Advanced Calculus (I)

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Definition

Let $\{x_n\}_{n\in\mathbf{N}}$ be a sequence of real numbers.

(i) $\{x_n\}$ is said to be *increasing* (respectively, *strictly increasing*) if and only if $x_1 \leq x_2 \leq \cdots$ (respectively, $x_1 < x_2 < \cdots$)

(ii) $\{x_n\}$ is said to be *decreasing* (respectively, *strictly decreasing*) if and only if $x_1 \geq x_2 \geq \cdots$ (respectively, $x_1 > x_2 > \cdots$)

(iii) $\{x_n\}$ is said to be *monotone* if and only if it is either increasing or decreasing.

2.3 Bolzano-Weierstrass Theorem

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We shall actually prove that an increasing sequence converges to its supremum, and a decreasing sequence converges to its infimum.

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Suppose that $\{x_n\}$ is increasing and bounded above. By the Completeness Axiom, the supremum a:$= \sup\{x_n : n \in \mathbf{N}\}$ exists and is finite. Let $\epsilon > 0$. By the Approximation Property for Supremum, choose $N \in \mathbf{N}$ such that

$$a - \epsilon < x_N \leq a.$$

Since $x_N \leq x_n$ for $n \geq N$. In particular, $x_n \uparrow a$ as $n \to \infty$.

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If $\{x_n\}$ is decreasing with infimum b:$=\inf\{x_n : n \in \mathbf{N}\}$, then $\{-x_n\}$ is increasing with supremum $-b$ (see Theorem 1.28). Hence, by part(i) and Theorem 2.12(ii)

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Theorem (Nested Interval Property)

If $\{I_n\}_{n\in\mathbf{N}}$ is a nested sequence of nonempty closed bounded intervals, then

$$E = \bigcap_{n\in\mathbf{N}} I_n := \{x : x \in I_n \ \ for \ \ all \ \ n \in \mathbf{N}\}$$

contains at least one number. Moreover, if the lengths of these intervals satisfy $|I_n| \to 0$ as $n \to \infty$, then E contains exactly one number.

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Let $I_n = [a_n, b_n]$. Since $\{I_n\}$ is nested, the real sequence $\{a_n\}$ is increasing and bounded above by b_1, and $\{b_n\}$ is decreasing and bounded below by a_1. Thus by M.C.T, there exist $a, b \in \mathbf{R}$ such that $a_n \uparrow a$ and $b_n \downarrow b$ as $n \to \infty$. Since $a_n \leq b_n$ for all $n \in \mathbf{N}$, it also follows from the Comparison Theorem that $a_n \leq a \leq b \leq b_n$. Hence, a number x belongs to I_n for all $n \in \mathbf{N}$ if and only if $a \leq x \leq b$. This proves that $E = [a, b]$.

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Suppose now that $|I_n| \to 0$ as $n \to \infty$. Then $b_n - a_n \to 0$ as $n \to \infty$, and we have by Theorem2.12 that $b - a = 0$. In particular, $E = [a, a] = \{a\}$ contains exactly one number. $\square$

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