Advanced Calculus (II)

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10.6: Continuous Functions

Recall:
Given $(X, \rho),\ (Y, \tau)$, a function $f : X \to Y$ is continuous.
$\Leftrightarrow$ Given $a \in X$, and $\varepsilon > 0$, there is a $\delta > 0$ such that $\rho(x, a) < \delta$ implies

$$\tau(f(x), f(a)) < \varepsilon.$$

(i.e., $\quad B_\delta(a) \subset f^{-1}(B_\varepsilon(f(a))).\quad$)

Theorem (10.58)

Let X and Y be metric spaces, and let $f : X \to Y$. Then f is continuous if and only if $f^{-1}(V)$ is open in X for every open V in Y.

Corollary (10.59)

Let $E \subseteq X$ and $f : E \to Y$. Then f is continuous on E if and only if $f^{-1}(V) \cap E$ is relatively open in E for all open sets V in Y.

Theorem (10.61)

If H is compact in X and $f : H \to Y$ is continuous on H, then $f(H)$ is compact in Y.

Proof.

Suppose that $\{V_\alpha\}_{\alpha \in A}$ is an open covering of $f(H)$. By Theorem 1.43,

$$H \subseteq f^{-1}(f(H)) \subseteq f^{-1}\left(\bigcup_{\alpha \in A} V_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(V_\alpha).$$

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Hence, by Corollary 10.59, $\{f^{-1}(V_\alpha)\}_{\alpha \in A}$ is a covering of H whose sets are all relatively open in H. since H is compact, there are indices $\alpha_1, \alpha_2, \ldots, \alpha_N$ such that

$$H \subseteq \bigcup_{j=1}^{N} f^{-1}(V_{\alpha_j})$$

(see Exercise 7, p.316). It follows from Theorem 1.43 that

$$f(H) \subseteq f\left(\bigcup_{j=1}^{N} f^{-1}(V_{\alpha_j})\right) = \bigcup_{j=1}^{N} (f \circ f^{-1})(V_{\alpha_j}) = \bigcup_{j=1}^{N} (V_{\alpha_j}).$$

Therefore, $f(H)$ is compact. □

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Theorem (10.62)

If E is connected in X and $f : E \to Y$ is continuous on E, then $f(E)$ is connected in Y.

Proof.

Suppose that $f(E)$ is not connected. By Definition 10.53, there exist a pair $U, V \subset Y$ of relatively open sets in $f(E)$ that separate $f(E)$. By Exercise 4, $f^{-1}(U) \cap E$ and $f^{-1}(V) \cap E$ are relatively open in E. Since $f(E) = U \cup V$, we have

$$E = (f^{-1}(U) \cap E) \cup (f^{-1}(V) \cap E).$$

Since $U \cap V = \emptyset$, we also have $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Thus $f^{-1}(U) \cap E$, $f^{-1}(V) \cap E$ is a pair of relatively open sets that separate E. Hence, by Definition 10.53, E is not connected, a contradiction.

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Theorem (10.63 Extreme Value Theorem)

Let H be a nonempty, compact set in a metric space X and suppose that $f : H \to \mathbf{R}$ is continuous. Then

$$M := \sup\{f(x) : x \in H\} \quad m := \inf\{f(x) : x \in H\}$$

are finite real numbers and there exist points $x_M, x_m \in H$ such that $M = f(x_M)$ and $m = f(x_m)$.

Proof.

By symmetry, it suffices to prove the result for M. Since H is compact, $f(H)$ is compact. Hence, by Theorem 10.46, $f(H)$ is closed and bounded. Since $f(H)$ is bounded, M is finite. By the Approximation Property, choose $x_k \in H$ such that $f(x_k) \to M$ as $k \to \infty$. Since $f(H)$ is closed, $M \in f(H)$. Therefore, there is an $x_M \in H$ such that $M = f(x_M)$. A similar argument shows that m is finite and attained on H □

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Theorem (10.64)

Let X and Y be metric spaces. If H is a compact subset of X and $f : H \to Y$ is 1-1 and continuous, then f^{-1} is continuous on $f(H)$.

Proof.

By Exercise 4a, it suffices to show that $(f^{-1})^{-1}$ takes closed sets in X to relatively closed sets in $f(H)$. Let E be closed in X. Then $E \cap H$ is a closed subset of H, so by Remark 10.45, $E \cap H$ is compact. Hence, by Theorem 10.61, $f(E \cap H)$ is compact, in particular closed. Since f is 1-1, $f(E \cap H) = f(E) \cap f(H)$ (see Exercise 6, p.33). Since $f(E \cap H)$ and $f(H)$ are closed, it follows that $f(E) \cap f(H)$ is relatively closed in $f(H)$. Since $(f^{-1})^{-1} = f$, we conclude that $(f^{-1})^{-1}(E) \cap f(H)$ is relatively closed in $f(H)$.

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Thank you.