Advanced Calculus (II)

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10.2: Limit Of Functions

Definition (10.22)

A point a is said to be a *cluster point* (of X) if and only if $B_\delta(a)$ contains infinitely many points for each $\delta > 0$.

Notation:
E is a subspace of X.
$B_r^E(a) := \{x \in E : \rho(x, a) < r\}$.(relative balls in E)

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Example (10.24)

Show that every point in the interval [0,1] is a cluster point of the open interval (0,1).

Proof.

Let $x_0 \in [0, 1]$ and $\delta > 0$. Then $x_0 + \delta > 0$ and $x_0 - \delta < 1$. In particular, $(x_0 - \delta, x_0 + \delta) \cap (0, 1)$ is itself a nondegenerate interval, say (a, b). But (a, b) contains infinitely many points, e.g., $(a + b)/2$, $(2a + b)/3$, $(3a + b)/4$, ... Therefore, x_0 is a cluster point of $(0, 1)$. □

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Definition (10.25)

Given (X,ρ), (Y,τ), let a be a cluster point of X and $f : X\backslash\{a\} \to Y$. Then $f(x)$ is said to *converge to L, as x approaches a*, if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$(1) \qquad 0 < \rho(x,a) < \delta \quad \text{implies} \quad \tau(f(x),L) < \varepsilon.$$

In this case we write

$$L = \lim_{x \to a} f(x)$$

and call L the *limit* of $f(x)$ as x approaches a.

Theorem (10.26)

Let a be a cluster point of X and $f, g : X\backslash\{a\} \to Y$.

(i) If $f(x) = g(x)$ *for all* $x \in X\backslash\{a\}$ *and* $f(x)$ *has a limit as* $x \to a$*, then* $g(x)$ *also has a limit as* $x \to a$ *and*

$$\lim_{x \to a} g(x) = \lim_{x \to a} f(x).$$

(ii) [Sequential Characterizations of Limits]. The limit

$$L := \lim_{x \to a} f(x)$$

exists if and only if $f(x_n) \to L$ *as* $n \to \infty$ *for every sequence* $x_n \in X\backslash\{a\}$ *that converges to a as* $n \to \infty$.

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(iii) Suppose that $Y = \mathbf{R}^n$. *If* $f(x)$ *and* $g(x)$ *have a limit as* x *approaches* a*, then so do* $(f+g)(x)$, $(f \cdot g)(x)$, $(\alpha f)(x)$, *and* $(f/g)(x)$ *(when* $Y = \mathbf{R}$ *and the limit of* $g(x)$ *is nonzero). In fact,*

$$\lim_{x \to a}(f+g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x),$$

$$\lim_{x \to a}(\alpha f)(x) = \alpha \lim_{x \to a} f(x),$$

$$\lim_{x \to a}(f \cdot g)(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x),$$

and (when $Y = \mathbf{R}$ *and the limit of* $g(x)$ *is nonzero)*

$$\lim_{x \to a}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}.$$

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(iv) [Squeeze Theorem for Functions] Suppose that $Y = \mathbf{R}$*. If* $h : X\backslash\{a\} \to \mathbf{R}$ *satisfies* $g(x) \le h(x) \le f(x)$ *for all* $x \in X\backslash\{a\}$*, and*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L,$$

then the limit of h exists, as $x \to a$*, and*

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(v) [Comparison Theorem for Functions] Suppose that $Y = \mathbf{R}$*. If* $f(x) \le g(x)$ *for all* $x \in X\backslash\{a\}$*, and f and g have a limit as x approaches a, then*

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(v) [Comparison Theorem for Functions] Suppose that $Y = \mathbf{R}$. *If* $f(x) \le g(x)$ *for all* $x \in X\backslash\{a\}$, *and* f *and* g *have a limit as* x *approaches* a, *then*

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Definition (10.27)

Let E be a nonempty subset of X and $f : E \to Y$.
(i) f is said to be *continuous at a point* $a \in E$ if and only if given ε there is a $\delta > 0$ such that

$$\rho(x, a) < \delta \quad \text{and} \quad x \in E \quad \text{imply} \quad \tau(f(x), f(a)) < \varepsilon.$$

(ii) f is said to be *continuous on* E (Notation: $f : E \to Y$ is continuous) if and only if f is continuous at every $x \in E$.

Theorem (10.28)

Let E be a nonempty subset of X and $f, g : E \to Y$.

(i) f is continuous at $a \in E$ if and only if $f(x_n) \to f(a)$, as $n \to \infty$, for all sequences $x_n \in E$ that converge to a.

(ii) Suppose that $Y = \mathbf{R}^n$. If f, g are continuous at a point $a \in E$ (respectively, continuous on a set E), then so are $f + g$, $f \cdot g$, and αf (for any $\alpha \in \mathbf{R}$). Moreover, in the case $Y = \mathbf{R}$, f/g is continuous at $a \in E$ when $g(a) \neq 0$ (respectively, on E when $g(x) \neq 0$ for all $a \in E$.)

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Theorem (10.29)

Suppose that X, Y, and Z are metric spaces, a is a cluster point of X, $f : X \to Y$, and $g : f(X) \to Z$. If $f(x) \to L$ as $x \to a$ and g is continuous at L, then

$$\lim_{x \to a} (g \circ f)(x) = g\left(\lim_{x \to a} (f)(x) \right).$$

Thank you.