Advanced Calculus (II)

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2009

11.3: Derivatives, Differentials, and Tangent Planes

Theorem (11.20)

Let $\alpha \in \mathbf{R}$, $\mathbf{a} \in \mathbf{R}^n$, and suppose that f and g are vector functions. If f and g are differentiable at $\mathbf{a}$, then $f+g$, αf, and $f \cdot g$ are all differentiable at $\mathbf{a}$. In fact,

$$D(f+g)(\mathbf{a}) = Df(\mathbf{a}) + Dg(\mathbf{a}), \tag{7}$$

$$D(\alpha f)(\mathbf{a}) = \alpha Df(\mathbf{a}), \tag{8}$$

and

$$D(f \cdot g)(\mathbf{a}) = g(\mathbf{a})Df(\mathbf{a}) + f(\mathbf{a})Dg(\mathbf{a}). \tag{9}$$

Definition (11.21 A tangent Hyperplane)

(1) Let S be a subset of $\mathbf{R}^m$ and $\mathbf{c} \in S$. A hyperplane Π with normal $\mathbf{n}$ is said to be tangent to S at $\mathbf{c}$ if and only if $\mathbf{c} \in \Pi$ and

$$\mathbf{n} \cdot \frac{\mathbf{c}_k - \mathbf{c}}{\|\mathbf{c}_k - \mathbf{c}\|} \to 0 \tag{11}$$

for all sequences $\mathbf{c}_k \in S \backslash \{\mathbf{c}\}$ that converge to $\mathbf{c}$.

(2) $\mathbf{n} \cdot (\mathbf{x} - \mathbf{c}) = 0$

Theorem (11.22)

Suppose that V is open in $\mathbf{R}^n$, that $\mathbf{a} \in V$, and that $f : V \to \mathbf{R}$. If f is differentiable at $\mathbf{a}$, then the surface

$$S := \{(\mathbf{x}, z) \in \mathbf{R}^{n+1} : z = f(\mathbf{x}) \text{ and } \mathbf{x} \in V\}$$

has a tangent hyperplane at $(\mathbf{a}, f(\mathbf{a}))$ with normal

$$(12) \quad \mathbf{n} = (\nabla f(\mathbf{a}), -1) := (f_{x_1}(\mathbf{a}), f_{x_2}(\mathbf{a}), \dots, f_{x_n}(\mathbf{a}), -1).$$

Proof.

Let $\mathbf{c}_k \in S$ with $\mathbf{c}_k \neq (\mathbf{a}, f(\mathbf{a}))$ and $\mathbf{c}_k \to (\mathbf{a}, f(\mathbf{a}))$. Then $\mathbf{c}_k = (\mathbf{a}_k, f(\mathbf{a}_k))$ for some $\mathbf{a}_k \in V$ and $\mathbf{a}_k \to \mathbf{a}$ as $k \to \infty$. For small $\mathbf{h}$, set $\varepsilon(\mathbf{h}) = f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{h})$ and define $\mathbf{n}$ by (12). Since

$$\|\mathbf{c}_k - \mathbf{c}\| = \sqrt{\|\mathbf{a}_k - \mathbf{a}\|^2 + |f(\mathbf{a}_k) - f(\mathbf{a})|^2} \geq \|\mathbf{a}_k - \mathbf{a}\|,$$

it is clear by (12) that

$$0 \leq \left| \mathbf{n} \cdot \frac{\mathbf{c}_k - \mathbf{c}}{\|\mathbf{c}_k - \mathbf{c}\|} \right| \leq \frac{|\varepsilon(\mathbf{a}_k - \mathbf{a})|}{\|\mathbf{a}_k - \mathbf{a}\|}.$$

Since $\frac{\varepsilon(\mathbf{h})}{\|\mathbf{h}\|} \to 0$ as $\mathbf{h} \to \mathbf{0}$, it follows from the Squeeze Theorem that $\mathbf{n}$ satisfies (11) for $\mathbf{c} := (\mathbf{a}, f(\mathbf{a}))$. □

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Note: If f is a near-valued function of two variables that is differentiable at (a, b), then the surface $z = f(x, y)$ has a tangent plane at $(a, b, f(a, b))$ with normal

$$\mathbf{n} =: (\nabla f(a, b), -1).$$

Notation (the first total differential):

$$z = f(\mathbf{x}), \qquad \Delta z := f(\mathbf{a} + \Delta \mathbf{x}) - f(\mathbf{a})$$

$$\Delta \mathbf{x} := (\Delta x_1, \ldots \Delta x_n)$$

$$dz := \nabla f(\mathbf{a}) \cdot \Delta \mathbf{x} := \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\mathbf{a}) dx_j.$$

Remark (11.23)

Let $f : \mathbf{R}^n \to \mathbf{R}$ be differentiable at $\mathbf{a}$ and $\Delta\mathbf{x} = (\Delta x_1, \ldots, \Delta x_n)$. Then

$$\frac{\Delta z - dz}{\|\Delta\mathbf{x}\|} \to 0 \quad as \quad \Delta\mathbf{x} \to \mathbf{0}.$$

In particular, the differential dz approximates Δz.

Proof.

By definition, if f is differentiable at $\mathbf{a}$, then $\varepsilon(\mathbf{h}) := f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot \mathbf{h}$ satisfies $\frac{\varepsilon(\mathbf{h})}{\|\mathbf{h}\|} \to 0$ as $\mathbf{h} \to \mathbf{0}$. Since $\Delta z = f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a})$ for $\mathbf{h} := \Delta\mathbf{x}$ and $dz = \nabla f(\mathbf{a}) \cdot \mathbf{h}$, it follows that $\frac{\Delta z - dz}{\|\Delta\mathbf{x}\|} \to 0$ as $\Delta\mathbf{x} \to \mathbf{0}$. □

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Thank you.