Advanced Calculus (II)

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11.1: Partial Derivatives and Partial Integrals

Notation:

(1) $E_1 \times E_2 \times \ldots \times E_n$

$$:= \{(x_1, x_2, \ldots, x_n) : x_j \in E_j,\ \textit{for } j = 1, \ldots, n\}$$

(2) The partial derivative f_{x_j} exists at a point $\mathbf{a}$ if and only if the limit

$$\frac{\partial f}{\partial x_j}(\mathbf{a}) = \lim_{h \to 0} \frac{f(\mathbf{a} + he_j) - f(\mathbf{a})}{h}$$

exists.

(3)

$$f_{x_j x_k} := \frac{\partial^2 f}{\partial x_k \partial x_j} := \frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_j}\right).$$

Definition (11.1)

Let V be a nonempty, open subset of $\mathbf{R}^n$, let $f : V \to \mathbf{R}^m$, and let $p \in \mathbf{N}$.
(i) f is said to be $\mathcal{C}^p$ on V if and only if each partial derivative of f of order $k \leq p$ exists and is continuous on V.

(ii) f is said to be $\mathcal{C}^\infty$ on V if and only if f is $\mathcal{C}^p$ on V for all $p \in \mathbf{N}$.

Theorem (11.2)

Suppose that V is open in $\mathbf{R}^2$, that $(a, b) \in V$, and that $f : V \to \mathbf{R}$. If f is $\mathcal{C}^1$ on V, and if one of the mixed second partial derivatives of f exists on V and is continuous at the point (a, b), then the other mixed second partial derivative exists at (a, b) and

$$\frac{\partial^2 f}{\partial y \partial x}(a, b) = \frac{\partial^2 f}{\partial x \partial y}(a, b).$$

Note: These hypotheses are met if $f \in \mathcal{C}^2(H)$.

Proof.

Suppose that f_{yx} exists on V and is continuous at the point (a, b). Consider $\Delta(h, k) := f(a+h, b+k) - f(a+h, b)$ $-f(a, b+k) + f(a, b)$, define for $|h|, |k| < r/\sqrt{2}$, where $r > 0$ is so small that $B_r(a, b) \subset V$. Apply the Mean Value Theorem twice to choose scalars $s, t \in (0, 1)$ such that

$$\Delta(h,k) = k\frac{\partial f}{\partial y}(a+h, b+tk) - k\frac{\partial f}{\partial y}(a, b+tk)$$

$$= hk\frac{\partial f^2}{\partial x \partial y}(a+sh, b+tk).$$

Since this last mixed partial derivative is continuous at the point (a, b), we have

$$(1) \qquad \lim_{k\to 0}\lim_{h\to 0}\frac{\Delta(h,k)}{hk} = \frac{\partial^2 f}{\partial x \partial y}(a, b).$$

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Proof.

On the other hand, the Mean Value Theorem also implies that there is a scalar $u \in (0, 1)$ such that

$$\Delta(h,k) = f(a+h, b+k) - f(a, b+k) - f(a+h, b) + f(a, b)$$

$$= h\frac{\partial f}{\partial x}(a+uh, b+k) - h\frac{\partial f}{\partial x}(a+uh, b).$$

Hence, it follows from (1) that

$$\lim_{k\to 0}\lim_{h\to 0}\frac{1}{k}\left(\frac{\partial f}{\partial x}(a+uh, b+k) - \frac{\partial f}{\partial x}(a+uh, b)\right)$$

$$= \lim_{k\to 0}\lim_{h\to 0}\frac{\Delta(h,k)}{hk} = \frac{\partial^2 f}{\partial x \partial y}(a, b).$$

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□

Proof.

Since f_x is continuous on $B_r(a,b)$, we can let $h = 0$ in the first expression. We conclude by definition that

$$\frac{\partial^2 f}{\partial y \partial x}(a,b) = \lim_{k \to 0} \frac{1}{k}\left(\frac{\partial f}{\partial x}(a, b+k) - \frac{\partial f}{\partial x}(a,b)\right) = \frac{\partial^2 f}{\partial x \partial y}(a,b).$$

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Example (11.3)

Prove that

$$f(x,y)=\begin{cases} xy\left(\frac{x^2-y^2}{x^2+y^2}\right) & (x,y)\neq 0 \\ 0 & (x,y)=0 \end{cases}$$

Theorem (11.4)

Let $H = [a, b] \times [c, d]$ *be a rectangle and suppose that* $f : H \to \mathbf{R}$ *is continuous. If*

$$F(y) = \int_a^b f(x, y)dx,$$

then F *is continuous on* $[c, d]$*; i.e.,*

$$\lim_{\substack{y \to y_0 \\ y \in [c,d]}} \int_a^b f(x, y)dx = \int_a^b \lim_{\substack{y \to y_0 \\ y \in [c,d]}} f(x, y)dx$$

for all $y_0 \in [c, d]$.

Proof.

For each $y \in [c, d]$, $f(\cdot, y)$ is continuous on $[a, b]$. Hence, by Theorem 5.10, $F(y)$ exists for $y \in [c, d]$.

Fix $y_0 \in [c, d]$ and let $\varepsilon > 0$. Since H is compact, f is uniformly continuous on H. Hence, choose $\delta > 0$ such that $\|(x, y) - (z, w)\| < \delta$ and $(x, y), (z, w) \in H$ imply

$$|f(x, y) - f(z, w)| < \frac{\varepsilon}{b - a}.$$

Since $|y - y_0| = \|(x, y) - (x, y_0)\|$, it follows that

$$|F(y) - F(y_0)| \leq \int_a^b |f(x, y) - f(x, y_0)| dx < \varepsilon$$

for all $y \in [c, d]$ that satisfy $|y - y_0| < \delta$. We conclude that F is continuous on $[c, d]$. □

Proof.

For each $y \in [c, d]$, $f(\cdot, y)$ is continuous on $[a, b]$. Hence, by Theorem 5.10, $F(y)$ exists for $y \in [c, d]$.

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For each $y \in [c, d]$, $f(\cdot, y)$ is continuous on $[a, b]$. Hence, by Theorem 5.10, $F(y)$ exists for $y \in [c, d]$.
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for all $y \in [c, d]$ that satisfy $|y - y_0| < \delta$. We conclude that F is continuous on $[c, d]$.

□

Theorem (11.5)

Let $H = [a, b] \times [c, d]$ be a rectangle in $\mathbf{R}^2$ and let $f : H \to \mathbf{R}$. Suppose that $f(\cdot, y)$ is integrable on $[a, b]$ for each $y \in [c, d]$ and that the partial derivative $f_y(x, \cdot)$ exists on $[c, d]$ for each $x \in [a, b]$. If the two-variable function $f_y(x, y)$ is continuous on H, then

$$\frac{d}{dy} \int_a^b f(x, y)dx = \int_a^b \frac{\partial f}{\partial y}(x, y)dx$$

for all $y \in [c, d]$.

Note: These hypotheses are met if $f \in \mathcal{C}^1(H)$.

Proof.

Recall that "$f_y(x,\cdot)$ exists on $[c,d]$" means that $f_y(x,\cdot)$ exists on (c,d), and

$$f_y(x,c) := \lim_{h\to 0+} \frac{f(x,c+h)-f(x,c)}{h},$$

$$f_y(x,d) := \lim_{h\to 0-} \frac{f(x,d+h)-f(x,d)}{h}$$

both exist (see Definition 4.6). Hence, it suffices to show that

$$\lim_{h\to 0+} \int_a^b \frac{f(x,y+h)-f(x,y)}{h}dx = \int_a^b \frac{\partial f}{\partial y}(x,y)dx$$

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Proof.

$$\lim_{h \to 0-} \int_a^b \frac{f(x, y+h) - f(x, y)}{h} dx = \int_a^b \frac{\partial f}{\partial y}(x, y) dx$$

for $y \in (c, d]$. The arguments are similar; we provide the details only for the first identity.

Fix $x \in [a, b]$ and $y \in [c, d)$, and let $h > 0$ be so small that $y + h \in [c, d)$. Let $\varepsilon > 0$. By uniform continuity, choose a $\delta > 0$ so small that $|y - c| < \delta$ and $x \in [a, b]$ imply $|f_y(x, y) - f_y(x, c)| < \varepsilon/(b - a)$. By the Mean Value Theorem, choose a point $c(x; h)$ between y and $y + h$ such that

$$F(x, y, h) := \frac{f(x, y+h) - f(x, y)}{h} = \frac{\partial f}{\partial y}(x, c(x; h)).$$

Proof.

$$\lim_{h\to 0-} \int_a^b \frac{f(x,y+h)-f(x,y)}{h}dx = \int_a^b \frac{\partial f}{\partial y}(x,y)dx$$

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$$F(x,y,h) := \frac{f(x,y+h)-f(x,y)}{h} = \frac{\partial f}{\partial y}(x,c(x;h)).$$

□

Proof.

Since $|c(x;h) - y| = c(x;h) - y \leq h$, it follows that if $0 < h < \delta$, then

$$\left| F(x,y,h) - \int_a^b \frac{\partial f}{\partial y}(x,y)dx \right|$$

$$\leq \int_a^b \left| \frac{\partial f}{\partial y}(x, c(x;h)) - \frac{\partial f}{\partial y}(x,y) \right| dx < \varepsilon.$$

Therefore,

$$\frac{d}{dy} \int_a^b f(x,y)dx = \int_a^b \frac{\partial f}{\partial y}(x,y)dx.$$

Proof.

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Therefore,

$$\frac{d}{dy} \int_a^b f(x,y)dx = \int_a^b \frac{\partial f}{\partial y}(x,y)dx.$$

$\square$

Definition (11.6)

Let $a < b$ be extended real numbers, let I be an interval in $\mathbf{R}$, and suppose that $f : (a, b) \times I \to \mathbf{R}$. The improper integral

$$\int_a^b f(x, y)dx$$

is said to *converge uniformly* on I if and only if $f(\cdot, y)$ is improperly integrable on (a, b) for each $y \in I$ and given $\varepsilon > 0$ there exist real numbers $A, B \in (a, b)$ such that

$$\left| \int_a^b f(x, y)dx - \int_\alpha^\beta f(x, y)dx \right| < \varepsilon$$

for all $a < \alpha < A$, $B < \beta < b$ and all $y \in I$.

Theorem (11.7 Weierstrass M-Test)

Suppose that $a < b$ are extended real numbers, that I is an interval in $\mathbf{R}$, *that* $f : (a, b) \times I \to \mathbf{R}$, *and that* $f(\cdot, y)$ *is locally integrable on the interval* (a, b) *for each* $y \in I$. *If there is a function* $M : (a, b) \to \mathbf{R}$, *absolutely integrable on* (a, b), *such that*

$$|f(x, y)| \leq M(x)$$

for all $x \in (a, b)$ *and* $y \in I$, *then*

$$\int_a^b f(x, y) dx$$

converges uniformly on I.

Proof.

Let $\varepsilon > 0$. By hypothesis and the Comparison Test for improper integrals, $\int_a^b f(x,y)dx$ exists and is finite for each $y \in I$. Moreover, since $M(x)$ is improperly integrable on (a,b), there exist real numbers A, B such that $a < A < B < b$ and

$$\int_\alpha^A M(x)dx + \int_B^b M(x)dx < \varepsilon.$$

Thus for each $a < \alpha < A < B < \beta < b$ and each $y \in I$, we have

$$\left| \int_a^b f(x,y)dx - \int_\alpha^\beta f(x,y)dx \right| \le \int_a^\alpha |f(x,y)|dx + \int_\beta^b |f(x,y)|dx$$

$$\le \int_a^A M(x)dx + \int_B^b M(x)dx < \varepsilon$$

Proof.

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$$\left| \int_a^b f(x,y)dx - \int_\alpha^\beta f(x,y)dx \right| \leq \int_a^\alpha |f(x,y)|dx + \int_\beta^b |f(x,y)|dx$$

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Thus for each $a < \alpha < A < B < \beta < b$ and each $y \in I$, we have

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$$\leq \int_a^A M(x)dx + \int_B^b M(x)dx < \varepsilon$$

Theorem (11.8)

Suppose that $a < b$ are extended real numbers, that $c < d$ are finite real numbers, and that $f : (a, b) \times [c, d] \to \mathbf{R}$ is continuous. If

$$F(y) = \int_a^b f(x, y)dx$$

converges uniformly on $[c, d]$; i.e.,

$$\lim_{\substack{y \to y_0 \\ y \in [c,d]}} \int_a^b f(x, y)dx = \int_a^b \lim_{\substack{y \to y_0 \\ y \in [c,d]}} f(x, y)dx$$

for all $y_0 \in [c, d]$.

Proof.

Let $\varepsilon > 0$ and $y_0 \in [c, d]$. Choose real numbers A, B such that $a < A < B < b$ and

$$\left| F(y) - \int_A^B f(x, y)dx \right| < \frac{\varepsilon}{3}$$

for all $y \in [c, d]$. By Theorem 11.4, choose $\delta > 0$ such that

$$\left| \int_A^B (f(x, y) - f(x, y_0))dx \right| < \frac{\varepsilon}{3}$$

for all $y \in [c, d]$ that satisfy $|y - y_0| < \delta$. Then

Proof.

Let $\varepsilon > 0$ and $y_0 \in [c, d]$. Choose real numbers A, B such that $a < A < B < b$ and

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Let $\varepsilon > 0$ and $y_0 \in [c, d]$. Choose real numbers A, B such that $a < A < B < b$ and

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for all $y \in [c, d]$. By Theorem 11.4, choose $\delta > 0$ such that

$$\left| \int_A^B (f(x, y) - f(x, y_0))dx \right| < \frac{\varepsilon}{3}$$

for all $y \in [c, d]$ that satisfy $|y - y_0| < \delta$. Then

□

Proof.

$$
\begin{aligned}
|F(y)-F(y_0)| &\le \left|F(y)-\int_A^B f(x,y)dx\right| \\
&+\left|\int_A^B (f(x,y)-f(x,y_0))dx\right| \\
&+\left|F(y_0)-\int_A^B f(x,y_0)dx\right| \\
&< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon
\end{aligned}
$$

for all $y \in [c,d]$ that satisfy $|y-y_0|<\delta$. □

Proof.

$$|F(y) - F(y_0)| \leq \left| F(y) - \int_A^B f(x,y)dx \right|$$

$$+ \left| \int_A^B (f(x,y) - f(x,y_0))dx \right|$$

$$+ \left| F(y_0) - \int_A^B f(x,y_0)dx \right|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for all $y \in [c,d]$ that satisfy $|y - y_0| < \delta$. □

Theorem (11.9)

Suppose that $a < b$ are extended real numbers, that $c < d$ are finite real numbers, that $f : (a, b) \times [c, d] \to \mathbf{R}$ is continuous, and that the improper integral

$$F(y) = \int_a^b f(x, y)dx$$

exists for all $y \in [c, d]$. If $f_y(x, y)$ exists and is continuous on $(a, b) \times [c, d]$ and if

$$\phi(y) = \int_a^b \frac{\partial f}{\partial y}(x, y)dx$$

converges uniformly on $[c, d]$, then F is differentiable on $[c, d]$, and $F'(y) = \phi(y)$; i.e.,

Theorem (11.9)

$$\frac{d}{dy}\int_a^b f(x,y)dx = \int_a^b \frac{\partial f}{\partial y}(x,y)dx$$

for all $y \in [c,d]$.

Thank you.