Advanced Calculus (II)

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12.1: Jordan Regions

Notations:

(1) $R = [a_1, b_1] \times \ldots \times [a_n, b_n]$, an n-dimensional rectangle.

(2) A grid $G = \{R_1, \ldots, R_p\}$ on R is a collection of n-dimensional rectangles obtained by subdividing the sides of R.

(3) The volume of R: $|R| = (b_1 - a_1) \ldots (b_n - a_n)$.

Definition (1)

Let E be a given set. The outer sum of E with respect to a grid G on a rectangle R is

$$V(E;G) := \sum_{R_j \cap \overline{E} \neq \emptyset} |R_j|.$$

Remark (12.1)

Let R be an n-dimensional rectangle.

(i) Let E be a subset of R, and let $\mathcal{G}, \mathcal{H}$ be grids on R. If $\mathcal{G}$ is finer than $\mathcal{H}$, then

$$V(E;\mathcal{G}) \leq V(E;\mathcal{H}).$$

(ii) If A and B are subsets of R and $A \subseteq B$, then

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Proof.

(i) Since $\mathcal{G}$ is finer than $\mathcal{H}$, each $Q \in \mathcal{H}$ is a finite union of R_j's in $\mathcal{G}$. If $Q \cap \overline{E} \neq \emptyset$, then some of the R_j's in Q intersect $\overline{E}$ and others might not (see Figure 12.3, when the darker lines represent the grid $\mathcal{H}$, the lighter lines represent $\mathcal{G} \backslash \mathcal{H}$, and the R_j's that intersect $\overline{E}$ are shaded).

Let $\mathcal{I}_1 = \{R \in \mathcal{G} : G \cap \overline{E} \neq \emptyset\}$ and $\mathcal{I}_2 = \{R \in \mathcal{G} \backslash \mathcal{I}_1 : R \subseteq Q$ for some $Q \in \mathcal{H}$ with $Q \cap \overline{E} \neq \emptyset\}$. Then

$$V(E; \mathcal{H}) = \sum_{R \in \mathcal{I}_1} |R| + \sum_{R \in \mathcal{I}_2} |R| \geq \sum_{R \in \mathcal{I}_1} |R| = V(E; \mathcal{G}).$$

(ii) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$ (see Exercise 3, p.254). Thus, every rectangle that appears in the sum $V(A; \mathcal{G})$ also appears in the sum $V(B; \mathcal{G})$. Since all $|R_j|$'s are nonnegative, it follows that $V(A; \mathcal{G}) \leq V(B; \mathcal{G})$.

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Example (12.2)

If $R = [0,1] \times [0,1]$, $A = \{(x,y) : x, y \in \mathbf{Q} \cap [0,1]\}$, and $B = R \backslash A$, then $V(A;\mathcal{G}) + V(B;\mathcal{G}) = 2V(R;\mathcal{G})$ no matter how fine $\mathcal{G}$ is.

Proof.

Let $\mathcal{G} = \{R_1, \ldots, R_p\}$ be a grid on R. Since each R_j is nondegenerate, it is clear by the Density of Rationals (Theorem 1.24) that $R_j \cap A \neq \emptyset$ for all $j \in [1,p]$. Hence $V(A;\mathcal{G}) = |R| = 1$. Similarly, the Density of the Irrationals (Exercise 3, p.23) implies $V(B;\mathcal{G}) = |R| = 1$. $\square$

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Definition (12.3)

Let E be a subset of $\mathbf{R}^n$. Then E is said to be a *Jordan region* if and only if given $\varepsilon > 0$ there is rectangle $R \supseteq E$, and a grid $\mathcal{G} = \{R_1, \ldots, R_p\}$ on R, such that

$$V(\partial E; \mathcal{G}) := \sum_{R_j \cap \partial E \neq \emptyset} |R_j| < \varepsilon.$$

(The last sum IS the outer sum of ∂E since $\overline{\partial E} = \partial E$ by Theorem 8.36.)

Definition (12.4)

Let E be a Jordan region in $\mathbf{R}^n$ and let R be an n-dimensional rectangle that satisfies $E \subseteq R$. The *volume* (or *Jordan content*) of E is defined by

$$\begin{aligned} Vol(E) &:= \inf_{\mathcal{G}} V(E;\mathcal{G}) \\ &:= \inf\{V(E;\mathcal{G}) : \mathcal{G} \text{ ranges over all grids on } R\} \end{aligned}$$

Remark (12.5)

If R is an n-dimensional rectangle, then R is a Jordan region in $\mathbf{R}^n$ and $Vol(R) = |R|$.

Proof.

Let $\varepsilon > 0$ and suppose that

$$R = [a_1, b_1] \times \ldots \times [a_n, b_n].$$

Since $b_j - a_j - 2\delta \to b_j - a_j$ as $\delta \to 0$, we can choose $\delta > 0$ so small that if

$$Q = [a_1 + \delta, b_1 - \delta] \times \ldots \times [a_n + \delta, b_n - \delta],$$

then $|R| - |Q| < \varepsilon$.
Let $\mathcal{G}_0 := \{H_1, \ldots, H_q\}$ be the grid on R determined by

$$P_j(\mathcal{G}) = \{a_j, a_j + \delta, b_j - \delta, b_j\}.$$

Proof.

Let $\varepsilon > 0$ and suppose that

$$R = [a_1, b_1] \times \ldots \times [a_n, b_n].$$

Since $b_j - a_j - 2\delta \to b_j - a_j$ as $\delta \to 0$, we can choose $\delta > 0$ so small that if

$$Q = [a_1 + \delta, b_1 - \delta] \times \ldots \times [a_n + \delta, b_n - \delta],$$

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Proof.

Then it is clear that an $H_j \in \mathcal{G}$ intersects ∂R if and only if $H_j \neq Q$. Hence,

$$V(\partial R; \mathcal{G}) := \sum_{H_j \cap \partial R \neq \emptyset} |H_j| = |R| - |Q| < \varepsilon.$$

This proves that R is a Jordan region.

To compute the volume of R by Definition 12.4, let $\mathcal{G} = \{R_1, \ldots, R_p\}$ be any grid on R. Since $R_j \cap R \neq \emptyset$ for all $R_j \in \mathcal{G}$, it follows from definition that $V(R; \mathcal{G}) = |R|$. Taking the infimum of this identity over all grids $\mathcal{G}$ on R, we find that $Vol(R) = |R|$.

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Remark (12.6)

Suppose that E is a bounded subset of $\mathbf{R}^n$.

(i) E is a Jordan region of volume zero if and only if there is an absolute constant C, that does not depend on E, such that for each $\varepsilon > 0$ one can find a grid $\mathcal{G}$ that satisfies $V(E;\mathcal{G}) < C\varepsilon$.

(ii) E is a Jordan region if and only if $Vol(\partial E) = 0$.

(iii) If E is a set of volume zero and $A \subseteq E$, then A is a Jordan region and $Vol(A) = 0$.

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Proof.

By Definition 12.3 and 12.4, and Remark 12.1ii, it suffices to prove (i). Let E be a Jordan region of volume zero, and let $\varepsilon > 0$. By the Approximation Property for Infima, there is a grid $\mathcal{G}$ such that $V(E;\mathcal{G}) < \varepsilon$. Hence set $C = 1$.

Conversely, let $\varepsilon > 0$ and suppose that there is a grid $\mathcal{G}$ such that $V(E;\mathcal{G}) < C\varepsilon$. Then $\partial E = \overline{E} \backslash E^o \subset \overline{E}$ implies

$$0 \leq \alpha := \inf_{\mathcal{G}} V(\partial E;\mathcal{G}) \leq \beta := V(E;\mathcal{G}) \leq C\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\alpha = \beta = 0$. Since $\alpha = 0$, we can use the Approximation Property for Infima to choose a grid $\mathcal{H}$ such that $V(\partial E;\mathcal{H}) < \varepsilon$. Thus E is a Jordan region. Since $\beta = 0$, we conclude by Definition 12.4 that $Vol(E) = 0$.

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Definition (12.7)

Let $\mathcal{E} := \{E_\ell\}_{\ell \in \mathbf{N}}$ be a collection of subsets of $\mathbf{R}^n$.

(i) $\mathcal{E}$ is said to be *nonoverlapping* if and only if $E_j \cap E_k$ is of volume zero for $j \neq k$.

(ii) $\mathcal{E}$ is said to be *pairwise disjoint* if and only if $E_j \cap E_k = \emptyset$ for $j \neq k$.

Notice that since $\emptyset$ is of volume zero, every collection of pairwise disjoint sets is nonoverlapping.

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Notice that since $\emptyset$ is of volume zero, every collection of pairwise disjoint sets is nonoverlapping.

Theorem (12.8)

Let E be a subset of $\mathbf{R}^n$. Then E is a Jordan region of volume zero if and only if for every $\varepsilon > 0$ there is a finite collection of cubes Q_k of the same size, i.e., all with sides of length s, such that

$$\overline{E} \subset \bigcup_{k=1}^{q} Q_k \quad \text{and} \quad \sum_{k=1}^{q} |Q_k| < \varepsilon.$$

Corollary (12.9)

If E_1 and E_2 are Jordan regions, then $E_1 \cup E_2$ is a Jordan region and

$$Vol(E_1 \cup E_2) \leq Vol(E_1) + Vol(E_2).$$

Corollary (12.10)

Suppose that V is a bounded, open set in $\mathbf{R}^n$ and that $\phi : V \to \mathbf{R}^n$ is 1-1 and $\mathcal{C}^1$ on V with $\Delta_\phi \neq 0$.

(i) If E is of volume zero and $\overline{E} \subset V$, then $\phi(E)$ is of volume zero.

(ii) If $\{E_k\}_{k\in\mathbf{N}}$ is a nonoverlapping collection of sets in $\mathbf{R}^n$ with $\overline{E}_k \subset V$ for all $k \in \mathbf{N}$, then $\{\phi(E_k)\}_{k\in\mathbf{N}}$ is a nonoverlapping collection of sets in $\mathbf{R}^n$.

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Definition (2)

The inner sum of E with E with respect to G is

$$\upsilon(E;G) := \sum_{R_j \subset E^o} |R_j|.$$

Remark (12.11)

Let R be an n-dimensional rectangle, let E be a subset of R, and let $\mathcal{G}, \mathcal{H}$ be grids on R. If $\mathcal{G}$ is finer than $\mathcal{H}$, then

$$0 \leq v(E;\mathcal{H}) \leq v(E;\mathcal{G}) \leq V(E;\mathcal{G}) \leq V(E;\mathcal{H}).$$

This leads us to the following fundamental principle.

Remark (12.12)

Let R be an n-dimensional rectangle and E be a subset of R. If $\mathcal{G}$ and $\mathcal{H}$ are grids on R, then

$$0 \le v(E;\mathcal{G}) \le V(E;\mathcal{H}).$$

Definition (12.13)

Let E be a bounded subset of $\mathbf{R}^n$ and let R be an n-dimensional rectangle that satisfies $E \subseteq R$. The *inner volume* of E is defined by

$$\underline{Vol}(E) := \sup\{v(E;\mathcal{G}) : \ \mathcal{G} \text{ ranges over all grids on R}\},$$

and the *outer volume* of E is defined by

$$\overline{Vol}(E) := \inf\{V(E;\mathcal{G}) : \ \mathcal{G} \text{ ranges over all grids on R}\}.$$

Theorem (12.14)

Let E be a bounded subset of $\mathbf{R}^n$. Then E is a Jordan region if and only if $\overline{Vol}(E) = \underline{Vol}(E)$.

Thank you.