Advanced Calculus (I)

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Definition

A sequence of real numbers $\{x_n\}$ is said to *converge* to a real number $a \in \mathbf{R}$ if and only if for every $\epsilon > 0$ there is an $N \in \mathbf{N}$ (which in general depends on ϵ) such that

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Prove that $\frac{1}{n} \to 0$ as $n \to \infty$.

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Remark:

If $\{x_n\}_{n\in\mathbf{N}}$ *converges* to a and $\{x_{n_k}\}_{k\in\mathbf{N}}$ is any subsequence of $\{x_n\}_{n\in\mathbf{N}}$, then x_{n_k} *converges* to a as $k \to \infty$.

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Definition

Let $\{x_n\}$ be a sequence of real numbers.

(i)$\{x_n\}$ is said to be *bounded above* if and only if there is an $M \in \mathbf{R}$ such that $x_n \leq M$ for all $n \in \mathbf{N}$

(ii) $\{x_n\}$ is said to be *bounded below* if and only if there is an $m \in \mathbf{R}$ such that $x_n \geq m$ for all $n \in \mathbf{N}$

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Exercise:

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2. Suppose that $\lim_{n\to\infty} x_n = 1$

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Thank you.