We start by considering the general form of an argument, one we wish to show is valid. So let us consider the implication

$$(p_1 \land p_2 \land p_3 \land \cdots \land p_n) \rightarrow q.$$  

Here $n$ is a positive integer, the statement $p_1, p_2, p_3, \ldots, p_n$ are called the **premises** of the argument, and the statement $q$ is the **conclusion** for the argument.
Example (2.19)

Let $p$, $q$, $r$ denote the primitive statements given as:

$p$: Roger studies.
$q$: Roger plays basketball.
$r$: Roger passes discrete mathematics.

Now let $p_1$, $p_2$, $p_3$ denote the premises:

$p_1$: If Roger studies, then he will be pass discrete mathematics.
$p_2$: If Roger doesn’t play basketball, then he’ll study.
$p_3$: Roger failed discrete mathematics.
Example (2.19)

We want to determine whether the argument

\[(p_1 \land p_2 \land p_3) \rightarrow q\]

is valid. To do so, we rewrite \(p_1, p_2, p_3\) as

\[p_1; \ p \rightarrow r \quad p_2 : \neg q \rightarrow p \quad p_3 : \neg r\]

and examine the truth table for the implication

\[\left[ (p \rightarrow r) \land (\neg q \rightarrow p) \land \neg r \right] \rightarrow q\]

given in Table 2.14.
Because the final column in Table 2.14 contains all 1’s, the implication is a tautology. Hence we can say that \((p_1 \land p_2 \land p_3) \rightarrow q\) is a valid argument.
2.3: Logic Implication: Rules of Inference.

Definition (2.4)
If $p, q$ are arbitrary statements such that $p \rightarrow q$ is a tautology, then we say that $p$ logically implies $q$ and we write $p \Rightarrow q$ to denote this situation.

When $p, q$ are statements and $p \Rightarrow q$, the implication $p \rightarrow q$ is a tautology and we refer to $p \rightarrow q$ as a logical implication.
We consider the rule of inference called *Modus Ponens*, or the *Rule of Detachment*. In symbolic form this rule is expressed by the logical implication

\[ [p \land (p \rightarrow q)] \rightarrow q, \]

which is verified in Table 2.16, where we find that the fourth row is the only one where both of the premises \( p \) and \( p \rightarrow q \) (and the conclusion \( q \)) are true.
Example (2.22 Cont.)

The actual rule will be written in the tabular form

\[
\begin{array}{c}
p \\
p \rightarrow q \\
\therefore q
\end{array}
\]

where the three dots (\(\therefore\)) stand for the word ”therefore,” indicating that \(q\) is the conclusion for the premises \(p\) and \(p \rightarrow q\), which appear above the horizontal line.

This rule arises when we argue that if (1) \(p\) is true, and (2) \(p \rightarrow q\) is true (or \(p \Rightarrow q\)), then the conclusion \(q\) must be true. (After all, if \(q\) were false and \(p\) were true, then we could not have \(p \rightarrow q\) is true.)
A second rule of inference is given by the logical implication

\[(p \rightarrow q) \land (q \rightarrow r) \rightarrow (p \rightarrow r),\]

where \(p\), \(q\), and \(r\) are any statements. In tabular form it is written

\[
\begin{array}{c}
p \rightarrow q \\
q \rightarrow r \\
\therefore p \rightarrow r
\end{array}
\]

This rule, which is referred to as the Law of the Syllogism, arises in many arguments.
Example (2.23 Cont.)
For example, we may use it as follows:

1. If the integer 35244 is divisible by 396, then the integer 35244 is divisible by 66. 
   \[ p \rightarrow q \]

2. If the integer 35244 is divisible by 66, then the integer 35244 is divisible by 3. 
   \[ q \rightarrow r \]

3. Therefore, if the integer 35244 is divisible by 396, then the integer 35244 is divisible by 3. 
   \[ p \rightarrow r. \]
2.3: Logic Implication: Rules of Inference.

<table>
<thead>
<tr>
<th>Rule of Inference</th>
<th>Related Logical Implication</th>
<th>Name of Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) ( \frac{p}{\therefore p \rightarrow q} ) ( \frac{p \land (p \rightarrow q)}{\therefore q} )</td>
<td>( p \rightarrow q )</td>
<td>Rule of Detachment (Modus Ponens)</td>
</tr>
<tr>
<td>2) ( \frac{p \rightarrow q}{\therefore q \rightarrow r} ) ( \frac{q \rightarrow r}{\therefore p \rightarrow r} )</td>
<td>( [(p \rightarrow q) \land (q \rightarrow r)] \rightarrow (p \rightarrow r) )</td>
<td>Law of the Syllogism</td>
</tr>
<tr>
<td>3) ( \frac{p \rightarrow q}{\therefore \neg q \rightarrow \neg p} ) ( \frac{\neg q}{\therefore \neg p} )</td>
<td>( [(p \rightarrow q) \land \neg q] \rightarrow \neg p )</td>
<td>Modus Tollens</td>
</tr>
<tr>
<td>4) ( \frac{p}{\therefore \neg p} ) ( \frac{q}{\therefore \neg q} )</td>
<td>( [p \lor q \land \neg p \land \neg q] \rightarrow \neg p \land \neg q )</td>
<td>Rule of Disjunctive Syllogism</td>
</tr>
<tr>
<td>5) ( \frac{p \lor q}{\therefore \neg p \lor \neg q} ) ( \frac{\neg p}{\therefore \neg q} )</td>
<td>( [(p \lor q) \land \neg p \lor \neg q] \rightarrow q )</td>
<td>Rule of Contradiction</td>
</tr>
<tr>
<td>6) ( \frac{\neg p \rightarrow \neg q}{\therefore (p \land \neg q) \rightarrow \neg q} ) ( \frac{\neg q \rightarrow \neg p}{\therefore \neg p \rightarrow \neg q} )</td>
<td>( [(p \land \neg q) \land \neg \neg q] \rightarrow \neg \neg p )</td>
<td>Rule of Simplification</td>
</tr>
<tr>
<td>7) ( \frac{p \land q}{\therefore p} ) ( \frac{p \land q}{\therefore q} )</td>
<td>( [(p \land q) \land \neg p \land \neg q] \rightarrow q )</td>
<td>Rule of Disjunctive Amplification</td>
</tr>
<tr>
<td>8) ( \frac{\neg p \rightarrow \neg q}{\therefore \neg q \rightarrow \neg p} ) ( \frac{\neg q \rightarrow \neg p}{\therefore \neg p \rightarrow \neg q} )</td>
<td>( [(\neg p \lor \neg q) \lor (p \land q)] \rightarrow (p \lor q) )</td>
<td>Rule of Conditional Proof</td>
</tr>
<tr>
<td>9) ( \frac{\neg p \lor \neg q}{\therefore \neg p \lor \neg q} ) ( \frac{\neg q \lor \neg p}{\therefore \neg q \lor \neg p} )</td>
<td>( [(p \lor q) \land (p \lor q)] \rightarrow r )</td>
<td>Rule for Proof by Cases</td>
</tr>
<tr>
<td>10) ( \frac{p \lor q}{\therefore (p \lor q) \lor (q \lor r)} ) ( \frac{(p \lor q) \lor (q \lor r)}{\therefore r} )</td>
<td>( [(p \lor q) \land (r \lor s) \land (p \lor q)] \rightarrow (q \lor s) )</td>
<td>Rule of the Constructive Dilemma</td>
</tr>
<tr>
<td>11) ( \frac{p \lor q}{\therefore (p \lor q) \lor (q \lor r)} ) ( \frac{(p \lor q) \lor (q \lor r)}{\therefore r} )</td>
<td>( [(p \lor q) \land (r \lor s) \land (p \lor q)] \rightarrow (q \lor s) )</td>
<td>Rule of the Destructive Dilemma</td>
</tr>
</tbody>
</table>
Example (2.31)

Establish the validity of the argument

\((\neg p \lor \neg q) \rightarrow (r \land s)\)

\(r \rightarrow t\)

\(\neg t\)

\(\therefore p\)
Example (2.32)

In this instance we shall use the method of Proof by Contradiction. Consider the argument

\[ \neg p \leftrightarrow q \]
\[ q \rightarrow r \]
\[ \neg r \]

\[ \therefore p \]

To establish the validity for this argument, we assume the negation \( \neg p \) of the conclusion \( p \) as another premise. The objective now is to use these four premises to derive a contradiction \( F_0 \). Our derivation follows