The Isoperimetric Problem in the Heisenberg group $\mathbb{H}^n$

First Taiwan Geometry Symposium, NCTS South

November 20, 2010
Talk outline

- The Euclidean Isoperimetric Problem.
- Relevant terminology, concepts, definitions in/of the Heisenberg group $H^n$. 
Talk outline

- The Euclidean Isoperimetric Problem.
- Relevant terminology, concepts, definitions in/of the Heisenberg group $\mathbb{H}^n$.
- The Isoperimetric Problem in $\mathbb{H}^n$–Pansu’s Isoperimetric inequality and conjecture.
The Euclidean Isoperimetric Problem.

Relevant terminology, concepts, definitions in/of the Heisenberg group $\mathbb{H}^n$.

The Isoperimetric Problem in $\mathbb{H}^n$–Pansu’s Isoperimetric inequality and conjecture.


The $C^2$ solution to the Isoperimetric problem in $\mathbb{H}^1$: Ritoré-Rosales.

The non-smooth cases: Monti, Monti-Rickly.

An improvement of Danielli-Garofalo-Nhieu due to Ritoré: calibration argument.
The Euclidean Isoperimetric Problem.

Relevant terminology, concepts, definitions in/of the Heisenberg group $\mathbb{H}^n$.

The Isoperimetric Problem in $\mathbb{H}^n$–Pansu’s Isoperimetric inequality and conjecture.


The Euclidean Isoperimetric Problem.

- Relevant terminology, concepts, definitions in/of the Heisenberg group $\mathbb{H}^n$.

- The Isoperimetric Problem in $\mathbb{H}^n$—Pansu’s Isoperimetric inequality and conjecture.


- Partial Symmetry Case: Danielli-Garofalo-Nhieu.
The Euclidean Isoperimetric Problem.

Relevant terminology, concepts, definitions in/of the Heisenberg group $\mathbb{H}^n$.

The Isoperimetric Problem in $\mathbb{H}^n$–Pansu’s Isoperimetric inequality and conjecture.


Partial Symmetry Case: Danielli-Garofalo-Nhieu.

The $C^2$ solution to the Isoperimetric problem in $\mathbb{H}^1$: Ritore-Rosales.
Talk outline

- The Euclidean Isoperimetric Problem.
- Relevant terminology, concepts, definitions in/of the Heisenberg group $\mathbb{H}^n$.
- The Isoperimetric Problem in $\mathbb{H}^n$—Pansu’s Isoperimetric inequality and conjecture.
- Partial Symmetry Case: Danielli-Garofalo-Nhieu.
- The $C^2$ solution to the Isoperimetric problem in $\mathbb{H}^1$: Ritore-Rosales.
- The non-smooth cases: Monti, Monti-Rickly.
Talk outline

- The Euclidean Isoperimetric Problem.
- Relevant terminology, concepts, definitions in/of the Heisenberg group $\mathbb{H}^n$.
- The Isoperimetric Problem in $\mathbb{H}^n$–Pansu’s Isoperimetric inequality and conjecture.
- Partial Symmetry Case: Danielli-Garofalo-Nhieu.
- The $C^2$ solution to the Isoperimetric problem in $\mathbb{H}^1$: Ritoré-Rosales.
- The non-smooth cases: Monti, Monti-Rickly.
- An improvement of Danielli-Garofalo-Nhieu due to Ritoré: calibration argument.
Talk outline

- The Euclidean Isoperimetric Problem.
- Relevant terminology, concepts, definitions in/of the Heisenberg group $\mathbb{H}^n$.
- The Isoperimetric Problem in $\mathbb{H}^n$—Pansu’s Isoperimetric inequality and conjecture.
- Partial Symmetry Case: Danielli-Garofalo-Nhieu.
- The $C^2$ solution to the Isoperimetric problem in $\mathbb{H}^1$: Ritoré-Rosales.
- The non-smooth cases: Monti, Monti-Rickly.
- An improvement of Danielli-Garofalo-Nhieu due to Ritoré: calibration argument.
The Euclidean Isoperimetric Problem.

We begin with the following folklore which attributed the Isoperimetric Problem to Queen Dido, founder of the city of Carthage in North Africa.

Figure: Dido, Queen of Carthage. Engraving by Mathäus Merian the Elder 1630.

According to Virgil’s saga “Fleeing the vengeance of her brother, Dido (356-260 BC) lands on the coast of North Africa. For the bargain which Dido agrees to with a local potentate is this: she may have that portion of land which she is able to enclose with the hide of a bull. She then cut the hide into a seris of long thin strips and marked out a vast circumference. This area then eventually became the city of Carthage”.

Queen Dido’s problem/solution is a variant of what is now known as isoperimetric type problems. In more precise term, Dido’s problem is formulated as follows.

Among all bounded, connected open regions in the plane with a fixed perimeter, determine the one(s) that has the maximum volume.
Queen Dido’s problem/solution is a variant of what is now known as isoperimetric type problems. In more precise term, Dido’s problem is formulated as follows.

Among all bounded, connected open regions in the plane with a fixed perimeter, determine the one(s) that has the maximum volume.

The above problem is also equivalent to: Among all bounded, connected open regions in the plane with a fixed volume, determine the one(s) that has the minimum perimeter.
Queen Dido’s problem/solution is a variant of what is now known as isoperimetric type problems. In more precise term, Dido’s problem is formulated as follows.

Among all bounded, connected open regions in the plane with a fixed perimeter, determine the one(s) that has the maximum volume.

The above problem is also equivalent to: Among all bounded, connected open regions in the plane with a fixed volume, determine the one(s) that has the minimum perimeter.

Dido’s solution is correct: (although part of her region is bounded by a sea shore which we assume it to be a straight line when compared to the relative length of the bull’s hide): the extremal regions are precisely one half of the open circular planar discs.
Queen Dido’s problem/solution is a variant of what is now known as isoperimetric type problems. In more precise term, Dido’s problem is formulated as follows.

Among all bounded, connected open regions in the plane with a fixed perimeter, determine the one(s) that has the maximum volume.

The above problem is also equivalent to: Among all bounded, connected open regions in the plane with a fixed volume, determine the one(s) that has the minimum perimeter.

Dido’s solution is correct: (although part of her region is bounded by a sea shore which we assume it to be a straight line when compared to the relative length of the bull’s hide): the extremal regions are precisely one half of the open circular planar discs.
Over the centuries, the isoperimetric problem (in various forms) has stimulated substantial mathematical research in numerous areas:

- Geometric measure theory: The precise setting for the study of classical questions in the calculus of variations and the proof of existence of an isoperimetric profile. The tools are compactness theorems for $BV$ functions. Consequently, a priori solutions are only guaranteed within the class of sets of finite perimeter.
Over the centuries, the isoperimetric problem (in various forms) has stimulated substantial mathematical research in numerous areas:

Geometric measure theory: The precise setting for the study of classical questions in the calculus of variations and the proof of existence of an isoperimetric profile. The tools are compactness theorems for BV functions. Consequently, a priori solutions are only guaranteed within the class of sets of finite perimeter.

Differential Geometry: (Smooth) isoperimetric solutions are surfaces of constant mean curvature. The classification of such surfaces provides a characterization of isoperimetric profiles.
Over the centuries, the isoperimetric problem (in various forms) has stimulated substantial mathematical research in numerous areas:

- Geometric measure theory: The precise setting for the study of classical questions in the calculus of variations and the proof of existence of an isoperimetric profile. The tools are compactness theorems for $BV$ functions. Consequently, a priori solutions are only guaranteed within the class of sets of finite perimeter.

- Differential Geometry: (Smooth) isoperimetric solutions are surfaces of constant mean curvature. The classification of such surfaces provides a characterization of isoperimetric profiles.
The Euclidean Isoperimetric Problem.

- PDE: The introduction of dynamic algorithms of volume-constrained curvature flows which provides a way to smoothly deform a given region so that the isoperimetric ratio $P(E) \frac{n}{n-1} / |E|$ decreases monotonically. If the flow exists for all time, the deformed regions converge, in a suitable sense, to a solution of the isoperimetric problem.

- Functional Analysis: An equivalent way of formulating the isoperimetric problem consists in viewing it as a best constant problem for a Sobolev inequality, relating mean values of a given smooth function with those of its derivatives.
The Euclidean Isoperimetric Problem.

- PDE: The introduction of dynamic algorithms of volume-constrained curvature flows which provides a way to smoothly deform a given region so that the isoperimetric ratio $P(E) \frac{n}{n-1} / |E|$ decreases monotonically. If the flow exists for all time, the deformed regions converge, in a suitable sense, to a solution of the isoperimetric problem.

- Functional Analysis: An equivalent way of formulating the isoperimetric problem consists in viewing it as a best constant problem for a Sobolev inequality, relating mean values of a given smooth function with those of its derivatives.

- Geometric function theory: Symmetrization procedures that replace a given mathematical object or region with one admitting a larger symmetry group while retaining certain properties.
• PDE: The introduction of dynamic algorithms of volume-constrained curvature flows which provides a way to smoothly deform a given region so that the isoperimetric ratio $P(E) \frac{n}{n-1} / |E|$ decreases monotonically. If the flow exists for all time, the deformed regions converge, in a suitable sense, to a solution of the isoperimetric problem.

• Functional Analysis: An equivalent way of formulating the isoperimetric problem consists in viewing it as a best constant problem for a Sobolev inequality, relating mean values of a given smooth function with those of its derivatives.

• Geometric function theory: Symmetrization procedures that replace a given mathematical object or region with one admitting a larger symmetry group while retaining certain properties.
The Euclidean Isoperimetric Problem.

We recall the classical isoperimetric inequality in the Euclidean space.

**Theorem 1**

For every Borel set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with finite perimeter $P(\Omega)$,

$$
\min \{|\Omega|, |\mathbb{R}^n \setminus \Omega|\} \leq C_{iso}(\mathbb{R}^n) \frac{P(\Omega)^n}{n^{n-1}},
$$

(1)

where

$$C_{iso}(\mathbb{R}^n) = \frac{1}{n \omega_n^{n-1}},$$

Here, $\omega_k$ is the surface measure of the unit sphere $S^k$ in $\mathbb{R}^{k+1}$. Equality holds in (1) if and only if almost everywhere $\Omega = B(x, R)$ (i.e. a ball) for some $x \in \mathbb{R}^n$ and $R > 0$. In the case where $\partial \Omega$ is smooth say $C^1$ then $P(\Omega)$ coincide with surface measure of $\partial \Omega$. In the non-smooth case $P(\Omega) = Var(\chi_\Omega, \mathbb{R}^n)$ where $\chi_\Omega$ is the indicator function of $\Omega$ and

$$Var(u) = \sup \left\{ \int_{\mathbb{R}^n} u(x) \sum_{i=1}^n \partial x_i G_i \, dx \bigg| G_i \in C_0^\infty(\mathbb{R}^n), G_1^2 + \cdots + C_n^2 \leq 1 \right\}$$

is the variation of $u$ in $\mathbb{R}^n$. Note that $\Omega$ need not to be bounded in a priori.
The Euclidean Isoperimetric Problem.

We recall the classical isoperimetric inequality in the Euclidean space.

**Theorem 1**

*For every Borel set* $\Omega \subset \mathbb{R}^n$, $n \geq 2$, *with finite perimeter* $P(\Omega)$,*

$$\min\{|\Omega|, |\mathbb{R}^n \setminus \Omega|\} \leq C_{iso}(\mathbb{R}^n) P(\Omega)^{\frac{n}{n-1}},$$  \hspace{1cm} (1)

*where*

$$C_{iso}(\mathbb{R}^n) = \frac{1}{n\omega_k^{\frac{n-1}{n}}},$$

*Here,* $\omega_k$ *is the surface measure of the unit sphere* $S^k$ *in* $\mathbb{R}^{k+1}$. *Equality holds in* (1) *if and only if almost everywhere* $\Omega = B(x, R)$ *(i.e. a ball)* *for some* $x \in \mathbb{R}^n$ *and* $R > 0$. *In the case where* $\partial \Omega$ *is smooth say* $C^1$ *then* $P(\Omega)$ *coincide with surface measure of* $\partial \Omega$. *In the non-smooth case* $P(\Omega) = \text{Var}(\chi_{\Omega}, \mathbb{R}^n)$ *where* $\chi_{\Omega}$ *is the indicator function of* $\Omega$ *and*

$$\text{Var}(u) = \sup \left\{ \int_{\mathbb{R}^n} u(x) \sum_{i=1}^{n} \partial x_i G_i \, dx \mid G_i \in C^\infty(\mathbb{R}^n), G_1^2 + \cdots + C_n^2 \leq 1 \right\}$$

*is the variation of* $u$ *in* $\mathbb{R}^n$. *Note that* $\Omega$ *need not to be bounded in a priori.*
Roughly speaking, the *isoperimetric problem* consists in finding the smallest constant $C_{iso}(\mathbb{R}^n)$ and classifying sets $\Omega$ such that inequality (1) becomes an equality. This problem is equivalent to the two following formulations:

- Among all bounded, connected open sets of fixed perimeter $L$, find one with largest volume $V$.
- Among all bounded, connected open sets with fixed volume $V$, find one with smallest perimeter $L$. 

Of course, the solution (in any case anyway) was known long long time ago. However, it was not until 1841 that Jacob Steiner gave the first proof (which contain gaps) but later on completed by many mathematicians. Steiner proved that if such a region exists in the plane, it must be a circle.
Roughly speaking, the *isoperimetric problem* consists in finding the smallest constant $C_{iso}(\mathbb{R}^n)$ and classifying sets $\Omega$ such that inequality (1) becomes an equality. This problem is equivalent to the two following formulations:

- Among all bounded, connected open sets of fixed perimeter $L$, find one with largest volume $V$.
- Among all bounded, connected open sets with fixed volume $V$, find one with smallest perimeter $L$.

Of course, the solution (in $\mathbb{R}^2$ anyway) was known long long time ago. However, it was not until 1841 that Jacob Steiner gave the first proof (which contain gaps) but later on completed by many mathematicians. Steiner proved that if such a region exists in the plane, it must be a circle.
Roughly speaking, the *isoperimetric problem* consists in finding the smallest constant $C_{iso}(\mathbb{R}^n)$ and classifying sets $\Omega$ such that inequality (1) becomes an equality. This problem is equivalent to the two following formulations:

- Among all bounded, connected open sets of fixed perimeter $L$, find one with largest volume $V$.
- Among all bounded, connected open sets with fixed volume $V$, find one with smallest perimeter $L$.

Of course, the solution (in $\mathbb{R}^2$ anyway) was known long long time ago. However, it was not until 1841 that Jacob Steiner gave the first proof (which contain gaps) but later on completed by many mathematicians. Steiner proved that if such a region exists in the plane, it must be a circle.
The Euclidean Isoperimetric Problem.

The idea of his proof can be outlined in the following three steps. Assume therefore that there is a region $G$ in the plane such that among all other regions with the same perimeter of $G$, then $G$ must be a disc.

Step I: The region $G$ must be convex. For if not, using reflection, we can construct another region with the same perimeter but enclose a larger area, this contradict our assumption on $G$. 
The Euclidean Isoperimetric Problem.

The idea of his proof can be outlined in the following three steps. Assume therefore that there is a region $\mathcal{G}$ in the plane such that among all other regions with the same perimeter of $\mathcal{G}$, then $\mathcal{G}$ must be a disc.

Step I: The region $\mathcal{G}$ must be convex. For if not, using reflection, we can construct another region with the same perimeter but enclose a larger area, this contradict our assumption on $\mathcal{G}$.

Figure: Steiner’s proof, step I.
The Euclidean Isoperimetric Problem.

The idea of his proof can be outlined in the following three steps. Assume therefore that there is a region $\mathcal{G}$ in the plane such that among all other regions with the same perimeter of $\mathcal{G}$, then $\mathcal{G}$ must be a disc.

**Step I:** The region $\mathcal{G}$ must be convex. For if not, using reflection, we can construct another region with the same perimeter but enclose a larger area, this contradict our assumption on $\mathcal{G}$.

*Figure:* Steiner’s proof, step I.
Step II: Any straight line that divides the perimeter of \( \mathcal{G} \) in half must also divide the area of \( \mathcal{G} \) in half. Since \( \mathcal{G} \) is convex, each half of the bounding curve lies entirely on one side of the line through \( A \) and \( B \) (see the figures). Suppose the line \( \overline{AB} \) does not divide the area of \( \mathcal{G} \) in half, reflect the larger area across \( \overline{AB} \) to obtain another region having the same perimeter of \( \mathcal{G} \) but with a larger area. Again, we obtain a contradiction.
Step II: Any straight line that divides the perimeter of $G$ in half must also divide the area of $G$ in half. Since $G$ is convex, each half of the bounding curve lies entirely on one side of the line through $A$ and $B$ (see the figures). Suppose the line $AB$ does not divide the area of $G$ in half, reflect the larger area across $AB$ to obtain another region having the same perimeter of $G$ but with a larger area. Again, we obtain a contradiction.
Step II: Any straight line that divides the perimeter of \(G\) in half must also divide the area of \(G\) in half. Since \(G\) is convex, each half of the bounding curve lies entirely on one side of the line through \(A\) and \(B\) (see the figures). Suppose the line \(AB\) does not divide the area of \(G\) in half, reflect the larger area across \(AB\) to obtain another region having the same perimeter of \(G\) but with a larger area. Again, we obtain a contradiction.

Figure: Steiner's proof, step II.

Figure: The argument only works for a convex region.
Step III: Now we concentrate on “half of the figure”. Pick any point $C$ on this half curve and join it to $A$ and $B$ to obtain two “lunes” $AMC$ and $BNC$ and a triangle $ACB$. The angle at $C$ can be either $<\pi/2$, $=\pi/2$ or $>\pi/2$. Imagine that the lunes are made of non-deformable material and they are hinged at $C$. Now the area of the region is the area of the two lunes plus the area of the triangle. The area of the triangle is computed by the base $\overline{AC}$ and height $\overline{BD}$. By adjusting the angle at $C$ by moving the lunes, we see that the largest height is obtained when $B = D$, that is the angle at $C$ is $\pi/2$. 

Hence, from elementary geometry, the angle $ACB$ is a right angle if and only if $C$ lies on a semi-circle with diameter $\overline{AB}$. 

First Taiwan Geometry Symposium, NCTS South
The Isoperimetric Problem in the Heisenberg gro
November 20, 2010 11 / 44
Step III: Now we concentrate on “half of the figure”. Pick any point $C$ on this half curve and join it to $A$ and $B$ to obtain two “lunes” $AMC$ and $BNC$ and a triangle $ACB$. The angle at $C$ can be either $<\pi/2$, $=\pi/2$ or $>\pi/2$. Imagine that the lunes are made of non-deformable material and they are hinged at $C$. Now the area of the region is the area of the two lunes plus the area of the triangle. The area of the triangle is computed by the base $AC$ and height $BD$. By adjusting the angle at $C$ by moving the lunes, we see that the largest height is obtained when $B = D$, that is the angle at $C$ is $\pi/2$.

Hence, from elementary geometry, the angle $ACB$ is a right angle if and only if $C$ lies on a semi-circle with diameter $AB$. 

The Euclidean Isoperimetric Problem.
Step III: Now we concentrate on “half of the figure”. Pick any point $C$ on this half curve and join it to $A$ and $B$ to obtain two “lunes” $AMC$ and $BNC$ and a triangle $ACB$. The angle at $C$ can be either $<\pi/2$, $=\pi/2$ or $>\pi/2$. Imagine that the lunes are made of non-deformable material and they are hinged at $C$. Now the area of the region is the area of the two lunes plus the area of the triangle. The area of the triangle is computed by the base $AC$ and height $BD$. By adjusting the angle at $C$ by moving the lunes, we see that the largest height is obtained when $B = D$, that is the angle at $C$ is $\pi/2$.

Hence, from elementary geometry, the angle $ACB$ is a right angle if and only if $C$ lies on a semi-circle with diameter $AB$. 
The Euclidean Isoperimetric Problem.

Since Steiner’s proof there are many many proofs for the isoperimetric inequality. We present a few for the $\mathbb{R}^2$ case.

- **Proof by complex function theory.** Let $z = x + iy$ and $dA = dx \wedge dy = \frac{1}{2} idz \wedge d\overline{z}$. Using the fact that winding number of $\partial \Omega$ is one, Green and Fubini’s theorem we find

  $$4\pi A = \int_{\Omega} 2\pi i dz \wedge d\overline{z} = \int_{\Omega} \int_{\partial \Omega} \frac{d\xi}{\xi - z} d\overline{z} = \int_{\partial \Omega} \int_{\partial \Omega} \frac{\overline{\xi} - \overline{z}}{\xi - z} dz d\xi \leq L^2.$$  

- The case of equality is easy to analyze in the above. The interplay between geometric extremal problems (e.g. isoperimetric problem) and sharp analytic inequalities is witnessed in the following analytic proof of the planar isoperimetric inequality. First, let’s recall Wirtinger’s inequality. If $f$ is in the Sobolev space $W^{1,2}([0, 2\pi])$ satisfying $\int_0^{2\pi} f(t) dt = 0$ then

  $$\int_0^{2\pi} |f(t)|^2 dt \leq \int_0^{2\pi} |f'(t)|^2 dt,$$

  with equality holds only when $f(t) = A\cos(t) + B\sin(t)$. The proof of Wirtinger’s inequality is an easy exercise in Fourier series.
The Euclidean Isoperimetric Problem.

Let $ds$ denote the element of arc length and assume that $\partial \Omega$ is a Lipschitz curve which is the boundary of a domain $\Omega \subset \mathbb{R}^2$. Denote by $x = (x_1, x_2)$ the position vector. By translating the region $\Omega$ which preserves area and perimeter, we may assume that $\int_{\partial \Omega} x ds = 0$. The divergence theorem and Wirtinger’s inequality applied to $x_1$, $x_2$ then gives

$$2A = \int_{\Omega} \text{div}(x) \, dA = \int_{\partial \Omega} < x, \hat{n} > \, ds \leq \int_{\partial \Omega} |x| \, ds \leq \sqrt{L} \left( \int_{\partial \Omega} |x|^2 \, ds \right)^{\frac{1}{2}}$$

$$\leq \sqrt{L} \left[ \left( \frac{L}{2\pi} \right)^2 \int_{\partial \Omega} \left| \frac{dx}{ds} \right|^2 \, ds \right]^{\frac{1}{2}} \leq \frac{L^2}{2\pi}.$$ 

Equality holds if and only if $\Omega$ is a disc.
The Euclidean Isoperimetric Problem.

- The proof for higher dimension Euclidean spaces are more technical and we will not recall them here. However a few remarks should be made.
- The case of $\mathbb{R}^3$ was proved by Schwarz in 1884. His argument can be described in two steps:
The Euclidean Isoperimetric Problem.

- The proof for higher dimension Euclidean spaces are more technical and we will not recall them here. However a few remarks should be made.

- The case of $\mathbb{R}^3$ was proved by Schwarz in 1884. His argument can be described in two steps:
  - A symmetrization process (known nowadays as Schwarz symmetrization) that reduce the problem to finding the solution among a class of rotationally symmetric objects.
The Euclidean Isoperimetric Problem.

- The proof for higher dimension Euclidean spaces are more technical and we will not recall them here. However a few remarks should be made.

- The case of $\mathbb{R}^3$ was proved by Schwarz in 1884. His argument can be described in two steps:
  - A symmetrization process (known nowadays as Schwarz symmetrization) that reduce the problem to finding the solution among a class of rotationally symmetric objects.
  - A geometric construction to rule out rotationally symmetric candidates different from the sphere.
The Euclidean Isoperimetric Problem.

- The proof for higher dimension Euclidean spaces are more technical and we will not recall them here. However a few remarks should be made.
- The case of $\mathbb{R}^3$ was proved by Schwarz in 1884. His argument can be described in two steps:
  - A symmetrization process (known nowadays as Schwarz symmetrization) that reduce the problem to finding the solution among a class of rotationally symmetric objects.
  - A geometric construction to rule out rotationally symmetric candidates different from the sphere.
- The full generality of Theorem 1 was established by de Giorgi in 1958.
The Euclidean Isoperimetric Problem.

- The proof for higher dimension Euclidean spaces are more technical and we will not recall them here. However a few remarks should be made.
- The case of $\mathbb{R}^3$ was proved by Schwarz in 1884. His argument can be described in two steps:
  - A symmetrization process (known nowadays as Schwarz symmetrization) that reduce the problem to finding the solution among a class of rotationally symmetric objects.
  - A geometric construction to rule out rotationally symmetric candidates different from the sphere.
- The full generality of Theorem 1 was established by de Giorgi in 1958.
- There are many proofs of the Euclidean isoperimetric problem. There are also analogues of the same type of problem an in different settings, i.e., Riemannian manifolds. Our goal is to investigate the Sub-Riemannian counterpart of this problem and we start with the Heisenberg group.
The Euclidean Isoperimetric Problem.

- The proof for higher dimension Euclidean spaces are more technical and we will not recall them here. However a few remarks should be made.
- The case of $\mathbb{R}^3$ was proved by Schwarz in 1884. His argument can be described in two steps:
  - A symmetrization process (known nowadays as Schwarz symmetrization) that reduce the problem to finding the solution among a class of rotationally symmetric objects.
  - A geometric construction to rule out rotationally symmetric candidates different from the sphere.
- The full generality of Theorem 1 was established by de Giorgi in 1958.
- There are many proofs of the Euclidean isoperimetric problem. There are also analogues of the same type of problem an in different settings, i.e., Riemannian manifolds. Our goal is to investigate the Sub-Riemannian counterpart of this problem and we start with the Heisenberg group.
The Heisenberg group $\mathbb{H}^n$ and relevant concept/quantities.

- The Heisenberg group $\mathbb{H}^n$ is a Lie group on $\mathbb{R}^{2n+1}$ with the following group law:

$$(x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(x' \cdot y - y' \cdot x)).$$

where $x = (x_1, ..., x_n), y = (y_1, ..., y_n), t \in \mathbb{R}$ and the dot product is the standard dot product on Euclidean spaces.

- Associated to this group law we work with the following standard left-invariant vector fields: $(i = 1..n)$

$$X_i = \frac{\partial}{\partial x_i} - \frac{y_i}{2} \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} + \frac{x_i}{2} \frac{\partial}{\partial t}, \quad T = [X_i, Y_i] = \frac{\partial}{\partial t},$$

together with an inner product $\langle , \rangle$ with respect to which these $2n+1$ vector fields form an orthonormal system. $\mathbb{H}^n$ equipped with $\langle , \rangle$ is then a Riemannian manifold.
The Heisenberg group $\mathbb{H}^n$ and relevant concept/quantities.

- The Heisenberg group $\mathbb{H}^n$ is a Lie group on $\mathbb{R}^{2n+1}$ with the following group law:

\[(x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(x' \cdot y - y' \cdot x))\]

where $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$, $t \in \mathbb{R}$ and the dot product is the standard dot product on Euclidean spaces.

- Associated to this group law we work with the following standard left-invariant vector fields: ($i = 1..n$)

\[X_i = \frac{\partial}{\partial x_i} - \frac{y_i}{2} \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} + \frac{x_i}{2} \frac{\partial}{\partial t}, \quad T = [X_i, Y_i] = \frac{\partial}{\partial t},\]

Together with an inner product $\langle , \rangle$ with respect to which these $2n+1$ vector fields form an orthonormal system. $\mathbb{H}^n$ equipped with $\langle , \rangle$ is then a Riemannian manifold.

- The Lebesgue measure on $\mathbb{R}^{2n+1}$ is both left and right translation measure and therefore a Haar measure on $\mathbb{H}^n$. For sets $E \subset \mathbb{H}^n$, the volume of $E$ is the Lebesgue measure of $E$ and will be denoted by $|E|$. 

First Taiwan Geometry Symposium, NCTS South
The Isoperimetric Problem in the Heisenberg group
The Heisenberg group $\mathbb{H}^n$ and relevant concept/quantities.

- The Heisenberg group $\mathbb{H}^n$ is a Lie group on $\mathbb{R}^{2n+1}$ with the following group law:

$$(x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(x' \cdot y - y' \cdot x)).$$

where $x = (x_1, ..., x_n), y = (y_1, ..., y_n), t \in \mathbb{R}$ and the dot product is the standard dot product on Euclidean spaces.

- Associated to this group law we work with the following standard left-invariant vector fields: ($i = 1..n$)

$$X_i = \frac{\partial}{\partial x_i} - \frac{y_i}{2} \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} + \frac{x_i}{2} \frac{\partial}{\partial t}, \quad T = [X_i, Y_i] = \frac{\partial}{\partial t},$$

together with an inner product $<, >$ with respect to which these $2n + 1$ vector fields form an orthonormal system. $\mathbb{H}^n$ equipped with $<, >$ is then a Riemannian manifold.

- The Lebesgue measure on $\mathbb{R}^{2n+1}$ is both left and right translation measure and therefore a Haar measure on $\mathbb{H}^n$. For sets $E \subset \mathbb{H}^n$, the volume of $E$ is the Lebesgue measure of $E$ and will be denoted by $|E|$.
The Heisenberg group $\mathbb{H}^n$ and relevant concept/quantities.

- Given an oriented $C^2$ embedded hypersurface $\mathcal{S} \subset \mathbb{H}^n$ (and after an orientation is chosen) we let $N$ be the Riemannian unit normal to $\mathcal{S}$ and we write $N = \sum_{i=1}^{n} (p_i X_i + q_i Y_i) + \omega T$
- The projection of $N$ onto the horizontal plane $\text{span}\{X_i, Y_i\mid i = 1, \ldots, n\}$ at each point $g \in \mathcal{S}$ is called the horizontal normal and is denoted by $N_H = \sum_{i=1}^{n} p_i X_i + q_i Y_i$. 
Given an oriented $C^2$ embedded hypersurface $\mathcal{S} \subset \mathbb{H}^n$ (and after an orientation is chosen) we let $N$ be the Riemannian unit normal to $\mathcal{S}$ and we write $N = \sum_{i=1}^n (p_i X_i + q_i Y_i) + \omega T$.

The projection of $N$ onto the horizontal plane $\text{span}\{X_i, Y_i\mid i = 1, \ldots, n\}$ at each point $g \in \mathcal{S}$ is called the horizontal normal and is denoted by $N_H = \sum_{i=1}^n p_i X_i + q_i Y_i$.

The set

$$\Sigma_{\mathcal{S}} \overset{\text{def}}{=} \{g \in \mathcal{S} \mid N_H(g) = 0\}$$

is called the characteristic set (singular set by some authors) of the hypersurface $\mathcal{S}$. For our purpose, it suffices to know that for any $C^2$ hypersurface $\mathcal{S}$ we have $\sigma(\Sigma_{\mathcal{S}}) = 0$ where $d\sigma$ is the Riemannian volume on $\mathcal{S}$. 

---

The Isoperimetric Problem in the Heisenberg group

---

November 20, 2010 16 / 44
The Heisenberg group $H^n$ and relevant concept/quantities.

- Given an oriented $C^2$ embedded hypersurface $\mathcal{S} \subset H^n$ (and after an orientation is chosen) we let $N$ be the Riemannian unit normal to $\mathcal{S}$ and we write $N = \sum_{i=1}^{n} (p_i X_i + q_i Y_i) + \omega T$
- The projection of $N$ onto the horizontal plane $\text{span}\{X_i, Y_i| i = 1, \ldots, n\}$ at each point $g \in \mathcal{S}$ is called the *horizontal normal* and is denoted by $N_H = \sum_{i=1}^{n} p_i X_i + q_i Y_i$.
- The set
  $$\Sigma_{\mathcal{S}} \overset{\text{def}}{=} \{ g \in \mathcal{S} | N_H(g) = 0 \}$$
  is called the *characteristic set* (*singular set* by some authors) of the hypersurface $\mathcal{S}$. For our purpose, it suffices to know that for any $C^2$ hypersurface $\mathcal{S}$ we have $\sigma(\Sigma_{\mathcal{S}}) = 0$ where $d\sigma$ is the Riemannian volume on $\mathcal{S}$.
- For points $g \notin \Sigma_{\mathcal{S}}$ we define the *horizontal Gauss map* (the horizontal unit normal) by setting
  $$\nu_H(g) = \frac{N_H(g)}{|N_H(g)|} = \sum_{i=1}^{n} \bar{p}_i X_i + \bar{q}_i Y_i.$$
The Heisenberg group $\mathbb{H}^n$ and relevant concept/quantities.

- Given an oriented $C^2$ embedded hypersurface $\mathcal{S} \subset \mathbb{H}^n$ (and after an orientation is chosen) we let $N$ be the Riemannian unit normal to $\mathcal{S}$ and we write $N = \sum_{i=1}^{n} (p_i X_i + q_i Y_i) + \omega T$.
- The projection of $N$ onto the horizontal plane $\text{span}\{X_i, Y_i \mid i = 1, \ldots, n\}$ at each point $g \in \mathcal{S}$ is called the horizontal normal and is denoted by $N_H = \sum_{i=1}^{n} p_i X_i + q_i Y_i$.
- The set
  \[ \Sigma_{\mathcal{S}} \overset{\text{def}}{=} \{ g \in \mathcal{S} \mid N_H(g) = 0 \} \]
  is called the characteristic set (singular set by some authors) of the hypersurface $\mathcal{S}$. For our purpose, it suffices to know that for any $C^2$ hypersurface $\mathcal{S}$ we have $\sigma(\Sigma_{\mathcal{S}}) = 0$ where $d\sigma$ is the Riemannian volume on $\mathcal{S}$.
- For points $g \notin \Sigma_{\mathcal{S}}$ we define the horizontal Gauss map (the horizontal unit normal) by setting
  \[ \nu_H(g) = \frac{N_H(g)}{|N_H(g)|} = \sum_{i=1}^{n} \overline{p}_i X_i + \overline{q}_i Y_i. \]
- The vector field $\nu_H$ plays an important role in the analysis of hypersurfaces in this setting.
The Heisenberg group $\mathbb{H}^n$ and relevant concept/quantities.

- Given an oriented $C^2$ embedded hypersurface $\mathcal{S} \subset \mathbb{H}^n$ (and after an orientation is chosen) we let $N$ be the Riemannian unit normal to $\mathcal{S}$ and we write $N = \sum_{i=1}^n (p_i X_i + q_i Y_i) + \omega T$
- The projection of $N$ onto the horizontal plane $\text{span}\{X_i, Y_i | i = 1,..,n\}$ at each point $g \in \mathcal{S}$ is called the *horizontal normal* and is denoted by $N_H = \sum_{i=1}^n p_i X_i + q_i Y_i$.
- The set
  \[ \Sigma_{\mathcal{S}} \overset{\text{def}}{=} \{ g \in \mathcal{S} | N_H(g) = 0 \} \]
  is called the *characteristic set* (*singular set* by some authors) of the hypersurface $\mathcal{S}$. For our purpose, it suffices to know that for any $C^2$ hypersurface $\mathcal{S}$ we have $\sigma(\Sigma_{\mathcal{S}}) = 0$ where $d\sigma$ is the Riemannian volume on $\mathcal{S}$.
- For points $g \notin \Sigma_{\mathcal{S}}$ we define the *horizontal Gauss map* (*the horizontal unit normal*) by setting
  \[ \nu_H(g) = \frac{N_H(g)}{|N_H(g)|} = \sum_{i=1}^n \bar{p}_i X_i + \bar{q}_i Y_i. \]
- The vector field $\nu_H$ plays an important role in the analysis of hypersurfaces in this setting.
...relevant concept... Recall $H$-Gauss map $\nu_H = \sum_{i=1}^{n} \overline{p}_i X_i + \overline{q}_i Y_i$.

An important geometric quantity that stems from $\nu_H$ is the so-called Horizontal mean curvature (H-mean curvature hereafter) $\mathcal{H}$.

- For $g_0 \in \mathcal{I} \setminus \Sigma\mathcal{I}$ define

$$\mathcal{H}(g_0) = \text{div}_H(\nu_H) = \sum_{i=1}^{n} X_i \overline{p}_i + Y_i \overline{q}_i,$$

and if $g_0 \in \Sigma\mathcal{I}$ \( \mathcal{H}(g_0) = \lim_{g \in \mathcal{I} \setminus \Sigma\mathcal{I}, g \to g_0} \mathcal{H}(g) \)

provided the limit exists.

- If $E \subset \mathcal{I}$ is a smooth portion of $\mathcal{I}$, the $H$-surface measure (or $H$-area) of $E$ is the quantity

$$\sigma_H(E) = \int_{E} |N_H(g)| \, d\sigma(g). \quad \text{recall} \ d\sigma = \text{Riemannian volume}$$
An important geometric quantity that stems from $\nu_H$ is the so-called Horizontal mean curvature (H-mean curvature hereafter) $\mathcal{H}$.

- For $g_0 \in \mathcal{I} \setminus \Sigma \mathcal{F}$ define
  \[
  \mathcal{H}(g_0) = div_H(\nu_H) = \sum_{i=1}^{n} X_i \bar{p}_i + Y_i \bar{q}_i ,
  \]
  and if $g_0 \in \Sigma \mathcal{F}$, $\mathcal{H}(g_0) = \lim_{g \in \mathcal{I} \setminus \Sigma \mathcal{F}, g \to g_0} \mathcal{H}(g)$ provided the limit exists.

- If $E \subset \mathcal{I}$ is a smooth portion of $\mathcal{I}$, the $H$-surface measure (or $H$-area) of $E$ is the quantity
  \[
  \sigma_H(E) = \int_{E} |N_H(g)| \, d\sigma(g) . \text{ recall } d\sigma = \text{Riemannian volume}
  \]

- However, the notion of $H$-surface measure can be extended to non-smooth sets as follows.
...relevant concept... Recall \( H \)-Gauss map \( \nu_H = \sum_{i=1}^{n} \overline{p}_i X_i + \overline{q}_i Y_i \).

An important geometric quantity that stems from \( \nu_H \) is the so called Horizontal mean curvature (H-mean curvature hereafter) \( \mathcal{H} \).

- For \( g_0 \in \mathcal{I} \setminus \Sigma \mathcal{F} \) define

\[
\mathcal{H}(g_0) = \text{div}_H(\nu_H) = \sum_{i=1}^{n} X_i \overline{p}_i + Y_i \overline{q}_i ,
\]

and if \( g_0 \in \Sigma \mathcal{F} \) \( \mathcal{H}(g_0) = \lim_{g \in \mathcal{I} \setminus \Sigma \mathcal{F}, g \to g_0} \mathcal{H}(g) \)

provided the limit exists.

- If \( E \subset \mathcal{I} \) is a smooth portion of \( \mathcal{I} \), the \textit{H-surface measure} (or \textit{H-area}) of \( E \) is the quantity

\[
\sigma_H(E) = \int_E |N_H(g)| \, d\sigma(g) . \quad \text{recall } d\sigma = \text{Riemannian volume}
\]

- However, the notion of \textit{H-surface measure} can be extended to non-smooth sets as follows.
For $u \in L^1_{loc}(\mathbb{H}^n, dg)$ where $dg$ is the Lebesgue measure on $\mathbb{H}^n$ we define the Horizontal variation of $u$ by

$$Var_H(u) = \sup \left\{ \int_{\mathbb{H}^n} u \sum_{i=1}^{n} X_i \xi_i + Y_i \eta_i \, dg \div \sum_{i=1}^{n} \xi_i^2 + \eta_i^2 \leq 1, \xi_i, \eta_i \in C_\infty(\mathbb{H}^n), i = 1..n \right\}$$

If $u \in L^1(\mathbb{H}^n, dg)$ is such that $Var_H(u) < \infty$ we say that $u$ is of bounded H-variation. For any sets $E \subset \mathbb{H}^n$, we define the horizontal perimeter of $E$ by

$$P_H(E) = Var_H(\chi_E) \quad \text{where} \quad \chi_E \text{ is the indicator function of } E.$$

If $P_H(E) < \infty$ we say that $E$ is of finite H-perimeter. We note here that if $E$ is smooth say ($\partial E$ is) $C^1$, then $P_H(E) = \sigma_H(\partial E)$. 
The Heisenberg group $\mathbb{H}^n$ and relevant concept/quantities.

For $u \in L^1_{\text{loc}}(\mathbb{H}^n, dg)$ where $dg$ is the Lebesgue measure on $\mathbb{H}^n$ we define the Horizontal variation of $u$ by

$$Var_H(u) = \sup \left\{ \int_{\mathbb{H}^n} u \sum_{i=1}^n X_i \xi_i + Y_i \eta_i \, dg \left| \sum_{i=1}^n \xi_i^2 + \eta_i^2 \leq 1, \xi_i, \eta_i \in C^\infty(\mathbb{H}^n), i = 1..n \right. \right\}$$

If $u \in L^1(\mathbb{H}^n, dg)$ is such that $Var_H(u) < \infty$ we say that $u$ is of bounded H-variation. For any sets $E \subset \mathbb{H}^n$, we define the horizontal perimeter of $E$ by

$$P_H(E) = Var_H(\chi_E) \quad \text{where } \chi_E \text{ is the indicator function of } E.$$ 

If $P_H(E) < \infty$ we say that $E$ is of finite H-perimeter. We note here that if $E$ is smooth say ($\partial E$ is) $C^1$, then $P_H(E) = \sigma_H(\partial E)$.
The Heisenberg group $\mathbb{H}^n$ and relevant concept/quantities.

Homogeneous structure on $\mathbb{H}^n$.

- For $\lambda \neq 0$, define a family of dilations on $\mathbb{H}^n$ to be a function $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$ given by $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$. Here $x, y \in \mathbb{R}^n, t \in \mathbb{R}$. 
Homogeneous structure on $\mathbb{H}^n$.

- For $\lambda \neq 0$, define a family of dilations on $\mathbb{H}^n$ to be a function $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$ given by $\delta_\lambda (x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$. Here $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$.

- For a fixed $g_0 \in \mathbb{H}^n$, the left translation is the map $\tau_{g_0} : \mathbb{H}^n \to \mathbb{H}^n$ given by $\tau_{g_0} (g) = g_0 \circ g$ where $\circ$ is the group law on $\mathbb{H}^n$. 

We note here that both the Lebesgue measure and the H-perimeter behave nicely with respect to left translation and dilation: For any measurable set $E \subset \mathbb{H}^n$:

- For any $g_0 \in \mathbb{H}^n$, $|\tau_{g_0}(E)| = |E|$, $P_{\mathbb{H}}(\tau_{g_0}(E)) = P_{\mathbb{H}}(E)$ for any $\lambda > 0$,
- $|\delta_\lambda(E)| = \lambda^Q |E|$, $P_{\mathbb{H}}(\delta_\lambda(E)) = \lambda^{Q-1} P_{\mathbb{H}}(E)$.

In the above, the number $Q = 2n + 2$ is called the homogeneous dimension of $\mathbb{H}^n$. 

The Heisenberg group $\mathbb{H}^n$ and relevant concept/quantities.

Homogeneous structure on $\mathbb{H}^n$.

- For $\lambda \neq 0$, define a family of dilations on $\mathbb{H}^n$ to be a function $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$ given by $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$. Here $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$.
- For a fixed $g_o \in \mathbb{H}^n$, the left translation is the map $\tau_{g_o} : \mathbb{H}^n \to \mathbb{H}^n$ given by $\tau_{g_o}(g) = g_o \circ g$ where $\circ$ is the group law on $\mathbb{H}^n$.
- We note here that both the Lebesgue measure and the $H$-perimeter behave nicely with respect to left translation and dilation: For any measurable set $E \subset \mathbb{H}^n$:

  for any $g_o \in \mathbb{H}^n$, $|\tau_{g_o}(E)| = |E|$, \quad $P_H(\tau_{g_o}(E)) = P_H(E)$

  for any $\lambda > 0$, $|\delta_\lambda(E)| = \lambda^Q |E|$, \quad $P_H(\delta_\lambda(E)) = \lambda^{Q-1} P_H(E)$.

In the above, the number $Q = 2n + 2$ is called the homogenous dimension of $\mathbb{H}^n$. 
The Heisenberg group $\mathbb{H}^n$ and relevant concept/quantities.

Homogeneous structure on $\mathbb{H}^n$.

- For $\lambda \neq 0$, define a family of dilations on $\mathbb{H}^n$ to be a function $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$ given by $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$. Here $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$.
- For a fixed $g_o \in \mathbb{H}^n$, the left translation is the map $\tau_{g_o} : \mathbb{H}^n \to \mathbb{H}^n$ given by $\tau_{g_o}(g) = g_o \circ g$ where $\circ$ is the group law on $\mathbb{H}^n$.
- We note here that both the Lebesgue measure and the H-perimeter behave nicely with respect to left translation and dilation: For any measurable set $E \subset \mathbb{H}^n$:

$$
\text{for any } g_o \in \mathbb{H}^n, \ |\tau_{g_o}(E)| = |E|, \quad P_H(\tau_{g_o}(E)) = P_H(E)
$$

$$
\text{for any } \lambda > 0, \ |\delta_\lambda(E)| = \lambda^Q|E|, \quad P_H(\delta_\lambda(E)) = \lambda^{Q-1} P_H(E).
$$

In the above, the number $Q = 2n + 2$ is called the homogenous dimension of $\mathbb{H}^n$. 
The Heisenberg group $\mathbb{H}^n$ and relevant concept/quantities.

Homogenous norm and distances on $\mathbb{H}^n$

- One can define a norm on $\mathbb{H}^n$ as follow. For $g = (x, y, t) \in \mathbb{H}^n$, $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$, we let $||g||_{\mathbb{H}^n} = ((|x|^2 + |y|^2)^2 + 16t^2)^{\frac{1}{4}}$. The fact that $||\cdot||_{\mathbb{H}^n}$ is a norm is proved by (1981). We also have $||\delta_\lambda (g)||_{\mathbb{H}^n} = |\lambda|||g||_{\mathbb{H}^n}$.

- With this homogeneous norm we define a distance on $\mathbb{H}^n$ as follows

$$d_{\mathbb{H}^n}(g, g') = ||g^{-1} \circ g'||_{\mathbb{H}^n}$$

where $g^{-1} = (-x, -y, -t)$ is the inverse of $g$ with respect to $\circ$. 

- A theorem due to W. L. Chow in 1939 guarantee that any two points $g, g' \in \mathbb{H}^n$ can be connected by a horizontal curve (actually, in $\mathbb{H}^n$ this fact can be easily established alone) and hence $d_{\mathbb{H}^n}(g, g')$ is well defined.
The Heisenberg group $\mathbb{H}^n$ and relevant concept/quantities.

Homogenous norm and distances on $\mathbb{H}^n$

- One can define a norm on $\mathbb{H}^n$ as follow. For $g = (x, y, t) \in \mathbb{H}^n$, $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$, we let $||g||_{\mathbb{H}^n} = ((|x|^2 + |y|^2)^2 + 16t^2)^{\frac{1}{4}}$. The fact that $||\cdot||_{\mathbb{H}^n}$ is a norm is proved by (1981). We also have $||\delta_\lambda(g)||_{\mathbb{H}^n} = |\lambda||g||_{\mathbb{H}^n}$.

- With this homogeneous norm we define a distance on $\mathbb{H}^n$ as follows

$$d_{\mathbb{H}^n}(g, g') = ||g^{-1} \circ g'||_{\mathbb{H}^n}$$

where $g^{-1} = (-x, -y, -t)$ is the inverse of $g$ with respect to $\circ$.

- There is however another distance from the Sub-Riemannian point of view. For any piecewise $C^1$ curve $\gamma : [a, b] \to \mathbb{H}^n$, we say that $\gamma$ is a horizontal curve if whenever $\gamma'(s)$ is defined then at the point $\gamma(s) \in \mathbb{H}^n$ we have $\gamma'(s) \in \text{Span}\{X_1, \ldots, Y_n\}$. 
The Heisenberg group $\mathbb{H}^n$ and relevant concept/quantities.

Homogenous norm and distances on $\mathbb{H}^n$

- One can define a norm on $\mathbb{H}^n$ as follow. For $g = (x, y, t) \in \mathbb{H}^n$, $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$, we let $||g||_{\mathbb{H}^n} = ((|x|^2 + |y|^2 + 16t^2)^{1/4}$. The fact that $|\cdot|_{\mathbb{H}^n}$ is a norm is proved by (1981). We also have $||\delta_\lambda(g)||_{\mathbb{H}^n} = |\lambda||g||_{\mathbb{H}^n}$.

- With this homogeneous norm we define a distance on $\mathbb{H}^n$ as follows

$$
\delta_{H^n}(g, g') = ||g^{-1} \circ g'||_{\mathbb{H}^n} \text{ where } g^{-1} = (-x, -y, -t) \text{ is the inverse of } g \text{ with respect to } \circ.
$$

- There is however another distance from the Sub-Riemannian point of view. For any piecewise $C^1$ curve $\gamma : [a, b] \to \mathbb{H}^n$, we say that $\gamma$ is a horizontal curve if whenever $\gamma'(s)$ is defined then at the point $\gamma(s) \in \mathbb{H}^n$ we have $\gamma'(s) \in \text{Span}\{X_1, \ldots, Y_n\}$.

- For any horizontal curve, its horizontal length is then

$$
l_H(\gamma) = \int_a^b <\gamma'(s), \gamma'(s)>^{1/2} \, ds
$$
The Heisenberg group $\mathbb{H}^n$ and relevant concept/quantities.

Homogenous norm and distances on $\mathbb{H}^n$

- One can define a norm on $\mathbb{H}^n$ as follows. For $g = (x, y, t) \in \mathbb{H}^n$, $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$, we let $||g||_{\mathbb{H}^n} = ((|x|^2 + |y|^2) + 16t^2)^{\frac{1}{4}}$. The fact that $||\cdot||_{\mathbb{H}^n}$ is a norm is proved by (1981). We also have $||\delta \lambda (g)||_{\mathbb{H}^n} = |\lambda|||g||_{\mathbb{H}^n}$.

- With this homogeneous norm we define a distance on $\mathbb{H}^n$ as follows

$$d_{\mathbb{H}^n}(g, g') = ||g^{-1} \circ g'||_{\mathbb{H}^n} \text{ where } g^{-1} = (-x, -y, -t) \text{ is the inverse of } g \text{ with respect to } \circ.$$

- There is however another distance from the Sub-Riemannian point of view. For any piecewise $C^1$ curve $\gamma : [a, b] \to \mathbb{H}^n$, we say that $\gamma$ is a horizontal curve if whenever $\gamma'(s)$ is defined then at the point $\gamma(s) \in \mathbb{H}^n$ we have $\gamma'(s) \in \text{Span}\{X_1, ..., Y_n\}$.

- For any horizontal curve, its horizontal length is then

$$l_H(\gamma) = \int_a^b <\gamma'(s), \gamma'(s)>^{\frac{1}{2}} \, ds$$

- For any $g, g' \in \mathbb{H}^n$ we define the Carnot-Caratheodory distance between $g, g'$ as

$$d_{\text{CC}}(g, g') = \inf\{l_H(\gamma) \mid \gamma \text{ is an horizontal curve joining } g \text{ and } g'\}.$$
The Heisenberg group $\mathbb{H}^n$ and relevant concept/quantities.

Homogenous norm and distances on $\mathbb{H}^n$

- One can define a norm on $\mathbb{H}^n$ as follow. For $g = (x, y, t) \in \mathbb{H}^n$, $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$, we let $||g||_{\mathbb{H}^n} = ((|x|^2 + |y|^2)^2 + 16t^2)^{1/4}$. The fact that $||\cdot||_{\mathbb{H}^n}$ is a norm is proved by (1981). We also have $||\delta_{\lambda}(g)||_{\mathbb{H}^n} = |\lambda|||g||_{\mathbb{H}^n}$.

- With this homogeneous norm we define a distance on $\mathbb{H}^n$ as follows

$$d_{\mathbb{H}^n}(g, g') = ||g^{-1} \circ g'||_{\mathbb{H}^n} \quad \text{where } g^{-1} = (-x, -y, -t) \text{ is the inverse of } g \text{ with respect to } \circ.$$ 

- There is however another distance from the Sub-Riemannian point of view. For any piecewise $C^1$ curve $\gamma : [a, b] \to \mathbb{H}^n$, we say that $\gamma$ is a horizontal curve if whenever $\gamma'(s)$ is defined then at the point $\gamma(s) \in \mathbb{H}^n$ we have $\gamma'(s) \in \text{Span}\{X_1, ..., Y_n\}$.

- For any horizontal curve, its horizontal length is then

$$l_H(\gamma) = \int_a^b <\gamma'(s), \gamma'(s)>^{1/2} \, ds$$

- For any $g, g' \in \mathbb{H}^n$ we define the Carnot-Caratheodory distance between $g, g'$ as

$$d_{CC}(g, g') = \inf \{l_H(\gamma) \mid \gamma \text{ is an horizontal curve joining } g \text{ and } g'.\}$$

- A theorem due to W.L. Chow in 1939 guarantee that any two points $g, g' \in \mathbb{H}^n$ can be connected by a horizontal curve (actually, in $\mathbb{H}^n$ this fact can be easily established alone) and hence $d_{CC}(g, g')$ is well defined.
The Heisenberg group $\mathbb{H}^n$ and relevant concept/quantities.

Homogenous norm and distances on $\mathbb{H}^n$

- One can define a norm on $\mathbb{H}^n$ as follow. For $g = (x, y, t) \in \mathbb{H}^n$, $x, y \in \mathbb{R}^n, t \in \mathbb{R}$ and $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$, we let $||g||_{\mathbb{H}^n} = ((|x|^2 + |y|^2)^2 + 16t^2)^{\frac{1}{4}}$. The fact that $||\cdot||_{\mathbb{H}^n}$ is a norm is proved by (1981). We also have $||\delta_{\lambda}(g)||_{\mathbb{H}^n} = |\lambda|||g||_{\mathbb{H}^n}$.
- With this homogeneous norm we define a distance on $\mathbb{H}^n$ as follows
  \[
  d_{\mathbb{H}^n}(g, g') = ||g^{-1} \circ g'||_{\mathbb{H}^n}
  \]
  where $g^{-1} = (-x, -y, -t)$ is the inverse of $g$ with respect to $\circ$.

- There is however another distance from the Sub-Riemannian point of view. For any piecewise $C^1$ curve $\gamma : [a, b] \to \mathbb{H}^n$, we say that $\gamma$ is a horizontal curve if whenever $\gamma'(s)$ is defined then at the point $\gamma(s) \in \mathbb{H}^n$ we have $\gamma'(s) \in \text{Span}\{X_1, \ldots, Y_n\}$.
- For any horizontal curve, its horizontal length is then
  \[
  l_H(\gamma) = \int_a^b <\gamma'(s), \gamma'(s)>^{\frac{1}{2}} \, ds
  \]
- For any $g, g' \in \mathbb{H}^n$ we define the Carnot-Caratheodory distance between $g, g'$ as
  \[
  d_{CC}(g, g') = \inf\{l_H(\gamma) \mid \gamma \text{ is an horizontal curve joining } g \text{ and } g'\}.
  \]
- A theorem due to W.L. Chow in 1939 guarantee that any two points $g, g' \in \mathbb{H}^n$ can be connected by a horizontal curve (actually, in $\mathbb{H}^n$ this fact can be easily established alone) and hence $d_{CC}(g, g')$ is well defined.
The Heisenberg group $\mathbb{H}^n$ and relevant concept/quantities.

- If $d$ denotes either $d_{CC}$ or $d_{\mathbb{H}^n}$, then it is easy to verify that $d$ is translation invariant and homogeneous of degree one, i.e.

  $$d(g'' \circ g, g'' \circ g') = d(g, g') \quad \text{and} \quad d(\delta_\lambda(g), \delta_\lambda(g')) = |\lambda| d(g, g').$$

- The two distances are comparable in the sense that there exist constants $c = c(\mathbb{H}^n), C = C(\mathbb{H}^n)$ such that for all $g, g' \in \mathbb{H}^n$ we have $c d_{\mathbb{H}^n}(g, g') \leq d_{CC}(g, g') \leq C d_{\mathbb{H}^n}(g, g')$. However, the shape of balls with respect to these two distances are quite different as we will see.

- Now we can consider the Isoperimetric problem in the Heisenberg group.
The Heisenberg group $\mathbb{H}^n$ and relevant concept/quantities.

- If $d$ denotes either $d_{CC}$ or $d_{\mathbb{H}^n}$, then it is easy to verify that $d$ is translation invariant and homogeneous of degree one, i.e.

$$d(g'' \circ g, g'' \circ g') = d(g, g') \quad \text{and} \quad d(\delta_\lambda(g), \delta_\lambda(g')) = |\lambda| d(g, g').$$

- The two distances are comparable in the sense that there exist constants $c = c(\mathbb{H}^n)$, $C = C(\mathbb{H}^n)$ such that for all $g, g' \in \mathbb{H}^n$ we have $c d_{\mathbb{H}^n}(g, g') \leq d_{CC}(g, g') \leq C d_{\mathbb{H}^n}(g, g')$. However, the shape of balls with respect to these two distances are quite different as we will see.

- Now we can consider the Isoperimetric problem in the Heisenberg group.
Pansu’s Isoperimetric inequality and conjecture.

The Isoperimetric problem began with Pansu’s work in 1982. Using an idea of Croke, Pansu established the following Isoperimetric inequality in $\mathbb{H}^1$: There exist a constant $C > 0$ so that $|\Omega|^\frac{3}{2} \leq CP_H(\Omega)$ for any bounded open set $\Omega \subset \mathbb{H}^1$ with $C^1$ boundary. To state Pansu’s conjecture, we introduce the following:

**Definition 2**

The isoperimetric constant of the Heisenberg group $\mathbb{H}^n$ is the best constant for which the isoperimetric inequality $\min \{ |\Omega|^{\frac{Q-1}{Q}}, |\mathbb{H}^1 \setminus \Omega|^{\frac{Q-1}{Q}} \} \leq C_{iso}(\mathbb{H}^n)P_H(\Omega)$, that is

$$C_{iso}(\mathbb{H}^n) = \sup \left\{ \frac{\min \{ |\Omega|^{\frac{Q-1}{Q}}, |\mathbb{H}^n \setminus \Omega|^{\frac{Q-1}{Q}} \}}{P_H(\Omega)} \left| 0 < P_H(\Omega) < \infty \right. \right\},$$

An isoperimetric profile of parameter $V > 0$ for $\mathbb{H}^n$ consists of a family of bounded sets $\mathcal{B}_V$ with $|\mathcal{B}_V| = V$ and $|\mathcal{B}_V|^{\frac{Q-1}{Q}} = C_{iso}(\mathbb{H}^n)P_H(\mathcal{B}_V)$. Since $|\cdot|$ and $P_H(\cdot)$ are invariant under left translation and scaling property hold for them, the class of isoperimetric profile is preserved under left translation and group dilation.
Pansu’s Isoperimetric inequality and conjecture.

- The Isoperimetric problem began with Pansu’s work in 1982. Using an idea of Croke, Pansu established the following Isoperimetric inequality in $\mathbb{H}^1$: There exist a constant $C > 0$ so that $|\Omega|^{\frac{3}{4}} \leq CP_H(\Omega)$ for any bounded open set $\Omega \subset \mathbb{H}^1$ with $C^1$ boundary. To state Pansu’s conjecture, we introduce the following

**Definition 2**

The isoperimetric constant of the Heisenberg group $\mathbb{H}^n$ is the best constant for which the isoperimetric inequality $\min\{ |\Omega|^\frac{Q-1}{Q}, |\mathbb{H}^n \setminus \Omega|^\frac{Q-1}{Q} \} \leq C_{iso}(\mathbb{H}^n) P_H(\Omega)$, that is

$$C_{iso}(\mathbb{H}^n) = \sup \left\{ \min\left\{ \frac{|\Omega|^\frac{Q-1}{Q}}{P_H(\Omega)}, \frac{|\mathbb{H}^n \setminus \Omega|^\frac{Q-1}{Q}}{P_H(\Omega)} \right\} \mid 0 < P_H(\Omega) < \infty \right\},$$

An isoperimetric profile of parameter $V > 0$ for $\mathbb{H}^n$ consists of a family of bounded sets $B_V$ with $|B_V| = V$ and $|B_V|^\frac{Q-1}{Q} = C_{iso}(\mathbb{H}^n) P_H(B_V)$. Since $|\cdot|$ and $P_H(\cdot)$ are invariant under left translation and scaling property hold for them, the class of isoperimetric profile is preserved under left translation and group dilation.
Pansu conjectured in 1984 that the isoperimetric profiles on $\mathbb{H}^1$ is obtained by revolving around the $t$-axis the geodesics (with respect to the Carnot-Caratheodory metric) joining the points $(0, 0, \pi R^2/8)$ and $(0, 0, -\pi R^2/8)$. For the moment, $R > 0$ is just a parameter. These geodesics can be easily obtained as: $\gamma : [-\pi, \pi] \rightarrow \mathbb{H}^1$

$$\gamma(s) = \left( \frac{R}{2}(\cos(s) + 1), \frac{R}{2}\sin(s), \frac{R^2}{8}(\sin(s) + s) \right)$$

With the explicit form of the geodesics, it is an easy exercise to compute and to obtain $C_{iso}(\mathbb{H}^1) = \frac{3}{4\sqrt{\pi}}$
We have to stress that until today, Pansu's conjecture has not completely been solved in the greatest generality, that is, if we consider the largest admissible sets: \( \Omega \subset \mathbb{H}^n \) for which \( P_H(\Omega) < \infty \) without any regularity assumption on \( \partial \Omega \). In the remaining time, we survey results and some ideas used in the pursue of Pansu’s conjecture. The first question that one may have is why the gauge balls or the CC-metric balls are not solution to the isoperimetric problem (i.e., not the isoperimetric profiles).

- The gauge balls are smooth and one can compute its H-mean curvature and it is not constant. Later on, we see that a necessary condition for a smooth set to be a solution to the isoperimetric problem, its H-mean curvature must be constant.
We have to stress that until today, Pansu’s conjecture has not completely been solved in the greatest generality, that is, if we consider the largest admissible sets: \( \Omega \subset \mathbb{H}^n \) for which \( P_H(\Omega) < \infty \) without any regularity assumption on \( \partial \Omega \). In the remaining time, we survey results and some ideas used in the pursue of Pansu’s conjecture. The first question that one may have is why the gauge balls or the CC-metric balls are not solution to the isoperimetric problem (i.e., not the isoperimetric profiles).

- The gauge balls are smooth and one can compute its H-mean curvature and it is not constant. Later on, we see that a necessary condition for a smooth set to be a solution to the isoperimetric problem, its H-mean curvature must be constant.

- Monti in 2001 showed that Given a CC-ball \( B \) one can slightly modify it slightly to obtain a set \( D \) with \(|B| = |D|\) but \( P_H(D) < P_H(B) \). This rules out the CC-balls as solution as well.
Pansu’s Isoperimetric inequality and conjecture.

We have to stress that until today, Pansu’s conjecture has not completely been solved in the greatest generality, that is, if we consider the largest admissible sets: \( \Omega \subset \mathbb{H}^n \) for which \( P_H(\Omega) < \infty \) without any regularity assumption on \( \partial \Omega \). In the remaining time, we survey results and some ideas used in the pursuit of Pansu’s conjecture. The first question that one may have is why the gauge balls or the CC-metric balls are not solution to the isoperimetric problem (i.e., not the isoperimetric profiles).

- The gauge balls are smooth and one can compute its H-mean curvature and it is not constant. Later on, we see that a necessary condition for a smooth set to be a solution to the isoperimetric problem, its H-mean curvature must be constant.

- Monti in 2001 showed that Given a CC-ball \( B \) one can slightly modify it slightly to obtain a set \( D \) with \(|B| = |D|\) but \( P_H(D) < P_H(B) \). This rules out the CC-balls as solution as well.
Balls are not solution to the isoperimetric problem.

The isoperimetric profile from Pansu’s conjecture can be described more explicitly as follows.

\[ \partial B_R(0) = \{(x, y, t) \in \mathbb{R}^{2n+1} | t = \pm u(x, y) = u(|z|)\} \]

where

\[ u(x, y) = u(|z|) = \frac{|z|\sqrt{R^2 - |z|^2}}{4} - \frac{R^2}{4} \arcsin \left( \frac{|z|}{R} \right) + \frac{\pi R^2}{8} \quad (3) \]

In the above \(|z|^2 = |x|^2 + |y|^2\).
Balls are not solution to the isoperimetric problem.

The gauge balls, CC-balls and the isoperimetric profile from Pansu’s conjecture and is called now a days the Heisenberg bubbles. Note that CC-balls are not smooth.

Figure: The Gauge ball with $R = 4$. 
Balls are not solution to the isoperimetric problem.

The gauge balls, CC-balls and the isoperimetric profile from Pansu’s conjecture and is called now a days the Heisenberg bubbles. Note that CC-balls are not smooth.

**Figure:** The Gauge ball with $R = 4$.

**Figure:** A quarter of the CC-ball with $R = 4$. 
Balls are not solution to the isoperimetric problem.

The gauge balls, CC-balls and the isoperimetric profile from Pansu’s conjecture and is called now a days the Heisenberg bubbles. Note that CC-balls are not smooth.

**Figure:** The Gauge ball with $R = 4$.  

**Figure:** A quarter of the CC-ball with $R = 4$.  

**Figure:** The isoperimetric bubble, $R = 4$.  

Balls are not solution to the isoperimetric problem.

The gauge balls, CC-balls and the isoperimetric profile from Pansu’s conjecture and is called now a days the Heisenberg bubbles. Note that CC-balls are not smooth.

Figure: The Gauge ball with $R = 4$.

Figure: A quarter of the CC-ball with $R = 4$.

Figure: The isoperimetric bubble, $R = 4$. 
We turn to the first positive result in this direction: The existence result.

**Theorem 3 (Leonardi-Rigot, 2003)**

For any $V > 0$, there exists a bounded set $\Omega \subset \mathbb{H}^n$ with $P_H(\Omega) < \infty$, $|\Omega| = V$ and $\frac{Q-1}{Q} |\Omega| = C_{iso}(\mathbb{H}^n) P_H(\Omega)$.

**Remark 4**

Leonardi and Rigot’s result continue to hold in all Carnot-groups where the Heisenberg groups $\mathbb{H}^n$, $n \geq 1$ are the simplest such examples of step 2.
The proof of Theorem 3 consists of the following ideas.

- An important ingredient due to Garofalo-Nhieu (1996) states that if \( \{\Omega_n\} \) is a sequence of measurable sets in \( \mathbb{H}^n \) (in fact, in more greater generality) with \( \sup \{P_H(\Omega_n)\} < \infty \) then there is a subsequence still denoted by \( \{\Omega_n\} \) and a measurable set \( \Omega \) with \( P_H(\Omega) < \infty \) and \( \chi_{\Omega_n} \to \chi_{\Omega} \) in \( L^1_{loc}(\mathbb{H}^n) \).

- They considered a sequence \( \{\Omega_n\} \) such that \( |\Omega_n| = 1 \) and that

\[
\frac{|\Omega_n|^{Q-1}}{P_H(\Omega_n)} = \frac{1}{P_H(\Omega_n)} \to C_{iso}(\mathbb{H}^n)
\]

with \( P_H(\Omega_n) \leq C_{iso}(\mathbb{H}^n)^{-1}(1 + 1/n) \). Using the above theorem, they obtain a subsequence \( \{\Omega_n\} \) converging in \( L^1_{loc}(\mathbb{H}^n) \) to \( \Omega \) with \( |\Omega| = 1 \).

The proof of Theorem 3 consists of the following ideas.

- An important ingredient due to Garofalo-Nhieu (1996) states that if \( \{\Omega_n\} \) is a sequence of measurable sets in \( \mathbb{H}^n \) (in fact, in more greater generality) with \( \sup \{P_H(\Omega_n)\} < \infty \) then there is a subsequence still denoted by \( \{\Omega_n\} \) and a measurable set \( \Omega \) with \( P_H(\Omega) < \infty \) and \( \chi_{\Omega_n} \rightarrow \chi_{\Omega} \) in \( L^1_{\text{loc}}(\mathbb{H}^n) \).

- They considered a sequence \( \{\Omega_n\} \) such that \( |\Omega_n| = 1 \) and that

\[
\frac{|\Omega_n|^{\frac{Q-1}{Q}}}{P_H(\Omega_n)} = \frac{1}{P_H(\Omega_n)} \rightarrow C_{iso}(\mathbb{H}^n)
\]

with \( P_H(\Omega_n) \leq C_{iso}(\mathbb{H}^n)^{-1}(1 + 1/n) \). Using the above theorem, they obtain a subsequence \( \{\Omega_n\} \) converging in \( L^1_{\text{loc}}(\mathbb{H}^n) \) to \( \Omega \) with \( |\Omega| = 1 \).
To show that $\Omega$ is bounded, they established the following lemma that prevents the possibility that the sets $\Omega_n$ become very thin, spread out and in the limit lose volume at infinity. That is, for each $\Omega_n$, a fixed amount of volume must lie within a ball of radius one.

**Lemma 1 ("Concentration-Compactness")**

Let $A$ be a set with $0 < |A|, P_H(A) < \infty$. If $m \in (0, |B(0, 1)|/2)$ is such that $|A \cap B(g, 1)| < m$ for all $g \in \mathbb{H}^n$ then there is a constant $c > 0$ so that

$$c \left( \frac{|A|}{P_H(A)} \right)^Q \leq m.$$
To show that that $\Omega$ is bounded, they established the following lemma that prevents the possibility that the sets $\Omega_n$ become very thin, spread out and in the limit lose volume at infinity. That is, for each $\Omega_n$, a fixed amount of volume must lie within a ball of radius one.

**Lemma 1 ("Concentration-Compactness")**

*Let $A$ be a set with $0 < |A|, P_H(A) < \infty$. If $m \in (0, |B(0, 1)|/2)$ is such that $|A \cap B(g, 1)| < m$ for all $g \in \mathbb{H}^n$ then there is a constant $c > 0$ so that

$$c \left( \frac{|A|}{P_H(A)} \right)^Q \leq m.$$*

The lemma plus a bit of work show that the limiting set $\Omega$ is essentially bounded. Finally, They established that $\partial \Omega$ satisfies (1) Ahlfors regularity condition and (2) interior and exterior corkscrew condition. Essential boundedness together with (1) and (2) imply that $\Omega$ is bounded.
To show that $\Omega$ is bounded, they established the following lemma that prevents the possibility that the sets $\Omega_n$ become very thin, spread out and in the limit lose volume at infinity. That is, for each $\Omega_n$, a fixed amount of volume must lie within a ball of radius one.

**Lemma 1 ("Concentration-Compactness")**

Let $A$ be a set with $0 < |A|, \mathcal{P}_H(A) < \infty$. If $m \in (0, |B(0, 1)|/2)$ is such that $|A \cap B(g, 1)| < m$ for all $g \in H^n$ then there is a constant $c > 0$ so that

$$c \left( \frac{|A|}{\mathcal{P}_H(A)} \right)^Q \leq m.$$

The lemma plus a bit of work show that the limiting set $\Omega$ is essentially bounded. Finally, they established that $\partial \Omega$ satisfies (1) Ahlfors regularity condition and (2) interior and exterior corkscrew condition. Essential boundedness together with (1) and (2) imply that $\Omega$ is bounded.
The next line of investigation proceeds to demonstrate that Pansu’s conjecture holds for restricted family of sets in $\mathbb{H}^n$: rotationally invariant around the t-axis (i.e. cylindrically symmetric) and smooth i.e. sets whose boundary is $C^2$. Leonardi and Masnou considered the following restricted class $\mathcal{F}$ where $F \in \mathcal{F}$ satisfies the following conditions: Up to left translations $\partial F = \partial^+ F \cup \partial^- F$ where $\partial^+ F$ and $\partial^- F$ are the graphs of smooth functions $f(|z|)$ and $-f(|z|)$ respectively defined on some Euclidean balls $B \subset \mathbb{R}^{2n}$ with $f = 0$ on $\partial B$. By considering the isoperimetric problem in variational form, they solve the Euler-Lagrange equation which takes the form (due to the cylindrical symmetry assumption)

$$\frac{d}{d\rho} \left( \frac{\rho^{2n-1}f'(\rho)}{\sqrt{4\rho^2 + f'(\rho)^2}} \right) = \lambda_n \rho^{2n-1}, \quad f'(0) = 0.$$
The next line of investigation proceeds to demonstrate that Pansu’s conjecture holds for restricted family of sets in $\mathbb{H}^n$: rotationally invariant around the t-axis (i.e. cylindrically symmetric) and smooth i.e. sets whose boundary is $C^2$. Leonardi and Masnou considered the following restricted class $\mathcal{F}$ where $F \in \mathcal{F}$ satisfies the following conditions: Up to left translations $\partial F = \partial^+ F \cup \partial^- F$ where $\partial^+ F$ and $\partial^- F$ are the graphs of smooth functions $f(|z|)$ and $-f(|z|)$ respectively defined on some Euclidean balls $B \subset \mathbb{R}^{2n}$ with $f = 0$ on $\partial B$. By considering the isoperimetric problem in variational form, they solve the Euler-Lagrange equation which takes the form (due to the cylindrical symmetry assumption)

$$
\frac{d}{d\rho} \left( \frac{\rho^{2n-1} f'(\rho)}{\sqrt{4 \rho^2 + f'(\rho)^2}} \right) = \lambda_n \rho^{2n-1} , \quad f'(0) = 0.
$$
Smooth Cylindrical case, properties ...

$$u(x, y) = u(|z|) = \frac{|z| \sqrt{R^2 - |z|^2}}{4} - \frac{R^2}{4} \arcsin \left( \frac{|z|}{R} \right) + \frac{\pi R^2}{8}.$$ 

By solving the above equation, they obtain the formula (3) i.e., the Heisenberg bubbles. They also establish some properties of such sets which can be summarized as follows

**Theorem 5** (Leonardi-Masnou, 2005)

There exists, up to dilations and left translations, sets $\Omega$ given by (3) are critical points of the isoperimetric problem in $\mathbb{H}^n$. Furthermore, the H-mean curvature of $\partial \Omega$ is constant and $\partial \Omega$ are foliated by the geodesics joining the North and South poles of $\Omega$ in Pansu's conjecture.
Smooth Cylindrical case, properties ...

\[ u(x, y) = u(|z|) = \frac{|z|\sqrt{R^2 - |z|^2}}{4} - \frac{R^2}{4} \arcsin \left( \frac{|z|}{R} \right) + \frac{\pi R^2}{8}. \]

By solving the above equation, they obtain the formula (3) i.e., the Heisenberg bubbles. They also establish some properties of such sets which can be summarized as follows

**Theorem 5 (Leonardi-Masnou, 2005)**

*There exists, up to dilations and left translations, sets \( \Omega \) given by (3) are critical points of the isoperimetric problem in \( \mathbb{H}^n \). Furthermore, the H-mean curvature of \( \partial \Omega \) is constant and \( \partial \Omega \) are foliated by the geodesics joining the North and South poles of \( \Omega \) in Pansu’s conjecture.*

In 2006, Ritoré and Rosales derived the first variation formula for variations with a volume constraint. As a consequence, critical points of the isoperimetric problem (from the point of view of a variational problem) must have constant H-mean curvature. We make precise this notion.
Smooth Cylindrical case, properties ...

\[ u(x, y) = u(|z|) = \frac{|z| \sqrt{R^2 - |z|^2}}{4} - \frac{R^2}{4} \arcsin \left( \frac{|z|}{R} \right) + \frac{\pi R^2}{8}. \]

By solving the above equation, they obtain the formula (3) i.e., the Heisenberg bubbles. They also establish some properties of such sets which can be summarized as follows

**Theorem 5 (Leonardi-Masnou, 2005)**

There exists, up to dilations and left translations, sets \( \Omega \) given by (3) are critical points of the isoperimetric problem in \( \mathbb{H}^n \). Furthermore, the H-mean curvature of \( \partial \Omega \) is constant and \( \partial \Omega \) are foliated by the geodesics joining the North and South poles of \( \Omega \) in Pansu’s conjecture.

In 2006, Ritoré and Rosales derived the first variation formula for variations with a volume constraint. As a consequence, critical points of the isoperimetric problem (from the point of view of a variational problem) must have constant H-mean curvature. We make precise this notion.
Smooth Cylindrical case, properties ...

\[ u(x, y) = u(|z|) = \frac{|z| \sqrt{R^2 - |z|^2}}{4} - \frac{R^2}{4} \arcsin \left( \frac{|z|}{R} \right) + \frac{\pi R^2}{8}. \]

By solving the above equation, they obtain the formula (3) i.e., the Heisenberg bubbles. They also establish some properties of such sets which can be summarized as follows

**Theorem 5 (Leonardi-Masnou, 2005)**

*There exists, up to dilations and left translations, sets \( \Omega \) given by (3) are critical points of the isoperimetric problem in \( \mathbb{H}^n \). Furthermore, the H-mean curvature of \( \partial \Omega \) is constant and \( \partial \Omega \) are foliated by the geodesics joining the North and South poles of \( \Omega \) in Pansu’s conjecture.*

In 2006, Ritoré and Rosales derived the first variation formula for variations with a volume constraint. As a consequence, critical points of the isoperimetric problem (from the point of view of a variational problem) must have constant H-mean curvature. We make precise this notion.
Some properties of the isoperimetric profile.

**Definition 6**

Let $\Omega \subset \mathbb{H}^n$ be a $C^2$ bounded set such that $\partial \Omega$ is an embedded surface with $P_H(\Omega) = \sigma_H(\partial \Omega) < \infty$ and $U$ a $C^1$ vector field with compact support on $\mathbb{H}^n$. For small $\epsilon$ denote by $S_\epsilon = \{\exp(\epsilon U_p) | p \in \partial \Omega\}$ the variation of $\partial \Omega$ induced by $U$. We let $\Omega_\epsilon$ the region enclosed by $S_\epsilon$ and define $P(\epsilon) = \sigma_H(S_\epsilon)$, $V(\epsilon) = |\Omega_\epsilon|$.

It is well known that

$$V'(0) = \int_{\Omega} \operatorname{div}(U) \, dg = -\int_{\partial \Omega} < U, N > \, d\sigma$$

(4)

where $N$ is the Riemannian unit normal pointing into $\Omega$, $d\sigma$ is the Riemannian volume on $\partial \Omega$. 
Some properties of the isoperimetric profile.

**Theorem 7** (Ritoré-Rosales, 2006 “First variation formula”)

If $\Omega$ is such that $V'(0) = 0$ and the H-mean curvature $H$ of $\partial \Omega$ is in $L^1_{loc}(\partial \Omega, d\sigma)$ then

$$P'(0) = \int_{\partial \Omega} <U, N> H \, d\sigma.$$ 

Since a $C^2$ solution $\Omega$ to the isoperimetric problem must be a critical point of the H-perimeter functional (i.e. $P'(0) = 0$) that preserves volume (i.e. $V'(0) = 0$ for all $C^1$ vector fields $U$ with compact support in $\mathbb{H}^n$) we have (in view of Theorem 11 and (4))
Some properties of the isoperimetric profile.

**Theorem 7 (Ritoré-Rosales, 2006 “First variation formula”)**

If $\Omega$ is such that $V'(0) = 0$ and the H-mean curvature $\mathcal{H}$ of $\partial \Omega$ is in $L^1_{loc}(\partial \Omega, d\sigma)$ then

$$P'(0) = \int_{\partial \Omega} < U, N > \mathcal{H} d\sigma.$$ 

Since a $C^2$ solution $\Omega$ to the isoperimetric problem must be a critical point of the H-perimeter functional (i.e. $P'(0) = 0$) that preserves volume (i.e. $V'(0) = 0$ for all $C^1$ vector fields $U$ with compact support in $\mathbb{H}^n$) we have (in view of Theorem 11 and (4))

**Corollary 8 (Ritoré-Rosales, 2006)**

If $\Omega$ is a $C^2$ solution to the isoperimetric problem, then the H-mean curvature $\mathcal{H}$ is constant outside the singular set $\Sigma_{\partial \Omega}$ of $\partial \Omega$. 
Some properties of the isoperimetric profile.

**Theorem 7 (Ritoré-Rosales, 2006 “First variation formula”)**

If \( \Omega \) is such that \( V'(0) = 0 \) and the H-mean curvature \( \mathcal{H} \) of \( \partial \Omega \) is in \( L^1_{loc}(\partial \Omega, d\sigma) \) then

\[
P'(0) = \int_{\partial \Omega} <U, N> \mathcal{H} \, d\sigma.
\]

Since a \( C^2 \) solution \( \Omega \) to the isoperimetric problem must be a critical point of the H-perimeter functional (i.e. \( P'(0) = 0 \)) that preserves volume (i.e. \( V'(0) = 0 \) for all \( C^1 \) vector fields \( U \) with compact support in \( \mathbb{H}^n \)) we have (in view of Theorem 11 and (4))

**Corollary 8 (Ritoré-Rosales, 2006)**

If \( \Omega \) is a \( C^2 \) solution to the isoperimetric problem, then the H-mean curvature \( \mathcal{H} \) is constant outside the singular set \( \Sigma_{\partial \Omega} \) of \( \partial \Omega \).
In 2008, Danielli-Garofalo-Nhieu improved Leonardi-Masnou’s result by relaxing some symmetry conditions. We also show that Pansu’s conjecture is not only a critical point but indeed a minimizer of the $H$-perimeter functional, a fact not established by Leonardi-Masnou. We begin by describing these geometric conditions. We let $\mathbb{H}_n^+ = \{(z, t) \in \mathbb{H}^n \mid t > 0\}$, $\mathbb{H}_n^- = \{(z, t) \in \mathbb{H}^n \mid t < 0\}$, and consider the collection $\mathcal{E} = \{E \subset \mathbb{H}^n \mid E \text{ satisfies } (i) - (iii)\}$, where

(i) $|E \cap \mathbb{H}_n^+| = |E \cap \mathbb{H}_n^-|;$
In 2008, Danielli-Garofalo-Nhieu improved Leonardi-Masnou’s result by relaxing some symmetry conditions. We also show that Pansu’s conjecture is not only a critical point but indeed a minimizer of the $H$-perimeter functional, a fact not established by Leonardi-Masnou. We begin by describing these geometric conditions. We let $H^n_+ = \{(z, t) \in H^n \mid t > 0\}$, $H^n_- = \{(z, t) \in H^n \mid t < 0\}$, and consider the collection $\mathcal{E} = \{E \subset H^n \mid E \text{ satisfies (i) – (iii)}\}$, where

(i) $|E \cap H^n_+| = |E \cap H^n_-|$;

(ii) there exist $R > 0$, and functions $u, v : \overline{B}(0, R) \to [0, \infty)$, with $u, v \in C^2(B(0, R)) \cap C(\overline{B}(0, R))$, $u = v = 0$ on $\partial B(0, R)$, and such that

$$\partial E \cap H^n_+ = \{(z, t) \in H^n_+ \mid |z| < R, \ t = u(z)\},$$

$$\partial E \cap H^n_- = \{(z, t) \in H^n_- \mid |z| < R, \ t = -v(z)\}.$$
In 2008, Danielli-Garofalo-Nhieu improved Leonardi-Masnou’s result by relaxing some symmetry conditions. We also show that Pansu’s conjecture is not only a critical point but indeed a minimizer of the H-perimeter functional, a fact not established by Leonardi-Masnou. We begin by describing these geometric conditions. We let $\mathbb{H}^n_+ = \{(z, t) \in \mathbb{H}^n | t > 0\}$, $\mathbb{H}^n_- = \{(z, t) \in \mathbb{H}^n | t < 0\}$, and consider the collection $\mathcal{E} = \{E \subset \mathbb{H}^n | E$ satisfies $(i)-(iii)\}$, where

(i) $|E \cap \mathbb{H}^n_+| = |E \cap \mathbb{H}^n_-|;$

(ii) there exist $R > 0$, and functions $u, v : \overline{B}(0, R) \to [0, \infty)$, with $u, v \in C^2(B(0, R)) \cap C(\overline{B}(0, R))$, $u = v = 0$ on $\partial B(0, R)$, and such that

$$\partial E \cap \mathbb{H}^n_+ = \{(z, t) \in \mathbb{H}^n_+ | |z| < R, \ t = u(z)\},$$

$$\partial E \cap \mathbb{H}^n_- = \{(z, t) \in \mathbb{H}^n_- | |z| < R, \ t = -v(z)\}.$$

(iii) $\{z \in B(0, R) | u(z) = 0\} \cap \{z \in B(0, R) | v(z) = 0\} = \emptyset.$
In 2008, Danielli-Garofalo-Nhieu improved Leonardi-Masnou’s result by relaxing some symmetry conditions. We also show that Pansu’s conjecture is not only a critical point but indeed a minimizer of the H-perimeter functional, a fact not established by Leonardi-Masnou. We begin by describing these geometric conditions. We let $H^n_+ = \{(z, t) \in H^n \mid t > 0\}$, $H^n_- = \{(z, t) \in H^n \mid t < 0\}$, and consider the collection $\mathcal{E} = \{E \subset H^n \mid E$ satisfies (i) – (iii)$\}$, where

(i) $|E \cap H^n_+| = |E \cap H^n_-|$;
(ii) there exist $R > 0$, and functions $u, v \colon \overline{B}(0, R) \to [0, \infty)$, with $u, v \in C^2(B(0, R)) \cap C(\overline{B}(0, R))$, $u = v = 0$ on $\partial B(0, R)$, and such that

$$\partial E \cap H^n_+ = \{(z, t) \in H^n_+ \mid |z| < R, \ t = u(z)\},$$
$$\partial E \cap H^n_- = \{(z, t) \in H^n_- \mid |z| < R, \ t = -v(z)\}.$$  

(iii) $\{z \in B(0, R) \mid u(z) = 0\} \cap \{z \in B(0, R) \mid v(z) = 0\} = \emptyset$. 

Figure: $E \in \mathcal{E}$
In 2008, Danielli-Garofalo-Nhieu improved Leonardi-Masnou’s result by relaxing some symmetry conditions. We also show that Pansu’s conjecture is not only a critical point but indeed a minimizer of the $H$-perimeter functional, a fact not established by Leonardi-Masnou. We begin by describing these geometric conditions. We let $H^n_+ = \{(z, t) \in H^n \mid t > 0\}$, $H^n_- = \{(z, t) \in H^n \mid t < 0\}$, and consider the collection $\mathcal{E} = \{E \subset H^n \mid E$ satisfies (i) – (iii)$\}$, where

(i) $|E \cap H^n_+| = |E \cap H^n_-|$;

(ii) there exist $R > 0$, and functions $u, v : \overline{B}(0, R) \rightarrow [0, \infty)$, with $u, v \in C^2(B(0, R)) \cap C(\overline{B}(0, R))$, $u = v = 0$ on $\partial B(0, R)$, and such that

$$\partial E \cap H^n_+ = \{(z, t) \in H^n_+ \mid |z| < R, \ t = u(z)\} ,$$

$$\partial E \cap H^n_- = \{(z, t) \in H^n_- \mid |z| < R, \ t = -v(z)\} .$$

(iii) $\{z \in B(0, R) \mid u(z) = 0\} \cap \{z \in B(0, R) \mid v(z) = 0\} = \emptyset$. 

\[ \text{Figure: } E \in \mathcal{E} \]
Remark 9

We note explicitly that condition

\[(iii) \{z \in B(0, R) \mid u(z) = 0\} \cap \{z \in B(0, R) \mid v(z) = 0\} = \emptyset\]

serves to guarantee that every \( E \in \mathcal{E} \) is a piecewise \( C^2 \) domain in \( \mathbb{H}^n \) (with possible discontinuities in the derivatives only on that part of \( E \) which intersects the hyperplane \( t = 0 \)). We also stress that the upper and lower portions of a set \( E \in \mathcal{E} \) can be described by possibly different \( C^2 \) graphs, and that, besides \( C^2 \) smoothness, and the fact that their common domain is a ball, no additional assumption is made on the functions \( u \) and \( v \). For instance, we do not require a priori that \( u \) and/or \( v \) are spherically symmetric. Here is our main result.
Partial Symmetry: (3) \( u(x, y) = u(|z|) = \frac{|z|\sqrt{R^2 - |z|^2}}{4} - R^2 \frac{\arcsin(|z|/R)}{4} + \frac{\pi R^2}{8} \).

Theorem 10 (Danielli-Garofalo-Nhieu, 2008)

Let \( V > 0 \), and define the number \( R > 0 \) by

\[
R = \left( \frac{(Q-2)\Gamma\left(\frac{Q+2}{2}\right)\Gamma\left(\frac{Q-2}{2}\right)}{\pi \frac{Q-1}{2} \Gamma\left(\frac{Q+1}{2}\right)} \right)^{1/Q} V^{1/Q}.
\]

Given such \( R \), then the variational problem \( \min_{E \in \mathcal{E}, |E|=V} P_H(E; \mathbb{H}^n) \) has a unique solution \( E_R \in \mathcal{E} \), where \( \partial E_R \) is described by the graph \( t = \pm u(x, y) \) i.e. (3) The boundary of \( E_R \) is only of class \( C^2 \), but not of class \( C^3 \), near its two singular points \( (0, \pm \frac{\pi R^2}{8}) \), it is \( C^\infty \) away from them, and \( \partial E_R \) has positive constant \( H \)-mean curvature and isoperimetric constant given respectively by

\[
\mathcal{H} = \frac{Q-2}{R}, \quad C(\mathbb{H}^n) = \frac{(Q-1)\Gamma\left(\frac{Q}{2}\right)^{\frac{2}{Q}}}{Q \frac{Q-1}{Q} (Q-2)\Gamma\left(\frac{Q+1}{2}\right)^{\frac{1}{Q}} \pi \frac{Q-1}{2Q}}.
\]
Our proof is based on the following ideas.

- Under the assumption of sets $E \in \mathcal{E}$, the variational problem
  $$\min_{E \in \mathcal{E}, |E| = V} P_H(E; \mathbb{H}^n)$$
  is equivalent to minimizing the unconstrained functional with a Lagrange multiplier $\lambda$ to be properly chosen:

  $$\mathcal{F}[u] = \int_{\text{supp}(u)} \left\{ \left| \nabla_z u(z) + \frac{z^\perp}{2} \right| + \lambda u(z) \right\} \, dz \quad z = (x, y) \in \mathbb{R}^{2n}$$  \hspace{1cm} (5)

- We easily recognize that the Euler-Lagrange equation of (5) is

  $$\text{div}_z \left[ \frac{\nabla_z u + \frac{z^\perp}{2}}{\sqrt{\left| \nabla_z u \right|^2 + \frac{|z|^2}{4} + \langle \nabla_z u, z^\perp \rangle}} \right] = \lambda \quad z^\perp = (-y, x) \in \mathbb{R}^{2n}.$$  \hspace{1cm} (6)
Partial Symmetry: (3) $u(x, y) = u(|z|) = \frac{|z|\sqrt{R^2-|z|^2}}{4} - \frac{R^2}{4} \arcsin\left(\frac{|z|}{R}\right) + \frac{\pi R^2}{8}$.

Our proof is based on the following ideas.

- Under the assumption of sets $E \in \mathcal{E}$, the variational problem
  \[
  \min_{E \in \mathcal{E}, |E|=V} P_H(E; \mathbb{H}^n)
  \]
  is equivalent to minimizing the unconstrained functional with a Lagrange multiplier $\lambda$ to be properly chosen:
  \[
  F[u] = \int_{\text{supp}(u)} \left\{ \left| \nabla_z u(z) + \frac{z^\perp}{2} \right| + \lambda u(z) \right\} dz \quad z = (x, y) \in \mathbb{R}^{2n} \quad (5)
  \]

- We easily recognize that the Euler-Lagrange equation of (5) is
  \[
  \text{div}_z \left[ \frac{\nabla_z u + \frac{z^\perp}{2}}{\sqrt{\left| \nabla_z u \right|^2 + \frac{|z|^2}{4} + <\nabla_z u, z^\perp>}} \right] = \lambda \quad z^\perp = (-y, x) \in \mathbb{R}^{2n} \quad (6)
  \]

- We do not solve (6) but rather verify that the candidate given by the formula (3) that has cylindrical symmetric is a solution.
Partial Symmetry: \( u(x, y) = u(|z|) = \frac{|z|\sqrt{R^2-|z|^2}}{4} - \frac{R^2}{4} \arcsin\left(\frac{|z|}{R}\right) + \frac{\pi R^2}{8}. \)

Our proof is based on the following ideas.

- Under the assumption of sets \( E \in \mathcal{E} \), the variational problem
  \[
  \min_{E \in \mathcal{E}, |E|=V} P_H(E; \mathbb{H}^n)
  \]
  is equivalent to minimizing the unconstrained functional with a Lagrange multiplier \( \lambda \) to be properly chosen:
  \[
  \mathcal{F}[u] = \int_{\text{supp}(u)} \left\{ \left| \nabla_z u(z) + \frac{z^\perp}{2} \right| + \lambda u(z) \right\} \, dz \quad z = (x, y) \in \mathbb{R}^{2n} \tag{5}
  \]

- We easily recognize that the Euler-Lagrange equation of (5) is
  \[
  \text{div}_z \left[ \frac{\nabla_z u + \frac{z^\perp}{2}}{\sqrt{\left| \nabla_z u \right|^2 + \frac{|z|^2}{4} + < \nabla_z u, z^\perp>}} \right] = \lambda \quad z^\perp = (-y, x) \in \mathbb{R}^{2n}. \tag{6}
  \]

- We do not solve (6) but rather verify that the candidate given by the formula (3) that has cylindrical symmetric is a solution.

- Finally, we show that the functional (5) is a convex functional and therefore, the critical point is the unique minimizer.
Our proof is based on the following ideas.

- Under the assumption of sets \( E \in \mathcal{E} \), the variational problem
  \[
  \min_{E \in \mathcal{E}, |E| = V} P_H(E; \mathbb{H}^n)
  \]
  is equivalent to minimizing the unconstrained functional with a Lagrange multiplier \( \lambda \) to be properly chosen:

  \[
  \mathcal{F}[u] = \int_{\text{supp}(u)} \left\{ \left| \nabla_z u(z) + \frac{z^\perp}{2} \right| + \lambda u(z) \right\} \, dz \quad z = (x, y) \in \mathbb{R}^{2n}
  \]

- We easily recognize that the Euler-Lagrange equation of (5) is

  \[
  \text{div}_z \left[ \frac{\nabla_z u + \frac{z^\perp}{2}}{\sqrt{|\nabla_z u|^2 + \frac{|z|^2}{4} + \langle \nabla_z u, z^\perp \rangle}} \right] = \lambda \quad z^\perp = (-y, x) \in \mathbb{R}^{2n}.
  \]

- We do not solve (6) but rather verify that the candidate given by the formula (3) that has cylindrical symmetric is a solution.

- Finally, we show that the functional (5) is a convex functional and therefore, the critical point is the unique minimizer.
The $C^2$ solution to the Isoperimetric problem in $\mathbb{H}^1$: Ritore-Rosales.

The best smooth $C^2$ result in the first Heisenberg group $\mathbb{H}^1$ is obtained by Ritoré and Rosales in 2008. They proved the remarkable theorem without any symmetry assumption.

**Theorem 11 (Ritoré-Rosales, 2008)**

*If $\Omega$ is an isoperimetric region in $\mathbb{H}^1$ which is bounded by a $C^2$ surface $\mathcal{S}$, then $\mathcal{S}$ is congruent (i.e. up to left translation and dilations on $\mathbb{H}^1$)*
The $C^2$ solution to the Isoperimetric problem in $\mathbb{H}^1$: Ritore-Rosales.

The best smooth $C^2$ result in the first Heisenberg group $\mathbb{H}^1$ is obtained by Ritoré and Rosales in 2008. They proved the remarkable theorem without any symmetry assumption.

**Theorem 11 (Ritoré-Rosales, 2008)**

*If $\Omega$ is an isoperimetric region in $\mathbb{H}^1$ which is bounded by a $C^2$ surface $\mathcal{S}$, then $\mathcal{S}$ is congruent (i.e. up to left translation and dilations on $\mathbb{H}^1$)*

To briefly sketch the proof we recall a few concepts and facts introduced earlier:

- The singular set of a $C^2$ smooth surface $\mathcal{S} \subset \mathbb{H}^n$ is $\Sigma_{\mathcal{S}} = \{ g \in \mathcal{S} \mid |N_H| = 0 \}$, or equivalently it is where the tangent plane coincide with the horizontal plane spaned by the $2n$ vector fields $X_1, ..., Y_n$. 
The $C^2$ solution to the Isoperimetric problem in $\mathbb{H}^1$: Ritore-Rosales.

The best smooth $C^2$ result in the first Heisenberg group $\mathbb{H}^1$ is obtained by Ritoré and Rosales in 2008. They proved the remarkable theorem without any symmetry assumption.

**Theorem 11 (Ritoré-Rosales, 2008)**

*If $\Omega$ is an isoperimetric region in $\mathbb{H}^1$ which is bounded by a $C^2$ surface $\mathcal{S}$, then $\mathcal{S}$ is congruent (i.e. up to left translation and dilations on $\mathbb{H}^1$)*

To briefly sketch the proof we recall a few concepts and facts introduced earlier

- The singular set of a $C^2$ smooth surface $\mathcal{S} \subset \mathbb{H}^n$ is $\Sigma_{\mathcal{S}} = \{ g \in \mathcal{S} \ | \ |N_H| = 0 \}$, or equivalently it is where the tangent plane coincide with the horizontal plane spaned by the $2n$ vector fields $X_1, ..., Y_n$.
- $\sigma(\Sigma_{\mathcal{S}}) = \sigma_H(\Sigma_{\mathcal{S}})$.
The $C^2$ solution to the Isoperimetric problem in $\mathbb{H}^1$: Ritore-Rosales.

The best smooth $C^2$ result in the first Heisenberg group $\mathbb{H}^1$ is obtained by Ritoré and Rosales in 2008. They proved the remarkable theorem without any symmetry assumption.

**Theorem 11 (Ritoré-Rosales, 2008)**

*If $\Omega$ is an isoperimetric region in $\mathbb{H}^1$ which is bounded by a $C^2$ surface $\mathcal{S}$, then $\mathcal{S}$ is congruent (i.e. up to left translation and dilations on $\mathbb{H}^1$)*

To briefly sketch the proof we recall a few concepts and facts introduced earlier

- The singular set of a $C^2$ smooth surface $\mathcal{S} \subset \mathbb{H}^n$ is $\Sigma_{\mathcal{S}} = \{ g \in \mathcal{S} | \|N_H\| = 0 \}$, or equivalently it is where the tangent plane coincide with the horizontal plane spaned by the $2n$ vector fields $X_1, ..., Y_n$.
- $\sigma(\Sigma_{\mathcal{S}}) = \sigma_H(\Sigma_{\mathcal{S}})$.
- At points $p \in \mathcal{S} \setminus \Sigma_{\mathcal{S}}, \mathcal{S} \subset \mathbb{H}^1$, the horizontal tangential vector is the unit vector that lies in the intersection of the tangent plane with the horizontal plane, we call this unit vector say $J(\nu_H)$ (since it is also orthogonal to the horizontal unit normal $\nu_H$).
The $C^2$ solution to the Isoperimetric problem in $\mathbb{H}^1$: Ritore-Rosales.

The best smooth $C^2$ result in the first Heisenberg group $\mathbb{H}^1$ is obtained by Ritoré and Rosales in 2008. They proved the remarkable theorem without any symmetry assumption.

**Theorem 11 (Ritoré-Rosales, 2008)**

*If $\Omega$ is an isoperimetric region in $\mathbb{H}^1$ which is bounded by a $C^2$ surface $\mathcal{S}$, then $\mathcal{S}$ is congruent (i.e. up to left translation and dilations on $\mathbb{H}^1$)*

To briefly sketch the proof we recall a few concepts and facts introduced earlier

- The singular set of a $C^2$ smooth surface $\mathcal{S} \subset \mathbb{H}^n$ is $\Sigma_{\mathcal{S}} = \{ g \in \mathcal{S} \mid |N_H| = 0 \}$, or equivalently it is where the tangent plane coincide with the horizontal plane spaned by the $2n$ vector fields $X_1, \ldots, Y_n$.
- $\sigma(\Sigma_{\mathcal{S}}) = \sigma_H(\Sigma_{\mathcal{S}})$.
- At points $p \in \mathcal{S} \setminus \Sigma_{\mathcal{S}}$, $\mathcal{S} \subset \mathbb{H}^1$, the horizontal tangential vector is the unit vector that lies in the intersection of the tangent plane with the horizontal plane, we call this unit vector say $J(\nu_H)$ (since it is also orthogonal to the horizontal unit normal $\nu_H$).
- On an orientable surface $\mathcal{S}$ and when an orientation is chosen, the flows (integral curves) of the vector field $J(\nu_H)$ foliate $\mathcal{S}$. We call these Legendrian curves or Legendrian foliation of $\mathcal{S}$.
The $C^2$ solution to the Isoperimetric problem in $\mathbb{H}^1$: Ritore-Rosales.

The best smooth $C^2$ result in the first Heisenberg group $\mathbb{H}^1$ is obtained by Ritoré and Rosales in 2008. They proved the remarkable theorem without any symmetry assumption.

**Theorem 11 (Ritoré-Rosales, 2008)**

If $\Omega$ is an isoperimetric region in $\mathbb{H}^1$ which is bounded by a $C^2$ surface $\mathcal{S}$, then $\mathcal{S}$ is congruent (i.e. up to left translation and dilations on $\mathbb{H}^1$)

To briefly sketch the proof we recall a few concepts and facts introduced earlier

- The singular set of a $C^2$ smooth surface $\mathcal{S} \subset \mathbb{H}^n$ is $\Sigma_{\mathcal{S}} = \{ g \in \mathcal{S} \mid |N_H| = 0 \}$, or equivalently it is where the tangent plane coincide with the horizontal plane spaned by the $2n$ vector fields $X_1, \ldots, Y_n$.
- $\sigma(\Sigma_{\mathcal{S}}) = \sigma_H(\Sigma_{\mathcal{S}})$.
- At points $p \in \mathcal{S} \setminus \Sigma_{\mathcal{S}}, \mathcal{S} \subset \mathbb{H}^1$, the horizontal tangential vector is the unit vector that lies in the intersection of the tangent plane with the horizontal plane, we call this unit vector say $J(\nu_H)$ (since it is also orthogonal to the horizontal unit normal $\nu_H$.
- On an orientable surface $\mathcal{S}$ and when an orientation is chosen, the flows (integral curves) of the vector field $J(\nu_H)$ foliate $\mathcal{S}$. We call these Legendrian curves or Legendrian foliation of $\mathcal{S}$. 
A fundamental ingredient in the proof of Theorem 11 is the important contribution by Cheng-Hwang-Malchiodi-Yang (2005) in the analysis of $C^2$ surface in the Heisenberg group $\mathbb{H}^1$ concerning the structure of characteristic/singular set $\Sigma$. We summarize and collect these results below, specializing to the case where the surface has constant H-mean curvature.

**Theorem 12 (Cheng-Hwang-Malchiodi-Yang, 2005)**

Let $\mathcal{S} \subset \mathbb{H}^1$ be a $C^2$ oriented immersed surface with constant H-mean curvature $\mathcal{H}$. Then the singular set $\Sigma_{\mathcal{S}}$ consists of isolated points and $C^1$ curves with non-vanishing tangent vector. Furthermore...
The $C^2$ solution to the Isoperimetric problem in $\mathbb{H}^1$: Ritore-Rosales.

A fundamental ingredient in the proof of Theorem 11 is the important contribution by Cheng-Hwang-Malchiodi-Yang (2005) in the analysis of $C^2$ surface in the Heisenberg group $\mathbb{H}^1$ concerning the structure of characteristic/singular set $\Sigma$. We summarize and collect these results below, specializing to the case where the surface has constant H-mean curvature.

**Theorem 12 (Cheng-Hwang-Malchiodi-Yang, 2005)**

Let $\mathcal{S} \subset \mathbb{H}^1$ be a $C^2$ oriented immersed surface with constant H-mean curvature $\mathcal{H}$. Then the singular set $\Sigma_{\mathcal{S}}$ consists of isolated points and $C^1$ curves with non-vanishing tangent vector. Furthermore

- If $p \in \Sigma_{\mathcal{S}}$ is isolated then there exists $r > 0$, $\lambda \in \mathbb{R}$ with $|\lambda| = |\mathcal{H}|$ such that $D_r(p) = \{\gamma^A_{p,v}(s) \mid v \in T_p\mathcal{S}, |v| = 1, s \in [0, r]\}$ (where $\gamma^A_{p,v}$ is the geodesic whose curvature is $\lambda$ originated from $p$ with $v$ the tangent vector at $p$) is an open neighborhood of $p$ in $\mathcal{S}$.
The $C^2$ solution to the Isoperimetric problem in $\mathbb{H}^1$: Ritore-Rosales.

A fundamental ingredient in the proof of Theorem 11 is the important contribution by Cheng-Hwang-Malchiodi-Yang (2005) in the analysis of $C^2$ surface in the Heisenberg group $\mathbb{H}^1$ concerning the structure of characteristic/singular set $\Sigma_{\mathcal{S}}$. We summarize and collect these results below, specializing to the case where the surface has constant H-mean curvature.

**Theorem 12 (Cheng-Hwang-Malchiodi-Yang, 2005)**

Let $\mathcal{S} \subset \mathbb{H}^1$ be a $C^2$ oriented immersed surface with constant H-mean curvature $H$. Then the singular set $\Sigma_{\mathcal{S}}$ consists of isolated points and $C^1$ curves with non-vanishing tangent vector. Furthermore

- If $p \in \Sigma_{\mathcal{S}}$ is isolated then there exists $r > 0$, $\lambda \in \mathbb{R}$ with $|\lambda| = |H|$ such that
  \[ D_r(p) = \{ \gamma_{p,v}^\lambda(s) \mid v \in T_p \mathcal{S}, |v| = 1, s \in [0, r) \} \]
  (where $\gamma_{-p,v}^\lambda$ is the geodesic whose curvature is $\lambda$ originated from $p$ with $v$ the tangent vector at $p$) is an open neighborhood of $p$ in $\mathcal{S}$.

- If $p$ is contained in a $C^1$ singular curve $\Gamma \subset \Sigma_{\mathcal{S}}$ then there is a neighborhood $B$ of $p$ in $\mathcal{S}$ such that $B \setminus \Gamma$ is the union of two disjoint connected open sets $B^+, B^-$ contained in $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$ and $\nu_H$ extends continuously to $\Gamma$ from both sides of $B \setminus \Gamma$, that is the limits
  \[ \nu_H^+(p) = \lim_{q \to p, q \in B^+} \nu_H(q), \quad \nu_H^-(p) = \lim_{q \to p, q \in B^-} \nu_H(q), \]
  exits for any $p \in B \cap \Gamma$. These extensions satisfy $\nu_H^+(p) = -\nu_H^-(p)$. Moreover, there are exactly two geodesics $\gamma_{1}^\lambda \subset B^+$ and $\gamma_{2}^\lambda \subset B^-$ starting from $p$ and meeting transversely $\Gamma$ at $p$ with $(\gamma_{1}^\lambda)'(0) = -(\gamma_{2}^\lambda)'(0)$. The curvature $\lambda$ does not depends on $p$ and satisfies $|\lambda| = |H|$. 

The $C^2$ solution to the Isoperimetric problem in $\mathbb{H}^1$: Ritore-Rosales.

A fundamental ingredient in the proof of Theorem 11 is the important contribution by Cheng-Hwang-Malchiodi-Yang (2005) in the analysis of $C^2$ surface in the Heisenberg group $\mathbb{H}^1$ concerning the structure of characteristic/singular set $\Sigma_{\mathcal{S}}$. We summarize and collect these results below, specializing to the case where the surface has constant H-mean curvature.

**Theorem 12 (Cheng-Hwang-Malchiodi-Yang, 2005)**

Let $\mathcal{S} \subset \mathbb{H}^1$ be a $C^2$ oriented immersed surface with constant H-mean curvature $\mathcal{H}$. Then the singular set $\Sigma_{\mathcal{S}}$ consists of isolated points and $C^1$ curves with non-vanishing tangent vector. Furthermore

- If $p \in \Sigma_{\mathcal{S}}$ is isolated then there exists $r > 0$, $\lambda \in \mathbb{R}$ with $|\lambda| = |\mathcal{H}|$ such that $D_r(p) = \{\gamma_{p,v}^\lambda(s) \mid v \in T_p\mathcal{S}, |v| = 1, s \in [0, r]\}$ (where $\gamma_{p,v}^\lambda$ is the geodesic whose curvature is $\lambda$ originated from $p$ with $v$ the tangent vector at $p$) is an open neighborhood of $p$ in $\mathcal{S}$.

- If $p$ is contained in a $C^1$ singular curve $\Gamma \subset \Sigma_{\mathcal{S}}$ then there is a neighborhood $B$ of $p$ in $\mathcal{S}$ such that $B \setminus \Gamma$ is the union of two disjoint connected open sets $B^+, B^-$ contained in $\mathcal{S} \setminus \Sigma_{\mathcal{S}}$ and $\nu_H$ extends continuously to $\Gamma$ from both sides of $B \setminus \Gamma$, that is the limits

$$\nu_H^+(p) = \lim_{q \to p, q \in B^+} \nu_H(q), \quad \nu_H^-(p) = \lim_{q \to p, q \in B^-} \nu_H(q),$$

exist for any $p \in B \cap \Gamma$. These extensions satisfy $\nu_H^+(p) = -\nu_H^-(p)$. Moreover, there are exactly two geodesics $\gamma_1^\lambda \subset B^+$ and $\gamma_2^\lambda \subset B^-$ starting from $p$ and meeting transversely $\Gamma$ at $p$ with $(\gamma_1^\lambda)'(0) = -(\gamma_2^\lambda)'(0)$. The curvature $\lambda$ does not depends on $p$ and satisfies $|\lambda| = |\mathcal{H}|$. 
We now describe the proof of Ritoré-Rosales’ theorem. In what follows, we let \( \Omega \) to be a critical point of the isoperimetric problem in \( \mathbb{H}^1 \) such that \( \partial \Omega \) is \( C^2 \), compact, orientable.

I The interaction between Legendrian foliation and singular curves: If \( \Sigma_{\partial \Omega} \) contains a \( C^1 \) curve \( C \), then the rules of Legendrian foliation of \( \partial \Omega \) meet \( C \) orthogonally.

II Improved regularity of singular curves: If \( \Sigma_{\partial \Omega} \) contains a \( C^1 \) curve \( C \), then in fact \( C \) is \( C^2 \). This result leads to a crucial property of the singular curves.
The $C^2$ solution to the Isoperimetric problem in $\mathbb{H}^1$: Ritore-Rosales.

We now describe the proof of Ritoré-Rosales’ theorem. In what follows, we let $\Omega$ to be a critical point of the isoperimetric problem in $\mathbb{H}^1$ such that $\partial \Omega$ is $C^2$, compact, orientable.

I The interaction between Legendrian foliation and singular curves: If $\Sigma_{\partial \Omega}$ contains a $C^1$ curve $C$, then the rules of Legendrian foliation of $\partial \Omega$ meet $C$ orthogonally.

II Improved regularity of singular curves: If $\Sigma_{\partial \Omega}$ contains a $C^1$ curve $C$, then in fact $C$ is $C^2$. This result leads to a crucial property of the singular curves.

III If $\partial \Omega$ is complete surface with non-vanishing H-mean curvature, then any connected curve in $\Sigma_\mathcal{F}$ is a geodesic.
We now describe the proof of Ritoré-Rosales’ theorem. In what follows, we let $\Omega$ to be a critical point of the isoperimetric problem in $\mathbb{H}^1$ such that $\partial\Omega$ is $C^2$, compact, orientable.

I The interaction between Legendrian foliation and singular curves: If $\Sigma_{\partial\Omega}$ contains a $C^1$ curve $C$, then the rules of Legendrian foliation of $\partial\Omega$ meet $C$ orthogonally.

II Improved regularity of singular curves: If $\Sigma_{\partial\Omega}$ contains a $C^1$ curve $C$, then in fact $C$ is $C^2$. This result leads to a crucial property of the singular curves.

III If $\partial\Omega$ is complete surface with non-vanishing H-mean curvature, then any connected curve in $\Sigma_{\mathcal{F}}$ is a geodesic.

IV By step III, any compact $C^2$ solution to the isoperimetric problem cannot have a nontrivial curve in its singular set since such curve would be a geodesic which will leave any bounded domain in a finite time. Thus $\Sigma_{\mathcal{F}}$ consists of isolated points.
The $C^2$ solution to the Isoperimetric problem in $\mathbb{H}^1$: Ritore-Rosales.

We now describe the proof of Ritoré-Rosales’ theorem. In what follows, we let $\Omega$ to be a critical point of the isoperimetric problem in $\mathbb{H}^1$ such that $\partial \Omega$ is $C^2$, compact, orientable.

I The interaction between Legendrian foliation and singular curves: If $\Sigma_{\partial \Omega}$ contains a $C^1$ curve $C$, then the rules of Legendrian foliation of $\partial \Omega$ meet $C$ orthogonally.

II Improved regularity of singular curves: If $\Sigma_{\partial \Omega}$ contains a $C^1$ curve $C$, then infact $C$ is $C^2$. This result leads to a crucial property of the singular curves.

III If $\partial \Omega$ is complete surface with non-vanishing $H$-mean curvature, then any connected curve in $\Sigma_{\mathcal{S}}$ is a geodesic.

IV By step III, any compact $C^2$ solution to the isoperimetric problem cannot have a nontrival curve in its singular set since such curve would be a geodesic which will leave any bounded domain in a finite time. Thus $\Sigma_{\mathcal{S}}$ consists of isolated points.

V If $\mathcal{S}$ is any $C^2$, connected, complete oriented and immerse surface in $\mathbb{H}^1$ with constant $H$-mean curvature and if $\Sigma_{\mathcal{S}}$ contains an isolated point, then $\mathcal{S}$ is congruent to the boundary of a bubble set.
The $C^2$ solution to the Isoperimetric problem in $\mathbb{H}^1$: Ritoré-Rosales.

We now describe the proof of Ritoré-Rosales’ theorem. In what follows, we let $\Omega$ to be a critical point of the isoperimetric problem in $\mathbb{H}^1$ such that $\partial \Omega$ is $C^2$, compact, orientable.

I The interaction between Legendrian foliation and singular curves: If $\Sigma_{\partial \Omega}$ contains a $C^1$ curve $C$, then the rules of Legendrian foliation of $\partial \Omega$ meet $C$ orthogonally.

II Improved regularity of singular curves: If $\Sigma_{\partial \Omega}$ contains a $C^1$ curve $C$, then infact $C$ is $C^2$. This result leads to a crucial property of the singular curves.

III If $\partial \Omega$ is complete surface with non-vanishing H-mean curvature, then any connected curve in $\Sigma_\mathcal{S}$ is a geodesic.

IV By step III, any compact $C^2$ solution to the isoperimetric problem cannot have a nontrival curve in its singular set since such curve would be a geodesic which will leave any bounded domain in a finite time. Thus $\Sigma_\mathcal{S}$ consists of isolated points.

V If $\mathcal{S}$ is any $C^2$, connected, complete oriented and immerse surface in $\mathbb{H}^1$ with constant H-mean curvature and if $\Sigma_\mathcal{S}$ contains an isolated point, then $\mathcal{S}$ is congruent to the boundary of a bubble set.
We now describe two results in the directions of finding the isoperimetric profile among non-smooth sets.

**Definition 13**

A set $E \subset \mathbb{H}^n$ is axially symmetric if $(z, t) \in E$ implies $(\xi, t) \in E$ for all $\xi \in \mathbb{R}^{2n}$ such that $|\xi| = |z|$. Let $\mathcal{A}$ denote the collections of axially symmetric sets in $\mathbb{H}^n$. 

Remark 14  

We point out that here, the sets $E \in \mathcal{A}$ are not necessarily smooth nor necessarily the graph of a function.
The non-smooth cases: Monti.

We now describe two results in the directions of finding the isoperimetric profile among non-smooth sets.

**Definition 13**

A set $E \subset \mathbb{H}^n$ is axially symmetric if $(z, t) \in E$ implies $(\xi, t) \in E$ for all $\xi \in \mathbb{R}^{2n}$ such that $|\xi| = |z|$. Let $\mathcal{A}$ denote the collections of axially symmetric sets in $\mathbb{H}^n$.

**Remark 14**

We point out that here, the sets $E \in \mathcal{A}$ are not necessary smooth nor necessary the graph of a function.
We now describe two results in the directions of finding the isoperimetric profile among non-smooth sets.

**Definition 13**

A set $E \subset \mathbb{H}^n$ is axially symmetric if $(z, t) \in E$ implies $(\xi, t) \in E$ for all $\xi \in \mathbb{R}^{2n}$ such that $|\xi| = |z|$. Let $\mathcal{A}$ denote the collections of axially symmetric sets in $\mathbb{H}^n$.

**Remark 14**

We point out that here, the sets $E \in \mathcal{A}$ are not necessary smooth nor necessary the graph of a function.
The non-smooth cases: Monti, Monti-Rickly.

**Theorem 15 (Monti, 2008)**

The isoperimetric profiles in $\mathbb{H}^n$ up to a vertical translation, a dilation and a $2n + 1$-Lebesgue measure negligible set, are still given by the bubble set described by (3) if restricted to the class $\mathcal{A}$.

The ideas in the proof consist of using several rearrangements to reduce the Theorem 15 to a one dimensional problem which can be solved by elementary methods.
The non-smooth cases: Monti, Monti-Rickly.

**Theorem 15 (Monti, 2008)**

The isoperimetric profiles in $\mathbb{H}^n$ up to a vertical translation, a dilation and a $2n+1$-Lebesgue measure negligible set, are still given by the bubble set described by (3) if restricted to the class $\mathcal{A}$.

The ideas in the proof consist of using several rearrangements to reduce the Theorem 15 to a one dimensional problem which can be solved by elementary methods.

**Theorem 16 (Monti-Rickly, 2009)**

The isoperimetric profile given by (3) is the unique solution to the isoperimetric problem when restricted to Euclidean convex sets in $\mathbb{H}^1$. 

Note that convex sets are not necessarily smooth. The aim is to show that patches of $\partial \Omega$ can be parameterized by lefts of circles. The convexity hypothesis is then used to conclude the argument.
The non-smooth cases: Monti, Monti-Rickly.

**Theorem 15 (Monti, 2008)**

*The isoperimetric profiles in $\mathbb{H}^n$ up to a vertical translation, a dilation and a $2n+1$-Lebesgue measure negligible set, are still given by the bubble set described by (3) if restricted to the class $\mathcal{A}$.*

The ideas in the proof consist of using several rearrangements to reduce the Theorem 15 to a one dimensional problem which can be solved by elementary methods.

**Theorem 16 (Monti-Rickly, 2009)**

*The isoperimetric profile given by (3) is the unique solution to the isoperimetric problem when restricted to Euclidean convex sets in $\mathbb{H}^1$.*

Note that convex sets are not necessarily smooth. The aim is to show that patches of $\partial \Omega$ can be parameterized by lefts of circles. The convexity hypothesis is then used to conclude the argument.
The non-smooth cases: Monti, Monti-Rickly.

**Theorem 15 (Monti, 2008)**

*The isoperimetric profiles in $\mathbb{H}^n$ up to a vertical translation, a dilation and a $2n+1$-Lebesgue measure negligible set, are still given by the bubble set described by (3) if restricted to the class $\mathcal{A}$.*

The ideas in the proof consist of using several rearrangements to reduce the Theorem 15 to a one dimensional problem which can be solved by elementary methods.

**Theorem 16 (Monti-Rickly, 2009)**

*The isoperimetric profile given by (3) is the unique solution to the isoperimetric problem when restricted to Euclidean convex sets in $\mathbb{H}^1$.*

Note that convex sets are not necessarily smooth. The aim is to show that patches of $\partial \Omega$ can be parameterized by lefts of circles. The convexity hypothesis is then used to conclude the argument.
The most recent result up to now for the isoperimetric problem in $\mathbb{H}^n$ is again due to Ritoré. He generalized the results of Danielli-Garofalo-Nhieu by removing that the upper and lower part of $\partial \Omega$ to be graphs. To be precise, he proved the following theorem. Let $D_r = \{(z,0) \mid |z| < r\} \subset \mathbb{R}^{2n}$ be the Euclidean ball centered at 0 with radius $r$. $C_r = \{(z,t) \mid z \in D_r, t \in \mathbb{R}\}$. We also denote the region enclosed by the Heisenberg bubbles in $\mathbb{H}^n$ by $B_r$.

**Theorem 17**

Let $\Sigma \subset \mathbb{H}^n$ be such that $P_H(\Sigma) < \infty$ and $D_r \subset \Sigma \subset C_r$ for some $r > 0$. Then $P_H(\Sigma) \geq P_H(B_r)$. Equality holds if and only if $\Sigma = B_r$. 

**Figure:** Conditions for the sets $\Sigma$. 
A rough sketch of the proof is the following. On the cylinder $C_r$, two foliations by vertically translating the upper and lower boundary of $B_r$ are constructed. Using these foliations, he proved that the bubble sets $\partial B_\lambda$ minimize the functional “$\text{H-perimeter} - n\lambda \text{ volume}$” in the class of sets $E$ mentioned above. Then minimize over all bubble sets $B_\mu$ the functional “$\text{H-perimeter} - n\mu(\text{volume} - |\Sigma|)$” to obtained the desired result. Another important result due to Monti-Vittone is used to deal with the issue of regularity: a set in $\mathbb{H}^n$ with locally finite H-perimeter with continuous horizontal unit normal has H-regular boundary.
Thank you for your attention!