Advanced Calculus (II)

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10.1: Introduction

Definition (10.1)

A *metric space* is a set X together with a function $\rho : X \times X \to \mathbf{R}$ (call the *metric* of X), that satisfies the following properties for all $x, y, z \in X$:

Positive Definite: $\rho(x, y) \geq 0$ with $\rho(x, y) = 0$ if and only if $x = y$,
Symmetric: $\rho(x, y) = \rho(y, x)$,
Triangle Inequality: $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

(Notice that by definition, $\rho(x, y)$ is finite-valued for all $x, y \in X$.)

Example (10.2)

For each $n \in \mathbf{N}$, $\mathbf{R}^n$ is a metric space with metric $\rho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$. (We shall call this the *usual metric* on $\mathbf{R}^n$. Unless specified otherwise, we shall always use the usual metric on $\mathbf{R}^n$.)

Example (10.3)

R is a metric space with metric

$$\sigma(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$

(This metric is called the *discrete metric*.)

Example (10.6)

Let $\mathcal{C}[a, b]$ represent the collection of continuous $f : [a, b] \to \mathbf{R}$ and

$$\|f\| := \sup_{x \in [a,b]} |f(x)|.$$

Then $\rho(f, g) := \|f - g\|$ is a metric on $\mathcal{C}[a, b]$.

Proof.

By the Extreme Value Theorem, $\|f\|$ is finite for each $f \in \mathcal{C}[a,b]$. By definition, $\|f\| \geq 0$ for all f, and $\|f\| = 0$ if and only if $f(x) = 0$ for every $x \in [a,b]$. Thus ρ is positive definite. Since ρ is obviously symmetric, it remains to verify the triangle inequality. But

$$\|f+g\| = \sup_{x\in[a,b]} |f(x)+g(x)|$$

$$\leq \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |g(x)| = \|f\| + \|g\|.$$

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Definition (10.7)

Let $a \in X$ and $r > 0$. The *open ball* (in X) with *center a* and *radius r* is the set

$$B_r(a) := \{x \in X : \rho(x, a) < r\},$$

and the *closed ball* (in X) with *center a* and *radius r* is the set

$$\{x \in X : \rho(x, a) \leq r\}.$$

Definition (10.8)

(i) A set $V \subseteq X$ is said to be *open* if and only if for every $x \in V$ there is an $\varepsilon > 0$ such that the open ball $B_\varepsilon(x)$ is contained in V.

(ii) A set $E \subseteq X$ is said to be *closed* if and only if $E^c := X \backslash E$ is open.

Remark (10.9)

Every open ball is open,and every closed ball is closed.

Proof.

Let $B_r(a)$ be an open ball. By definition, we must prove that given $x \in B_r(a)$ there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq B_r(a)$. Let $x \in B_r(a)$ and set $\varepsilon = r - \rho(x, a)$. (Look at Figure 8.5 to see why this choice of ε should work.) If $y \in B_\varepsilon(x)$, then by the triangle inequality, assumption, and the choice of ε,

$$\rho(y, a) \leq \rho(y, x) + \rho(x, a) < \varepsilon + \rho(x, a) = r.$$

Thus by Definition 10.7, $y \in B_r(a)$. In particular, $B_\varepsilon(x) \subseteq B_r(a)$. Similarly, we can show that $\{x \in X : \rho(x, a) > r\}$ is also open. Hence, every closed ball is closed. □

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Remark (10.11)

In an arbitrary metric space, the empty set $\emptyset$ and the whole space X are both open and closed.

Definition (10.13)

Let $\{x_n\}$ be a sequence in a metric space X.
(i) $\{x_n\}$ *converges* (in X) if there is a point $a \in X$ (called the *limit* of x_n) such that for every $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$n \geq N \quad \text{implies} \quad \rho(x_n, a) < \varepsilon.$$

(ii) $\{x_n\}$ is *Cauchy* if for every $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$n, m \geq N \quad \text{implies} \quad \rho(x_n, x_m) < \varepsilon.$$

(iii) $\{x_n\}$ is *bounded* if there is an $M > 0$ and $a, b \in X$ such that $\rho(x_n, b) \leq M$ for all $n \in \mathbf{N}$.

Theorem (10.14)

Let X be a metric space.
(i) A sequence in X can have at most one limit.

(ii) If $x_n \in X$ converges to a and $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$, then x_{n_k} converges to a as $k \to \infty$.

(iii) Every convergent sequence in X is bounded.

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Remark (10.15)

Let $x_n \in X$. Then $x_n \to a$ as $n \to \infty$ if and only if for every open set V that contains a, there is an $N \in \mathbf{N}$ such that $n \geq N$ implies $x_n \in V$.

Proof.

Suppose that $x_n \to a$, and let V be an open set that contains a. By Definition 10.8, there is an $\varepsilon > 0$ such that $B_\varepsilon(a) \subseteq V$. Given this ε, use Definition 10.13 to choose an $N \in \mathbf{N}$ such that $n \geq N$ implies $x_n \in B_\varepsilon(a)$. By the choice of ε, $x_n \in V$ for all $n \geq N$.

Conversely, let $\varepsilon > 0$ and set $V = B_\varepsilon(a)$. Then V is an open set that contains a; hence, by hypothesis, there is an $N \in \mathbf{N}$ such that $n \geq N$ implies $x_n \in V$. In particular, $\rho(x_n, a) < \varepsilon$ for all $n \geq N$. □

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Let $x_n \in X$. Then $x_n \to a$ as $n \to \infty$ if and only if for every open set V that contains a, there is an $N \in \mathbf{N}$ such that $n \geq N$ implies $x_n \in V$.

Proof.

Suppose that $x_n \to a$, and let V be an open set that contains a. By Definition 10.8, there is an $\varepsilon > 0$ such that $B_\varepsilon(a) \subseteq V$. Given this ε, use Definition 10.13 to choose an $N \in \mathbf{N}$ such that $n \geq N$ implies $x_n \in B_\varepsilon(a)$. By the choice of ε, $x_n \in V$ for all $n \geq N$.

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Theorem (10.16)

Let $E \subseteq X$. Then E is closed if and only if the limit of every convergent sequence $x_k \in E$ satisfies

$$\lim_{k\to\infty} x_k \in E.$$

Proof.

The theorem is vacuously satisfied if E is the empty set. Suppose hat $E \neq \emptyset$ is closed but some sequence $x_n \in E$ converges to a point $x \in E^c$. Since E is closed, E^c is open. Thus, by Remark 10.15, there is an $N \in \mathbf{N}$ such that $n \geq N$ implies $x_n \in E^c$, a contradiction.

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Definition (10.19)

A metric space X is said to be *complete* if and only if every Cauchy sequence $x_n \in X$ converges to some point in X.

Theorem (10.21)

Let X be a complete metric space and E be a subset of X. Then E (as a subspace) is complete if and only if E (as a subset) is closed.

Proof.

Suppose that E is complete and $x_n \in E$ converges. By Theorem 10.14iv, $\{x_n\}$ is Cauchy. Since E is complete, it follows from Definition 10.19 that the limit of $\{x_n\}$ belongs to E. Thus, by Theorem 10.16, E is closed.

Conversely, suppose that E is closed and $x_n \in E$ is Cauchy in E. Since the metric on X and E are identical, $\{x_n\}$ is Cauchy in X. Since X is complete, it follows that $x_n \to x$, as $n \to \infty$, for some $x \in X$. But E is closed, so x must belong to E. Thus E is complete by definition. □

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Suppose that E is complete and $x_n \in E$ converges. By Theorem 10.14iv, $\{x_n\}$ is Cauchy. Since E is complete, it follows from Definition 10.19 that the limit of $\{x_n\}$ belongs to E. Thus, by Theorem 10.16, E is closed.

Conversely, suppose that E is closed and $x_n \in E$ is Cauchy in E. Since the metric on X and E are identical, $\{x_n\}$ is Cauchy in X. Since X is complete, it follows that $x_n \to x$, as $n \to \infty$, for some $x \in X$. But E is closed, so x must belong to E. Thus E is complete by definition. □

Theorem (10.21)

Let X be a complete metric space and E be a subset of X. Then E (as a subspace) is complete if and only if E (as a subset) is closed.

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Thank you.