Advanced Calculus (II)

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12.4: Change-of-Variables

(1)

$$\int_{\phi(E)} f(\mathbf{u})d\mathbf{u} = \int_E f(\phi(\mathbf{x}))|\Delta_\phi(\mathbf{x})|d\mathbf{x}.$$

(2) When $f = 1$,

$$|R| = \int_{\phi^{-1}(R)} |\Delta_\phi(\mathbf{x})|d\mathbf{x}.$$

Lemma (12.43)

Let W be open in $\mathbf{R}^n$, let $\phi : W \to \mathbf{R}^n$ be 1-1 and continuously differentiable on W with $\Delta_\phi \neq 0$ on W, and suppose that ϕ^{-1} is continuously differentiable on $\phi(W)$ with $\Delta_{\phi^{-1}} \neq 0$ on $\phi(W)$. Suppose further that (2) holds for every n-dimensional rectangle $R \subset \phi(W)$. If E is a Jordan region with $\overline{E} \subset W$, if f is integrable on $\phi(E)$, and if $f \circ \phi$ is integrable on E, then

$$\int_{\phi(E)} f(\mathbf{u})d\mathbf{u} = \int_E (f \circ \phi)(\mathbf{x})|\Delta_\phi(\mathbf{x})|d\mathbf{x}.$$

Lemma (12.44)

Let V be open in $\mathbf{R}^n$ and $\phi : V \to \mathbf{R}^n$ be 1-1 and continuously differentiable on V. If $\Delta_\phi(\mathbf{a}) \neq 0$ for some $\mathbf{a} \in V$, then there exists an open rectangle W such that $\mathbf{a} \in W \subset V$, Δ_ϕ is nonzero on W, ϕ^{-1} is $\mathcal{C}^1$ and its Jacobian is nonzero on $\phi(W)$, and such that if R is an n-dimensional rectangle contained in $\phi(W)$, then $\phi^{-1}(R)$ is Jordan and (2) holds.

Lemma (12.45)

Suppose that V is open in $\mathbf{R}^n$, $\mathbf{a} \in V$, and $\phi : V \to \mathbf{R}^n$ is continuously differentiable on V. If $\Delta_\phi(\mathbf{a}) \neq 0$, then there exists an open rectangle $W \subset V$ containing $\mathbf{a}$ such that if E is Jordan with $\overline{E} \subset W$, if $f \circ \phi$ is integrable on E, and if f is integrable on $\phi(E)$, then

$$\int_{\phi(E)} f(\mathbf{u}) d\mathbf{u} = \int_E f(\phi(\mathbf{x}))|\Delta_\phi(\mathbf{x})| d\mathbf{x}. \tag{37}$$

Theorem (12.46)

Suppose that V is open in $\mathbf{R}^n$, and that $\phi : V \to \mathbf{R}^n$ is 1-1 and continuously differentiable on V. If $\Delta_\phi \neq 0$ on V, if E is Jordan region with $\overline{E} \subset V$, if $f \circ \phi$ is integrable on E, and if f is integrable on $\phi(E)$, then

$$\int_{\phi(E)} f(\mathbf{u})d\mathbf{u} = \int_E f(\phi(\mathbf{x}))|\Delta_\phi(\mathbf{x})|d\mathbf{x}. \tag{38}$$

Example (spherical coordinates)

$$x = \rho \sin\varphi \cos\theta,$$
$$y = \rho \sin\varphi \sin\theta,$$
$$z = \rho \cos\varphi.$$

Example (12.50)

Find

$$\iiint_Q x\, dV,$$

where $Q = B_3(0,0,0) \backslash B_2(0,0,0)$.

Thank you.