Advanced Calculus (I)

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Theorem (Squeeze Theorem)

Suppose that $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$ are real sequences.

(i) If $x_n \to a$ and $y_n \to a$ (the SAME a) as $n \to \infty$, and if there is an $N_0 \in \mathbf{N}$ such that

$$x_n \leq w_n \leq y_n \quad \text{for} \quad n \geq N_0,$$

then $w_n \to a$ as $n \to \infty$.

(ii) If $x_n \to 0$ as $n \to \infty$ and $\{y_n\}$ is bounded, then $x_n y_n \to 0$ as $n \to \infty$.

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Proof:

(i)

Let $\epsilon > 0$. Since x_n and y_n converge to a, use Definition 2.1 and Theorem 1.6 to choose $N_1, N_2 \in \mathbf{N}$ such that $n \geq N_1$ implies $-\epsilon \leq x_n - a \leq \epsilon$ and $n \geq N_2$ implies $-\epsilon \leq y_n - a \leq \epsilon$. Set $N = \max\{N_0, N_1, N_2\}$. If $n \geq N$ we have by hypothesis and the choice of N_1 and N_2 that

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Theorem

Let $E \subset \mathbf{R}$. If E has a finite supermum (respectively, a finite infimum), then there is a sequence $x_n \in E$ such that $x_n \to \sup E$ (respectively, $x_n \to \inf E$) as $n \to \infty$

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Suppose that E has a finite supermum. For each $n \in \mathbf{N}$, choose (by the Approximation Property for Supermum) an $x_n \in E$ such that $\sup E - 1/n < x_n \leq \sup E$. Then by the Squeeze Theorem and Example 2.2, $x_n \to \sup E$ as $n \to \infty$. Similarly, there is a sequence $y_n \in E$ such that $y_n \to \inf E$. $\square$

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Theorem

Suppose that $\{x_n\}$ *and* $\{y_n\}$ *are real sequences and* $\alpha \in \mathbf{R}$. *If* $\{x_n\}$ *and* $\{y_n\}$ *are convergent, then*

(i)

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n,$$

(ii)

$$\lim_{n\to\infty}(\alpha x_n) = \alpha \lim_{n\to\infty} x_n$$

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$$\lim_{n\to\infty}(x_n y_n) = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n).$$

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Definition

Let $\{x_n\}$ be a sequence of real numbers.

(i) $\{x_n\}$ is said to *diverge* to $+\infty$ (notation: $x_n \to +\infty$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = +\infty$) if and only if for each $M \in \mathbf{R}$ there is an $N \in \mathbf{N}$ such that

$$n \geq N \quad implies \quad x_n > M$$

(ii) $\{x_n\}$ is said to *diverge* to $-\infty$ (notation: $x_n \to -\infty$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = -\infty$) if and only if for each $M \in \mathbf{R}$ there is an $N \in \mathbf{N}$ such that

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$$\lim_{n\to\infty}(x_n + y_n) = +\infty \quad (\textit{respectively}, \lim_{n\to\infty}(x_n + y_n) = -\infty).$$

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$$\lim_{n\to\infty}(x_n y_n) = +\infty \quad (\textit{respectively}, \lim_{n\to\infty}(x_n y_n) = -\infty).$$

(iv) If $\{y_n\}$ is bounded and $x_n \neq 0$, then

$$\lim_{n\to\infty}\frac{y_n}{x_n} = 0.$$

Corollary:

Let $\{x_n\}, \{y_n\}$ be real sequences and α, x, y be extended real numbers. If $x_n \to x$ and $y_n \to y$, as $n \to \infty$, then

$$\lim_{n\to\infty} (x_n + y_n) = x + y$$

(provided that the right side is not of the form $\infty - \infty$), and

$$\lim_{n\to\infty} (\alpha x_n) = \alpha x, \quad \lim_{n\to\infty} (x_n y_n) = xy$$

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Theorem (Comparison Theorem)

Suppose that $\{x_n\}$ and $\{y_n\}$ are convergence sequences. If there is an $N_0 \in \mathbf{N}$ such that

$$x_n \leq y_n \quad n \geq N_0, \tag{1}$$

then

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

In particular, if $x_n \in [a, b]$ converges to some point c, then c must belong to $[a, b]$.

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Proof:

Suppose that the first statement is false, i.e., that (1) holds but $x := \lim_{n\to\infty} x_n$ is great than $y := \lim_{n\to\infty} y_n$. Set $\epsilon = (x-y)/2$. Choose $N_1 > N_0$ such that $|x_n - x| < \epsilon$ and $|y_n - y| < \epsilon$ for $n \geq N_1$. Then for such an n,

$$x_n > x - \epsilon = x - \frac{x-y}{2} = y + \frac{x-y}{2} = y + \epsilon > y_n,$$

which contradicts (1). This prove the first statement.

We conclude by noting that the second statement follows from the first, since $a \leq x_n \leq b$ implies $a \leq c \leq b$. $\square$

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