WRITING EXERCISES

1. After substituting a power series representation into a differential equation, the next step is always to rewrite one or more of the series so that all series have the same exponent. (Typically, we want x^n.) Explain why this is an important step. For example, what would we be unable to do if the exponents were not the same?

2. The recurrence relation is typically solved for the coefficient with the largest subscript. Explain why this is an important step.

3. Explain why you can't solve equations with non-constant coefficients, like

$$y'' + 2xy' + 2y = 0$$

by looking for a solution in the form $y = e^{rx}$.

4. The differential equations solved in this section are actually of a special type, where we find power series solutions centered at what is called an **ordinary point.** For the equation

$x^2y'' + y' + 2y = 0$, the point $x = 0$ is not an ordinary point. Discuss what goes wrong here if you look for a power series solution of the form $\sum_{n=0}^{\infty} a_n x^n$.

In exercises 1–8, find the recurrence relation and general power series solution of the form $\sum_{n=0}^{\infty} a_n x^n$.

1. $y'' + 2xy' + 4y = 0$ **2.** $y'' + 4xy' + 8y = 0$

3. $y'' - xy' - y = 0$ **4.** $y'' - xy' - 2y = 0$

5. $y'' - xy' = 0$ **6.** $y'' + 2xy = 0$

7. $y'' - x^2y' = 0$ **8.** $y'' + xy' - 2y = 0$

9. Find a series solution of $y'' + (1 - x)y' - y = 0$, in the form $y = \sum_{n=0}^{\infty} a_n (x-1)^n$.

10. Find a series solution of $y'' + y' + (x - 2)y = 0$ in the form $\sum_{n=0}^{\infty} a_n (x-2)^n$.

11. Find a series solution of Airy's equation $y'' - xy = 0$ in the form $\sum_{n=0}^{\infty} a_n (x-1)^n$. [Hint: First rewrite the equation in the form $y'' - (x - 1)y - y = 0$.]

12. Find a series solution of Airy's equation $y'' - xy = 0$ in the form $\sum_{n=0}^{\infty} a_n (x-2)^n$.

13. Solve the initial value problem $y'' + 2xy' + 2y = 0$, $y(0) = 5$, $y'(0) = -7$. (See exercise 1.)

14. Solve the initial value problem $y'' + 4xy' + 8y = 0$, $y(0) = 2$, $y'(0) = \pi$. (See exercise 2.)

15. Solve the initial value problem $y'' + (1 - x)y' - y = 0$, $y(1) = -3$, $y'(1) = 12$. (See exercise 9.)

16. Solve the initial value problem $y'' + y' + (x - 2)y = 0$, $y(2) = 1$, $y'(2) = -1$. (See exercise 10.)

17. Determine the radius of convergence of the power series solutions about $x_0 = 0$ of $y'' - xy' - y = 0$. (See exercise 3.)

18. Determine the radius of convergence of the power series solutions about $x_0 = 0$ of $y'' - xy' - 2y = 0$. (See exercise 12.)

19. Determine the radius of convergence of the power series solutions about $x_0 = 1$ of $y'' + (1 - x)y' - y = 0$. (See exercise 9.)

20. Determine the radius of convergence of the power series solutions about $x_0 = 1$ of $y'' - xy = 0$ (See exercise 11.)

21. Find a series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ to the equation $x^2y'' + xy' + x^2y = 0$ (Bessel's equation of order 0).

22. Find a series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ to the equation $x^2y'' + xy' + (x^2 - 1)y = 0$ (Bessel's equation of order 1).

23. Determine the radius of convergence of the series solution found in example 10.3.

24. Determine the radius of convergence of the series solution found in problem 11.

25. For the initial value problem $y'' + 2xy' - xy = 0$, $y(0) = 2$, $y'(0) = -5$ substitute in $x = 0$ and show that $y''(0) = 0$. Then take $y'' = -2xy' + xy$ and show that $y''' = -2xy'' + (x - 2)y' + y$. Conclude that $y'''(0) = 12$. Then compute $y^{(4)}(x)$ and find $y^{(4)}(0)$. Finally, compute $y^{(5)}(x)$ and find $y^{(5)}(0)$. Write out the fifth-degree Taylor polynomial for the solution, $P_5(x) = y(0) + y'(0)x + y''(0)\frac{x^2}{2} + y'''(0)\frac{x^3}{3!} + y^{(4)}(0)\frac{x^4}{4!} + y^{(5)}(0)\frac{x^5}{5!}$.

26. Use the technique of exercise 25 to find the fifth-degree Taylor polynomial for the solution of the initial value problem $y'' + x^2y' - (\cos x)y = 0$, $y(0) = 3$, $y'(0) = 2$.

27. Use the technique of exercise 25 to find the fifth-degree Taylor polynomial for the solution of the initial value problem $y'' + e^x y' - (\sin x)y = 0$, $y(0) = -2$, $y'(0) = 1$.

28. Use the technique of exercise 25 to find the fifth-degree Taylor polynomial for the solution of the initial value problem $y'' + y' - (e^x)y = 0$, $y(0) = 2$, $y'(0) = 0$.

29. Use the technique of exercise 25 to find the fifth-degree Taylor polynomial for the solution of the initial value problem $y'' + xy' + (\sin x)y = 0$, $y(\pi) = 0$, $y'(\pi) = 4$.

30. Use the technique of exercise 25 to find the fifth-degree Taylor polynomial for the solution of the initial value problem $y'' + (\cos x)y' + xy = 0$, $y(\frac{\pi}{2}) = 3$, $y'(\frac{\pi}{2}) = 0$.

EXPLORATORY EXERCISES

1. The equation $y'' - 2xy' + 2ky = 0$ for some integer $k \geq 0$ is known as **Hermite's equation.** Following our procedure for finding series solutions in powers of *x,* show that in fact one of the series solutions is simply a polynomial of degree k. For this polynomial solution, choose the arbitrary constant such that the leading term of the polynomial is $2^k x^k$. The polynomial is called the **Hermite polynomial** $H_k(x)$. Find the Hermite polynomials $H_0(x), H_1(x), \ldots, H_5(x)$.

2. The Chebyshev polynomials are polynomial solutions of the equation $(1 - x^2)y'' - xy' + k^2y = 0$ for some integer $k \geq 0$. Find polynomial solutions for $k = 0, 1, 2,$ and 3.

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REVIEW EXERCISES

CONCEPTS

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Sequence	Limit of sequence	Squeeze Theorem
Infinite series	Partial sum	Series converges
Series diverges	Geometric series	kth-term test for divergence
Harmonic series	Integral Test	p-Series
Comparison Test	Limit Comparison Test	Alternating Series Test
Conditional convergence	Absolute convergence	Alternating harmonic series
Ratio Test	Root Test	Power series
Radius of convergence	Taylor series	Taylor polynomial
Taylor's Theorem	Fourier series	Recurrence relation

TRUE OR FALSE

State whether each statement is true or false and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to a new statement that is true.

1. An increasing sequence diverges to infinity.
2. As n increases, $n!$ increases faster than 10^n.
3. If the sequence a_n diverges, then the series $\sum_{k=1}^{\infty} a_k$ diverges.
4. If a_n decreases to 0 as $n \to \infty$, then $\sum_{k=1}^{\infty} a_k$ diverges.
5. If $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{k=1}^{\infty} a_k$ converges for $a_k = f(k)$.
6. If the Comparison Test can be used to determine the convergence or divergence of a series, then the Limit Comparison Test can also determine the convergence or divergence of the series.
7. Using the Alternating Series Test, if $\lim_{k\to\infty} a_k \neq 0$, then you can conclude that $\sum_{k=1}^{\infty} a_k$ diverges.
8. The difference between a partial sum of a convergent series and its sum is less than the first neglected term in the series.
9. If a series is conditionally convergent, then the Ratio Test will be inconclusive.
10. A series with all negative terms cannot be conditionally convergent.
11. If $\sum_{k=1}^{\infty} |a_k|$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.
12. A series may be integrated term-by-term, and the interval of convergence will remain the same.
13. A Taylor series of a function f is simply a power series representation of f.
14. The more terms in a Taylor polynomial, the better the approximation.
15. The Fourier series of x^2 converges to x^2 for all x.
16. A recurrence relation can always be solved to find the solution of a differential equation.

In exercises 1–8, determine whether the sequence converges or diverges. If it converges, give the limit.

1. $a_n = \dfrac{4}{3+n}$
2. $a_n = \dfrac{3n}{1+n}$
3. $a_n = (-1)^n \dfrac{n}{n^2+4}$
4. $a_n = (-1)^n \dfrac{n}{n+4}$
5. $a_n = \dfrac{4^n}{n!}$
6. $a_n = \dfrac{n!}{n^n}$
7. $a_n = \cos \pi n$
8. $a_n = \dfrac{\cos n\pi}{n}$

In exercises 9–18, answer with "converges" or "diverges" or "can't tell."

9. If $\lim_{k\to\infty} a_k = 1$, then $\sum_{k=1}^{\infty} a_k$ ______.
10. If $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=1}^{\infty} a_k$ ______.
11. If $\lim_{k\to\infty} \left|\dfrac{a_{k+1}}{a_k}\right| = 1$, then $\sum_{k=1}^{\infty} a_k$ ______.
12. If $\lim_{k\to\infty} \left|\dfrac{a_{k+1}}{a_k}\right| = 0$, then $\sum_{k=1}^{\infty} a_k$ ______.
13. If $\lim_{k\to\infty} a_k = \dfrac{1}{2}$, then $\sum_{k=1}^{\infty} a_k$ ______.
14. If $\lim_{k\to\infty} \left|\dfrac{a_{k+1}}{a_k}\right| = \dfrac{1}{2}$, then $\sum_{k=1}^{\infty} a_k$ ______.

REVIEW EXERCISES

15. If $\lim_{k\to\infty} \sqrt[k]{|a_k|} = \frac{1}{2}$, then $\sum_{k=1}^{\infty} a_k$ ______.

16. If $\lim_{k\to\infty} k^2 a_k = 0$, then $\sum_{k=1}^{\infty} a_k$ ______.

17. If $p > 1$, then $\sum_{k=1}^{\infty} \frac{8}{k^p}$ ______.

18. If $r > 1$, then $\sum_{k=1}^{\infty} ar^k$ ______.

In exercises 19–22, find the sum of the convergent series.

19. $\sum_{k=0}^{\infty} 4\left(\frac{1}{2}\right)^k$

20. $\sum_{k=1}^{\infty} \frac{4}{k(k+2)}$

21. $\sum_{k=0}^{\infty} 4^{-k}$

22. $\sum_{k=0}^{\infty} (-1)^k \frac{3}{4^k}$

T **In exercises 23 and 24, estimate the sum of the series to within 0.01.**

23. $\sum_{k=0}^{\infty} (-1)^k \frac{k}{k^4+1}$

24. $\sum_{k=0}^{\infty} (-1)^{k+1} \frac{3}{k!}$

In exercises 25–44, determine if the series converges or diverges.

25. $\sum_{k=0}^{\infty} \frac{2k}{k+3}$

26. $\sum_{k=0}^{\infty} (-1)^k \frac{2k}{k+3}$

27. $\sum_{k=0}^{\infty} (-1)^k \frac{4}{\sqrt{k+1}}$

28. $\sum_{k=0}^{\infty} \frac{4}{\sqrt{k+1}}$

29. $\sum_{k=1}^{\infty} 3k^{-7/8}$

30. $\sum_{k=1}^{\infty} 2k^{-8/7}$

31. $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^3+1}$

32. $\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^3+1}}$

33. $\sum_{k=1}^{\infty} (-1)^k \frac{4^k}{k!}$

34. $\sum_{k=1}^{\infty} (-1)^k \frac{2^k}{k}$

35. $\sum_{k=1}^{\infty} (-1)^k \ln\left(1+\frac{1}{k}\right)$

36. $\sum_{k=1}^{\infty} \frac{\cos k\pi}{\sqrt{k}}$

37. $\sum_{k=1}^{\infty} \frac{2}{(k+3)^2}$

38. $\sum_{k=2}^{\infty} \frac{4}{k \ln k}$

39. $\sum_{k=1}^{\infty} \frac{k!}{3^k}$

40. $\sum_{k=1}^{\infty} \frac{k}{3^k}$

41. $\sum_{k=1}^{\infty} \frac{e^{1/k}}{k^2}$

42. $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{\ln k + 1}}$

43. $\sum_{k=1}^{\infty} \frac{4^k}{(k!)^2}$

44. $\sum_{k=1}^{\infty} \frac{k^2+4}{k^3+3k+1}$

In exercises 45–48, determine if the series converges absolutely, converges conditionally or diverges.

45. $\sum_{k=1}^{\infty} (-1)^k \frac{k}{k^2+1}$

46. $\sum_{k=1}^{\infty} (-1)^k \frac{3}{k+1}$

47. $\sum_{k=1}^{\infty} \frac{\sin k}{k^{3/2}}$

48. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3}{\ln k + 1}$

In exercises 49 and 50, find all values of p for which the series converges.

49. $\sum_{k=1}^{\infty} \frac{2}{(3+k)^p}$

50. $\sum_{k=1}^{\infty} e^{kp}$

In exercises 51 and 52, determine the number of terms necessary to estimate the sum of the series to within 10^{-6}.

51. $\sum_{k=1}^{\infty} (-1)^k \frac{3}{k^2}$

T **52.** $\sum_{k=1}^{\infty} (-1)^k \frac{2^k}{k!}$

In exercises 53–56, find a power series representation for the function. Find the radius of convergence.

53. $\frac{1}{4+x}$

54. $\frac{2}{6-x}$

55. $\frac{3}{3+x^2}$

56. $\frac{2}{1+4x^2}$

In exercises 57 and 58, use the series from exercises 53 and 54 to find a power series and its radius of convergence.

57. $\ln(4+x)$

58. $\ln(6-x)$

In exercises 59–66, find the interval of convergence.

59. $\sum_{k=0}^{\infty} (-1)^k 2x^k$

60. $\sum_{k=0}^{\infty} (-1)^k (2x)^k$

61. $\sum_{k=1}^{\infty} (-1)^k \frac{2}{k} x^k$

62. $\sum_{k=1}^{\infty} \frac{-3}{\sqrt{k}} \left(\frac{x}{2}\right)^k$

63. $\sum_{k=0}^{\infty} \frac{4}{k!}(x-2)^k$

64. $\sum_{k=0}^{\infty} k^2 (x+3)^k$

65. $\sum_{k=0}^{\infty} 3^k (x-2)^k$

66. $\sum_{k=0}^{\infty} \frac{k}{4^k}(x+1)^k$

In exercises 67 and 68, derive the Taylor series of $f(x)$ about the center $x = c$.

67. $f(x) = \sin x, c = 0$

68. $f(x) = \frac{1}{x}, c = 1$

REVIEW EXERCISES

In exercises 69 and 70, find the Taylor polynomial $P_4(x)$. Graph $f(x)$ and $P_4(x)$.

69. $f(x) = \ln x, c = 1$

70. $f(x) = \frac{1}{\sqrt{x}}, c = 1$

In exercises 71 and 72, use the Taylor polynomials from exercises 69 and 70 to estimate the given values. Determine the order of the Taylor polynomial needed to estimate the value to within 10^{-8}.

71. $\ln 1.2$

72. $\frac{1}{\sqrt{1.1}}$

In exercises 73 and 74, use a known Taylor series to find a Taylor series of the function and find its radius of convergence.

73. e^{-3x^2}

74. $\sin 4x$

In exercises 75 and 76, use the first five terms of a known Taylor series to estimate the value of the integral.

75. $\int_0^1 \tan^{-1} x\, dx$

76. $\int_0^2 e^{-3x^2}\, dx$

In exercises 77 and 78, derive the Fourier series of the function.

77. $f(x) = x, -2 \le x \le 2$

78. $f(x) = \begin{cases} 0 & \text{if } -\pi < x \le 0 \\ 1 & \text{if } 0 < x \le \pi \end{cases}$

In exercises 79–82, graph at least three periods of the function to which the Fourier series expansion of the function converges.

79. $f(x) = x^2, -1 \le x \le 1$

80. $f(x) = 2x, -2 \le x \le 2$

81. $f(x) = \begin{cases} -1 & \text{if } -1 < x \le 0 \\ 1 & \text{if } 0 < x \le 1 \end{cases}$

82. $f(x) = \begin{cases} 0 & \text{if } -2 < x \le 0 \\ x & \text{if } 0 < x \le 2 \end{cases}$

83. Suppose you and your friend take turns tossing a coin. The first one to get a head wins. Obviously, the person who goes first has an advantage, but how much of an advantage is it? If you go first, the probability that you win on your first toss is $\frac{1}{2}$, the probability that you win on your second toss is $\frac{1}{8}$, the probability that you win on your third toss is $\frac{1}{32}$ and so on. Sum a geometric series to find the probability that you win.

84. In a game similar to that of exercise 83, the first one to roll a 4 on a six-sided die wins. Is this game more fair than the previous game? The probabilities of winning on the first, second and third roll are $\frac{1}{6}$, $\frac{25}{216}$ and $\frac{625}{7776}$, respectively. Sum a geometric series to find the probability that you win.

In exercises 85 and 86, find the recurrence relation and a general power series solution of the form $\sum_{n=0}^{\infty} a_n x^n$.

85. $y'' - 2xy' - 4y = 0$

86. $y'' + (x-1)y' = 0$

In exercises 87 and 88, find the recurrence relation and a general power series solution of the form $\sum_{n=0}^{\infty} a_n (x-1)^n$.

87. $y'' - 2xy' - 4y = 0$

88. $y'' + (x-1)y' = 0$

In exercises 89 and 90, solve the initial value problem.

89. $y'' - 2xy' - 4y = 0$, $y(0) = 4$, $y'(0) = 2$

90. $y'' - 2xy' - 4y = 0$, $y(1) = 2$, $y'(1) = 4$

CONNECTIONS

1. The challenge here is to determine $\sum_{k=1}^{\infty} \frac{x^k}{k(k+1)}$ as completely as possible. Start by finding the interval of convergence. Find the sum for the special cases (a) $x = 0$ and (b) $x = 1$. For $0 < x < 1$, do the following. (c) Rewrite the series using the partial fractions expansion of $\frac{1}{k(k+1)}$. (d) Because the series converges absolutely, it is legal to rearrange terms. Do so and rewrite the series as $x + \frac{x-1}{x}\left[\frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots\right]$. (e) Identify the series in brackets as $\int \left(\sum_{k=1}^{\infty} x^k\right) dx$, evaluate the series and then integrate term-by-term. (f) Replace the term in brackets in part (d) with its value obtained in part (e). (g) The next case is for $-1 < x < 0$. Use the technique in parts (c)–(f) to find the sum. (h) Evaluate the sum at $x = -1$ using the fact that the alternating harmonic series sums to ln 2. (Used by permission of Virginia Tech Mathematics Contest. Solution suggested by Gregory Minton.)

2. You have used Fourier series to show that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. Here, you will use a version of **Vièta's formula** to give an alternative derivation. Start by using a Maclaurin series for $\sin x$ to derive a series for $f(x) = \frac{\sin\sqrt{x}}{\sqrt{x}}$. Then find the zeros of $f(x)$. Vièta's formula states that the sum of the reciprocals of the zeros of $f(x)$ equals the negative of the coefficient of the linear term in the Maclaurin series of $f(x)$ divided by the constant term. Take this equation and multiply by π^2 to get the desired formula.

REVIEW EXERCISES

Use the same method with a different function to show that $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$.

3. Recall the Fibonacci sequence defined by $a_0 = 1, a_1 = 1, a_2 = 2$ and $a_{n+1} = a_n + a_{n-1}$. Prove the following fact: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1+\sqrt{5}}{2}$. (This number, known to the ancient Greeks, is called the **golden ratio.**) (Hint: Start with $a_{n+1} = a_n + a_{n-1}$ and divide by a_n. If $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$, argue that $\lim_{n\to\infty} \frac{a_{n-1}}{a_n} = \frac{1}{r}$ and then solve the equation $r = 1 + \frac{1}{r}$.) The Fibonacci sequence can be visualized with the following construction. Start with two side-by-side squares of side 1 (Figure A). Above them, draw a square (Figure B), which will have side 2. To the left of that, draw a square (Figure C), which will have side 3. Continue to spiral around, drawing squares which have sides given by the Fibonacci sequence. For each bounding rectangle in Figures A–C, compute the ratio of the sides of the rectangle. (Hint: Start with $\frac{2}{1}$ and then $\frac{3}{2}$.) Find the limit of the ratios as the construction process continues. The Greeks proclaimed this to be the most "pleasing" of all rectangles, building the Parthenon and other important buildings with these proportions (see *The Divine Proportion* by H. E. Huntley).

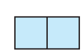
FIGURE A

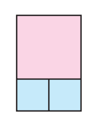
FIGURE B

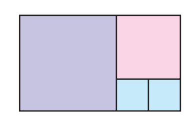
FIGURE C

4. Another type of sequence studied by mathematicians is the **continued fraction.** Numerically explore the sequence $1 + \frac{1}{1}, 1 + \frac{1}{1+\frac{1}{1}}, 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1}}}$ and so on. This is yet another occurence of the golden ratio. Viscount Brouncker, a seventeenth-century English mathematician, showed that the sequence $1 + \frac{1^2}{2}, 1 + \frac{1^2}{2+\frac{3^2}{2}}, 1 + \frac{1^2}{2+\frac{3^2}{2+\frac{5^2}{2}}}$ and so on, converges to $\frac{4}{\pi}$ (see *A History of Pi* by Petr Beckmann). Explore this sequence numerically.