Lebesgue's Criterion for Riemann integrability

Here we give Henri Lebesgue's characterization of those functions which are Riemann integrable.

Recall the example of the he Dirichlet function, defined on [0,1] by

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ is rational in lowest terms} \\ 0, & \text{otherwise} \end{cases}.$$

This function is continuous at all irrational numbers and discontinuous at the rational numbers. It is also Riemann-integrable (with integral 0). It turns out that there is a connection here. It is the nature of the set of discontinuities that determines integrability.

For a real-valued function f defined on a set X, and $I \subset X$, let $\omega f(I) = \sup_{s,t \in I} |f(s) - f(t)|$, the **oscillation of** f **on** I, as usual. The **oscillation of** f **at a point** x is defined as

$$\omega_f(x) = \inf\{\omega f(B(x, \delta)) : \delta > 0\}.$$

It is easy to prove that f is continuous at x if and only if $\omega_f(x) = 0$.

Lemma. *Let $f : [a, b] \to \mathbb{R}$. Then, for every $\alpha > 0$, $\{x : \omega_f(x) < \alpha\}$ is open in $[a, b]$ and $\{x : \omega_f(x) \geq \alpha\}$ is a closed set (in $\mathbb{R}$).*

Proof. Let $G = \{x \in [a, b] : \omega_f(x) < \alpha\}$. Let $c \in G$. Then, $\omega_f(c) < \alpha$ and by definition, there is a $\delta > 0$ such that $\omega f(B(c, \delta) \cap [a, b]) < \alpha$. If $x \in B(c, \delta) \cap [a, b]$, and U is a neighbourhood of x contained in $B(c, \delta)$, then $\omega f(U) < \alpha$, so $\omega_f(x) \leq \omega f(U) < \alpha$, also. Thus, G is open in $[a, b]$.

Since $[a, b]$ is closed and G is open in $[a, b]$, $\{x : \omega_f(x) \geq \alpha\} = [a, b] \setminus G$, is closed in $[a, b]$ and in $\mathbb{R}$. □

Let $\ell(I)$ denote the length of the interval I. A subset N of $\mathbb{R}$ is said to have **measure 0**, if for each $\varepsilon > 0$, there exists countable family $\mathcal{H} = \{I_1, I_2, \dots\}$ of open intervals covering N, with total length $\sum_k \ell(I_k) < \varepsilon$.

Lemma.

(1) *Every countable set of reals has measure* 0.
(2) *If B has measure 0 and $A \subset B$, then A also has measure 0.*
(3) *If A_k has measure* 0*, for all $k \in \mathbb{N}$, then $\bigcup_{k \in \mathbb{N}} A_k$ also has measure 0.*

.

Proof. (1) Let $A = \{a_1, a_2, \dots\}$ be countable, $\varepsilon > 0$, and for every k, let I_k be the interval $(a - \varepsilon/2^{k+1}, a + \varepsilon/2^{k+1})$. Then, $A \subset \bigcup_k I_k$ — that is these intervals cover A. For each k, the length of I_k is $\varepsilon/2^k$, and the total length is $\sum_k \ell(I_k) \leq \sum_{k=1}^{\infty} \varepsilon/2^k = \varepsilon$. Thus, A has measure 0.

(2) is obvious, because a family of intervals that covers B also covers A.

To prove (3), one uses a modification of the proof of (1). Let $\varepsilon > 0$. For each k, let $\mathcal{H}_k$ be a countable family of intervals whose total length is less than $\varepsilon/2^k$. Then, $\bigcup_k \mathcal{H}_k$ is still a countable family of intervals, and their total length is less than $\sum_k \varepsilon/2^k = \varepsilon$. □

Theorem. (Lebesgue's Criterion for integrablility) *Let $f : [a, b] \to \mathbb{R}$. Then, f is Riemann integrable if and only if f is bounded and the set of discontinuities of f has measure 0.*

Notice that the Dirichlet function satisfies this criterion, since the set of discontinuities is the set of rationals in $[0, 1]$, which is countable.

Proof. Let f be Riemann integrable on $[a, b]$. Then, f is certainly bounded. Let D be the set of points of discontinuity of F. Then $D = \{x : \omega_f(x) > 0\}$. We are to show that D has measure 0. For each $\alpha > 0$, let $N(\alpha) = \{x \in [a, b] : \omega_f(x) \geq \alpha\}$. Then, $D = \bigcup_{k=1}^{\infty} N(1/k)$. Thus, we need only prove that each $N(\alpha)$ has measure 0.

Fix such an α and let $\varepsilon > 0$. By the Basic Integrability Criterion, we can choose a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with

$$\sum_{i=1}^{n} \omega f([x_{i-1}, x_i])(x_i - x_{i-1}) < \alpha\varepsilon/2.$$

Assume, as we may, that the x_i are distinct. Let F be the set of all i for which (x_{i-1}, x_i) intersects $N(\alpha)$. Then for each $i \in F$, $\omega f([x_{i-1}, x_i]) \geq \alpha$. Thus,

$$\alpha \sum_{i \in F} \Delta x_i \leq \sum_{i \in F} \omega f([x_{i-1}, x_i]) \Delta x_i < \alpha\varepsilon/2,$$

so that the sum of the lengths of the intervals (x_{i-1}, x_i) is less than $\varepsilon/2$. These cover $N(\alpha)$ except for the elements of $\{x_0, x_1, \ldots, x_n\}$. But these can be covered by intervals whose lengths total less than $\varepsilon/2$, so that $N(\alpha)$ can be covered with open intervals of total length less than ε, as required.

For the converse, let f be bounded and suppose that the set D of discontinuities of f is of measure 0.

Fix $\varepsilon > 0$ and let $E = \{x : \omega_f(x) \geq \varepsilon\}$. Since $E \subset D$, E has measure 0. Thus, E can be covered by a countable family of open intervals, whose total length is less than ε. Since E is closed and bounded, it is compact, so a finite family of such intervals will do, say $E \subset \bigcup_{i=1}^{m} U_i$. For each i, let I_i be the closure of U_i. For simplicity, by replacing pairs that intersect, we may assume that no two I_i intersect. Let $\mathcal{D} = \{I_1, \ldots, I_m\}$.

The set $K = [a, b] \setminus \bigcup_{i=1}^{m} U_i$ is compact (in fact, is the union of a finite number of disjoint closed intervals) and consists of points where $\omega_f(x) < \varepsilon$. For each $x \in K$, there is a closed interval J with $x \in \operatorname{int} J$ and $\omega f([J]) < \varepsilon$. By compactness, a finite number of such intervals covers K. By intersecting with K, we can assume that they are all subsets of K. Thus, let $\mathcal{C} = \{J_1, \dots, J_k\}$, be closed intervals whose union is K and such that $\omega f([J_j]) < \varepsilon$, for all j. We can (and do) assume that the intervals J_k do not overlap.

The family $\mathcal{D} \cup \mathcal{C} = \{[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]\}$ partitions $[a, b]$ and

$$\begin{aligned}
\sum_{i=1}^{n} \omega f([x_{i-1}, x_i])(x_i - x_{i-1}) &= \sum_{i=1}^{m} \omega f(I_i)\ell(I_i) + \sum_{j=1}^{k} \omega f(J_j)\ell(J_j) \\
&\leq \sum_{i} 2\|f\|\ell(I_i) + \sum_{j=1}^{k} \varepsilon\ell(J_j) \\
&= 2\|f\| \sum_{i} \ell(I_i) + \varepsilon(b-a) \\
&\leq 2\|f\|\varepsilon + \varepsilon(b-a),
\end{aligned}$$

which is arbitrarily small. Thus, the Basic Integrablity Criterion is satisfied and f is integrable. □

You may have noticed that part of this argument is similar to that in the proof that the composition $g \circ f$ of a continuous function g with an integrable function f is integrable. We see now that the composition result is an immediate consequence of Lebesgue's criterion.

Lemma. *Let $f : [a, b] \to [c, d]$ be integrable and $g : [c, d] \to \mathbb{R}$ be continuous. Then, $g \circ f$ is integrable.*

Proof. The set of points of discontinuity of f has measure 0, since f is integrable. But $g \circ f$ is continuous wherever f is, so the set of discontinuities of $g \circ f$ is contained in that of f, so has measure 0 also. □