Advanced Calculus (II)

WEN-CHING LIEN

Department of Mathematics
National Cheng Kung University

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11.2: Definition of Differentiability

Definition (11.12)

Let f be a vector function from n variables to m variables.
(i) f is said to be differentiable at a point $\mathbf{a} \in \mathbf{R}^n$ if and only if there is an open set V containing $\mathbf{a}$ such that $f : V \to \mathbf{R}^m$ and there is a $T \in \mathcal{L}(\mathbf{R}^n; \mathbf{R}^m)$ such that the function

$$\varepsilon(\mathbf{h}) := f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - T(\mathbf{h})$$

(defined for $\mathbf{h}$ sufficiently small) satisfies $\frac{\varepsilon(\mathbf{h})}{\|\mathbf{h}\|} \to 0$ as $h \to 0$.

(ii) f is said to be differentiable on a set E if and only if E is nonempty, and f is differentiable at every point in E.

Theorem (11.13)

Let f be a vector function. If f is differentiable at $\mathbf{a}$*, then f is continuous at* $\mathbf{a}$*.*

Proof.

Suppose that f is differentiable at $\mathbf{a}$. Then by $\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h})-f(\mathbf{a})-B\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0}$, there is an $m \times n$ matrix B and a $\delta > 0$ such that $\|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - B\mathbf{h}\| \leq \|\mathbf{h}\|$ for all $\|\mathbf{h}\| < \delta$. By the triangle inequality (see Theorem 8.6iii) and the definition of the operator norm, it follows that

$$\|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a})\| \leq \|B\|\|\mathbf{h}\| + \|\mathbf{h}\|$$

for $\|\mathbf{h}\| < \delta$. Since $\|B\|$ is a finite real number, we conclude from the Squeeze Theorem that $f(\mathbf{a}+\mathbf{h}) \to f(\mathbf{a})$ as $\mathbf{h} \to \mathbf{0}$; i.e., f is continuous at $\mathbf{a}$. □

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Theorem (11.14)

Let f be a vector function. If f is differentiable at $\mathbf{a}$*, then all first-order partial derivatives of f exists at* $\mathbf{a}$*.*

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Let $B = [b_{ij}]$ be an $m \times n$ matrix that satisfies $\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h})-f(\mathbf{a})-B\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0}$. Fix $1 \le j \le n$ and set $\mathbf{h} = t\mathbf{e}_j$ for some $t > 0$. Since $\|\mathbf{h}\| = t$, we have

$$\frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - B\mathbf{h}}{\|\mathbf{h}\|} := \frac{f(\mathbf{a}+t\mathbf{e}_j) - f(\mathbf{a})}{t} - B\mathbf{e}_j.$$

Take the limit of this identity as $t \to 0+$, using $\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h})-f(\mathbf{a})-B\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0}$ and the definition of matrix multiplication. We obtain

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A similar argument shows that the limit of this quotient as $t \to 0-$ also exists and equals $(b_{1j}, \ldots b_{mj})$. Since a vector function converges if and only if each of its components converges (see Theorem 9.15), it follows that the first-order partial derivative of each component f_i with respect to x_j exists at $\mathbf{a}$ and satisfies

$$\frac{\partial f_i}{\partial x_j}(\mathbf{a}) = b_{ij}$$

for $i = 1, 2, \ldots, m$. In particular,

$$(4) \qquad B = \left[\frac{\partial f_i}{\partial x_j}(\mathbf{a})\right]_{m\times n} := \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{bmatrix}.$$

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Notation:

$$Df(\mathbf{a}) := [\frac{\partial f_i}{\partial x_j}(\mathbf{a})]_{m\times n}, \quad \begin{matrix} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n \end{matrix}$$

Theorem (11.15)

Let V be open in $\mathbf{R}^n$, let $\mathbf{a} \in V$, and suppose that $f : V \to \mathbf{R}^m$. If all first-order partial derivatives of f exist in V and are continuous at $\mathbf{a}$, then f is differentiable at $\mathbf{a}$.

Note: These hypotheses are met if f is $\mathcal{C}^1$ on V.

Proof.

Since a function converges if and only if its components converge (see Theorem 9.15), we may suppose that $m = 1$. By definition, then, it suffices to show that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot \mathbf{h}}{\|\mathbf{h}\|} = 0.$$

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Proof.

Let $\mathbf{a} = (a_1, \ldots, a_n)$. Choose $r > 0$ so small that $B_r(\mathbf{a}) \subset V$. Fix $\mathbf{h} = (h_1, \ldots, h_n) \neq \mathbf{0}$ in $B_r(\mathbf{0})$. By telescoping and using the one-dimensional Mean Value Theorem, we can choose numbers c_j between a_j and $a_j + h_j$ such that

$$
\begin{aligned}
f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) &= f(a_1+h_1, \ldots, a_n+h_n) - f(a_1, a_2+h_2, \ldots, a_n+h_n) \\
&\quad + \ldots + f(a_1, \ldots, a_{n-1}, a_n+h_n) - f(a_1, \ldots, a_n) \\
&= \sum_{j=1}^{n} h_j \frac{\partial f}{\partial x_j}(a_1, \ldots, a_{j-1}, c_j, a_{j+1}+h_{j+1}, \ldots, a_n+h_n).
\end{aligned}
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Therefore,

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(5) \qquad f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot \mathbf{h} = \mathbf{h} \cdot \delta,
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$$f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) = f(a_1+h_1, \ldots, a_n+h_n) - f(a_1, a_2+h_2, \ldots, a_n+h_n)$$

$$+ \ldots + f(a_1, \ldots, a_{n-1}, a_n + h_n) - f(a_1, \ldots, a_n)$$

$$= \sum_{j=1}^{n} h_j \frac{\partial f}{\partial x_j}(a_1, \ldots, a_{j-1}, c_j, a_{j+1} + h_{j+1}, \ldots, a_n + h_n).$$

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$$f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) = f(a_1+h_1, \ldots, a_n+h_n) - f(a_1, a_2+h_2, \ldots, a_n+h_n)$$

$$+ \ldots + f(a_1, \ldots, a_{n-1}, a_n + h_n) - f(a_1, \ldots, a_n)$$

$$= \sum_{j=1}^{n} h_j \frac{\partial f}{\partial x_j}(a_1, \ldots, a_{j-1}, c_j, a_{j+1} + h_{j+1}, \ldots, a_n + h_n).$$

Therefore,

$$(5) \qquad f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot \mathbf{h} = \mathbf{h} \cdot \delta,$$

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$$(5) \qquad f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot \mathbf{h} = \mathbf{h} \cdot \delta,$$

Proof.

where $\delta \in \mathbf{R}^n$ is the vector with components

$$\delta_j = \frac{\partial f}{\partial x_j}(a_1, \ldots, a_{j-1}, c_j, a_{j+1}+h_{j+1}, \ldots, a_n+h_n) - \frac{\partial f}{\partial x_j}(a_1, \ldots, a_n).$$

Since the first-order partial derivatives of f are continuous at $\mathbf{a}$, $\delta_j \to 0$ for each $1 \le j \le n$; i.e., $\|\delta\| \to 0$ as $\mathbf{h} \to \mathbf{0}$. Moreover, by the Cauchy-Schwarz Inequality and (5),

$$(6) \qquad 0 \le \frac{|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot \mathbf{h}|}{\|\mathbf{h}\|} = \frac{\mathbf{h} \cdot \delta}{\|\mathbf{h}\|} \le \|\delta\|.$$

It follows from the Squeeze Theorem that first quotient in (6) converges to 0 as $\mathbf{h} \to \mathbf{0}$. Thus f is differentiable at $\mathbf{a}$ by definition.

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It follows from the Squeeze Theorem that first quotient in (6) converges to 0 as $\mathbf{h} \to \mathbf{0}$. Thus f is differentiable at $\mathbf{a}$ by definition. □

Example (11.18)

Prove that

$$f(x,y) = \begin{cases} (x^2+y^2)\sin \frac{1}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is differentiable on $\mathbf{R}^2$ but not continuously differentiable at $(0,0)$.

Thank you.