Advanced Calculus (II)

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12.2: Riemann Integration on Jordan Regions

Definition (12.15)

Let E be a Jordan region in $\mathbf{R}^n$, let $f : E \to \mathbf{R}$ be a bounded function, let R be an n-dimensional rectangle such that $E \subseteq R$, and let $\mathcal{G} = \{R_1, \ldots, R_p\}$ be a grid on R. Extend f to $\mathbf{R}^n$ by setting $f(\mathbf{x}) = 0$ for $\mathbf{x} \in \mathbf{R}^n \setminus E$.

(i) The *upper sum* of f on E with respect to $\mathcal{G}$ is

$$U(f, \mathcal{G}) := \sum_{R_j \cap E \neq \emptyset} M_j |R_j|,$$

where $M_j = \sup_{\mathbf{x} \in R_j} f(\mathbf{x})$.

Definition (12.15)

(ii) The *lower sum* of f on E with respect to $\mathcal{G}$ is

$$L(f,\mathcal{G}) := \sum_{R_j \cap E \neq \emptyset} m_j |R_j|,$$

where $m_j = \inf_{\mathbf{x} \in R_j} f(\mathbf{x})$.

(iii) The *upper* and *lower integrals* of f on E are defined by

$$(L)\int_E f(\mathbf{x})d\mathbf{x} := (L)\int_E f dV := \sup_{\mathcal{G}} L(f,\mathcal{G})$$

and

$$(U)\int_E f(\mathbf{x})d\mathbf{x} := (U)\int_E f dV := \inf_{\mathcal{G}} U(f,\mathcal{G}),$$

where the supremum and infimum are taken over all grids $\mathcal{G}$ on R.

Remark (12.16)

Let E be a nonempty Jordan region in $\mathbf{R}^n$, let $f : E \to \mathbf{R}$ be bounded, and let R be a rectangle that contains E.

(i) If $\mathcal{G}$ and $\mathcal{H}$ are grids on R, then $L(f, \mathcal{G}) \leq U(f, \mathcal{H})$.

(ii) The upper and lower integrals of f over E exist, do not depend on the choice of R, and satisfy

$$(L) \int_E f(\mathbf{x}) d\mathbf{x} \leq (U) \int_E f(\mathbf{x}) d\mathbf{x}. \quad (6)$$

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Definition (12.17)

A real-valued bounded function f defined on a Jordan region E is said to be (*Riemann*) *integrable* on E if and only if for every $\varepsilon > 0$ there is a grid $\mathcal{G}$ such that

$$U(f, \mathcal{G}) - L(f, \mathcal{G}) < \varepsilon.$$

By modifying the proof of Theorem 5.15, we can establish the following result.

Remark (12.18)

Let E be a Jordan region in $\mathbf{R}^n$ and suppose that $f : E \to \mathbf{R}$ is bounded. Then f is integrable on E if and only if

$$(L) \int_E f(\mathbf{x}) d\mathbf{x} = (U) \int_E f(\mathbf{x}) d\mathbf{x}. \tag{7}$$

Theorem (12.20)

Let E be a Jordan region and suppose that $f : E \to \mathbf{R}$ is bounded. Then given $\varepsilon > 0$ there is a grid $\mathcal{G}_0$ such that if $\mathcal{G} := \{R_1, \ldots, R_p\}$ is any grid finer than $\mathcal{G}_0$ and M_j, m_j are defined as in Definition 12.15, then

$$\left| (U) \int_E f(\mathbf{x}) d\mathbf{x} - \sum_{R_j \subset E^o} M_j |R_j| \right| < \varepsilon$$

and

$$\left| (L) \int_E f(\mathbf{x}) d\mathbf{x} - \sum_{R_j \subset E^o} m_j |R_j| \right| < \varepsilon.$$

Theorem (12.21)

If E is a closed Jordan region in $\mathbf{R}^n$ and $f : E \longrightarrow \mathbf{R}$ is continuous on E, then f is integrable on E.

Theorem (12.22)

If E is a closed Jordan region, then

$$Vol(E) = \int_E 1 d\mathbf{x}.$$

Theorem (12.23 Linear Properties)

Let E be a Jordan region in $\mathbf{R}^n$, let $f, g : E \to \mathbf{R}$, and let α be a scalar.

(i) If f, g are integrable on E, then so are αf and $f+g$. In fact,

$$\int_E \alpha f(\mathbf{x}) d\mathbf{x} = \alpha \int_E f(\mathbf{x}) d\mathbf{x} \tag{10}$$

and

$$\int_E (f(\mathbf{x}) + g(\mathbf{x})) d\mathbf{x} = \int_E f(\mathbf{x}) d\mathbf{x} + \int_E g(\mathbf{x}) d\mathbf{x}. \tag{11}$$

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Theorem (12.23 Linear Properties)

(ii) If $E_1, E_2 \subseteq E$ are nonoverlapping Jordan regions and f is integrable on both E_1 and E_2, then f is integrable on $E_1 \cup E_2$ and

$$\int_{E_1 \cup E_2} f(\mathbf{x})d\mathbf{x} = \int_{E_1} f(\mathbf{x})d\mathbf{x} + \int_{E_2} f(\mathbf{x})d\mathbf{x}. \tag{12}$$

Theorem (12.24)

Let E be a Jordan region in $\mathbf{R}^n$, and suppose that $f, g : E \to \mathbf{R}$ are bounded functions.

(i) If E_0 is of volume zero, then g is integrable on E_0 and

$$\int_{E_0} g(\mathbf{x})d\mathbf{x} = 0.$$

(ii) If f is integrable on E and if there is a subset E_0 of E such that $Vol(E_0) = 0$ and $f(\mathbf{x}) = g(\mathbf{x})$ for all $\mathbf{x} \in E \setminus E_0$, then g is integrable on E and

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Theorem (12.25 Comparison Theorem for Multiple Integrals)

Let E be a Jordan region in $\mathbf{R}^n$, and suppose that $f, g : E \to \mathbf{R}$ are integrable on E.

(i) If $f(\mathbf{x}) \le g(\mathbf{x})$ for $\mathbf{E}$, then

$$\int_E f(\mathbf{x})d\mathbf{x} \le \int_E g(\mathbf{x})d\mathbf{x}.$$

(ii) If m, M are scalars that satisfy $m \le f(\mathbf{x}) \le M$ for $\mathbf{x} \in E$, then

$$mVol(E) \le \int_E f(\mathbf{x})d\mathbf{x} \le MVol(E).$$

(iii) The function $|f|$ is integrable on E and

$$\left|\int_E f(\mathbf{x})d\mathbf{x}\right| \le \int_E |f(\mathbf{x})|d\mathbf{x}. \quad (17)$$

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Theorem (12.26 Mean Value Theorem for Multiple Integrals)

Let E be a Jordan region in $\mathbf{R}^n$ and let $f, g : E \to \mathbf{R}$ be integrable on E with $g(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in E$.

(i) There is a number c satisfying

$$\inf_{\mathbf{x}\in E} f(\mathbf{x}) \leq c \leq \sup_{\mathbf{x}\in E} f(\mathbf{x}) \tag{19}$$

such that

$$c \int_E g(\mathbf{x})d\mathbf{x} = \int_E f(\mathbf{x})g(\mathbf{x})d\mathbf{x}. \tag{20}$$

(ii) There is a number c satisfying (19) such that

$$cVol(E) = \int_E f(\mathbf{x})d\mathbf{x}.$$

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Thank you.