Advanced Calculus (II)

WEN-CHING LIEN

Department of Mathematics
National Cheng Kung University

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10.3: Interior, Closure, and boundary

Theorem (10.31)

Let X be metric space.

(i) If $\{V_\alpha\}_{\alpha\in A}$ is any collection of open sets in X, then

$$\bigcup_{\alpha\in A} V_\alpha \quad \text{is open.}$$

(ii) If $\{V_k : k = 1, 2, \ldots, n\}$ is a finite collection of open sets in X, then

$$\bigcap_{k=1}^{n} V_k := \bigcap_{k\in\{1,2,\ldots,n\}} V_k \quad \text{is open.}$$

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Theorem (10.31)

(iii) If $\{E_\alpha\}_{\alpha\in A}$ *is any collection of closed sets in* X*, then*

$$\bigcap_{\alpha\in A} E_\alpha$$

is closed.

(iv) If $\{E_k : k = 1, 2, \ldots, n\}$ *is a finite collection of closed sets in* X*, then*

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is closed.

(v) If V *is open in* X *and* E *is closed in* X*, then* $V\setminus E$ *is open and* $E\setminus V$ *is closed.*

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Example

(1)

$$\bigcap_{k\in\mathbf{N}}(-\frac{1}{k},\frac{1}{k})$$

(2)

$$\bigcup_{k\in\mathbf{N}}[\frac{1}{k+1},\frac{k}{k+1}]$$

Definition (10.33)

Let E be a subset of a metric Space X.
(i) The *interior* of E is the set

$$E^o := \bigcup \{V : V \subseteq E \text{ and } V \text{ is open in } X\}.$$

(ii) The *closure* of E is the set

$$\overline{E} := \bigcap \{B : B \supseteq E \text{ and } B \text{ is closed in } X\}.$$

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Theorem (10.34)

Let $E \subseteq X$. Then
(i) $E^o \subseteq E \subseteq \overline{E}$,

(ii) if V is open and $V \subseteq E$, then $V \subseteq E^o$, and

(iii) if C is closed and $C \supseteq E$, then $V \supseteq \overline{E}$.

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Example

(1) $E = \{(x,y):\ -1 \leq x \leq 1,\ -|x| < y < |x|\}$

$E^o = ?\quad \overline{E} = ?$

(2) $E = B_1(-2,0) \cup B_1(2,0) \cup \{(x,0) : -1 \leq x \leq 1\}$

$E^o = ?\quad \overline{E} = ?$

Definition (10.37)

Let $E \subseteq X$. The *boundary* of E is the set

$$\partial E := \{x \in X : \text{for all } r > 0,\ B_r(x) \cap E \neq \emptyset \text{ and } B_r(x) \cap E^c \neq \emptyset\}.$$

Theorem (10.39)

Let $E \subseteq X$. *Then* $\partial E = \overline{E} \backslash E^o$.

Theorem (10.40)

Let $A, B \subseteq X$. Then

(i)

$$(A \cup B)^o \supseteq A^o \cup B^o, \quad (A \cap B)^o = A^o \cap B^o,$$

(ii)

$$\overline{A \cup B} = \overline{A} \cup \overline{B}, \quad \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$$

(iii)

$$\partial(A \cup B) \subseteq \partial A \cup \partial B, \text{ and}$$

$$\partial(A \cap B) \subseteq (A \cap \partial B) \cup (B \cap \partial A) \cup (\partial A \cap \partial B).$$

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Thank you.