Advanced Calculus (II)

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Ch10: Metric Spaces

10.4: Compact Sets

Definition (10.41)

Let $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$ be a collection of subsets of a metric space X and suppose that E is a subset of X.

(i) $\mathcal{V}$ is said to *cover* E (or be a *covering* of E) if and only if

$$E \subseteq \bigcup_{\alpha \in A} V_\alpha.$$

(ii) $\mathcal{V}$ is said to be an *open covering* of E if and only if $\mathcal{V}$ covers E and each V_α is open.

(iii) Let $\mathcal{V}$ be a covering of E. $\mathcal{V}$ is said to have a *finite* (respectively, *countable*) *subcovering* if and only if there is a finite (respectively, countable) subset A_0 of A such that $\{V_\alpha\}_{\alpha \in A}$ covers E.

Definition (10.42)

A subset H of a metric space X is said to be *compact* if and only if every open covering of H has a finite subcover.

Remark (10.44)

A compact set is always closed.

Proof.

Suppose that H is compact but not closed. Then H is nonempty and (by Theorem 10.16) there is a convergent sequence $x_k \in H$ whose limit x does not belong to H. For each $y \in H$, set $r(y) := \frac{\rho(x,y)}{2}$. Since x does not belong to H, $r(y) > 0$; hence, each $B_{r(y)}r(y)$ is open and contains y; i.e., $\{B_{r(y)}r(y) : y \in H\}$ is an open covering of H. Since H is compact, we can choose points y_j and radii is $r_j := r(y_j)$ such that $\{B_{r_j}(y_j) : j = 1, 2, \ldots, N\}$ covers H.

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Set $r := \min\{r_1, \dots, r_N\}$. (This is a finite set of positive numbers, so r is also positive.) Since $x_k \to x$ as $k \to \infty$, $x_k \in B_r(x)$ for large k. But $x_k \in B_r(x) \cap H$ implies $x_k \in B_{r_j}(y_j)$ for some $j \in \mathbf{N}$. Therefore, it follows from the choices of r_j and r, and from the triangle inequality, that

$$r_j \geq \rho(x_k, y_j) \geq \rho(x, y_j) - \rho(x_k, x)$$

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a contradiction. □

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Remark (10.45)

A closed subset of H of a compact set is compact.

Theorem (10.46)

Let H be a subset of metric space X. If H is compact, then H is closed and bounded.

Proof.

Suppose that H is compact. By Remark 10.44, H is closed. It is also bounded. Indeed, fix $b \in X$ and observe that $\{B_n(b) : n \in \mathbf{N}\}$ covers X. Since H is compact, it follows that

$$H \subset \bigcup_{n=1}^{N} B_n(b)$$

for some $N \in \mathbf{N}$. Since these balls are nested, we conclude that $H \subset B_N(b)$; i.e., H is bounded. □

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Remark (10.47)

The converse of Theorem 10.46 is false for arbitrary metric spaces.

Proof.

Let $X = \mathbf{R}$ be the discrete metric space introduced in Example 10.3. Since $\sigma(0, x) \leq 1$ for all $x \in \mathbf{R}$, every subset of X is bounded. Since $x_k \to x$ in X implies $x_k \to x$ for large k, every subset of X is closed. Thus $[0, 1]$ is a closed bounded subset of X. Since $\{x\}_{x \in [0,1]}$ is an uncountable open covering of $[0, 1]$ that has no proper finite subcover, we conclude that $[0, 1]$ is closed and bounded, but not compact. □

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Definition (10.48)

A metric space X is said to be *separable* if and only if it contains a countable dense subset; i.e., there is a countable set Z of X such that for every point $a \in X$ there is a sequence $x_k \in Z$ such that $x_k \to a$ as $x \to \infty$.

Theorem (10.49 Lindelöf)

Let E be a subset of separable metric space X. If $\{V_\alpha\}_\alpha$ is a collection of open sets and $E \subseteq \cup_{\alpha\in A} V_\alpha$, then there is a countable subset A_0 of A such that

$$E \subseteq \bigcup_{\alpha\in A_0} V_\alpha.$$

Theorem (10.50 Heine-Borel)

Let X be a separable metric space that satisfies the Bolzano-Weierstrass Property and H be a subset of X. Then H is compact if and only if it is closed and bounded.

Definition (10.51)

Let X be a metric space, E be a nonempty subset of X, and $f : E \to Y$. Then f is said to be *uniformly continuous* on E (notation: $f : E \to Y$ is uniformly continuous) if and only if given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\rho(x, a) < \delta \quad \text{and} \quad x, a \in E \quad \text{imply} \quad \tau(f(x), f(a)) < \varepsilon.$$

Theorem (10.52)

Suppose that E is a compact subset of X and $f : X \to Y$. Then F is uniformly continuous on E if and only if f is continuous on E.

Thank you.