Advanced Calculus (I)

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2.5 Limits Superemum and Infimum

Definition

Let $\{x_n\}$ ba a real sequence. Then the *limit supremum* of $\{x_n\}$ is the extended real number

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} (\sup_{k\geq n} x_k),$$

and the *limit infimum* of $\{x_n\}$ is the extended real number

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Theorem

Let $\{x_n\}$ be a real sequence of numbers, $s = \limsup_{n \to \infty} x_n$, and $t = \liminf_{n \to \infty} x_n$. Then there are subsequences $\{x_{n_k}\}_{k \in \mathbf{N}}$ and $\{x_{l_j}\}_{j \in \mathbf{N}}$ such that $x_{n_k} \to s$ as $k \to \infty$ and $x_{l_j} \to t$ as $j \to \infty$.

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Proof:

We will prove the result for the limit supremum. A similar argument establishes the result for the limit infimum. Let $s_n = \sup_{k \geq n} x_k$ and observe that $s_n \downarrow s$ as $n \to \infty$.

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Case 1.

$s = \infty$. Then by definition $s_n = \infty$ for all $n \in \mathbf{N}$. Since $s_1 = \infty$, there is an $n_1 \in \mathbf{N}$ such that $x_{n_1} > 1$. Since $s_{n_1} = \infty$, there is an $n_2 \geq n_1 + 1 > n_1$ such that $x_{n_2} > 2$. Continuing in this manner, we can choose a subsequence $\{x_{n_k}\}$ such that $x_{n_k} > k$ for all $k \in \mathbf{N}$. Hence, it follows from the Squeeze Theorem (see Exersise 6, p. 44) that $x_{n_k} \to \infty = s$ as $k \to \infty$.

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$-\infty < s < \infty$. Set $n_0 = 0$. By Theorem 1.20 (the Approximation Property for Supremum), there is an integer $n_1 \in \mathbf{N}$ such that $s_{n_0+1} - 1 < x_{n_1} \leq s_{n_0} + 1$. Simiarly, there is an integer $n_2 \geq n_1 + 1 > n_1$ such that $s_{n_1+1} - 1/2 < x_{n_2} \leq s_{n_1+1}$. Continuing in this manner, we can choose integers $n_1 < n_2 < \cdots$ such that

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Theorem

Let $\{x_n\}$ be a real sequence and x be an extended real number. Then $x_n \to x$ as $n \to \infty$ if and only if

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x. \tag{6}$$

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Proof:

Suppose that $x_n \to x$ as $n \to \infty$. Then $x_{n_k} \to x$ as $k \to \infty$ for all subsequences $\{x_{n_k}\}$. Hence by Theorem 2.35, $\limsup_{n\to\infty} x_n = x$ and $\liminf_{n\to\infty} x_n = x$; i.e., (6) holds.

Conversely, suppose that (6) holds.

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Conversely, suppose that (6) holds.

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$x = \pm\infty$. By considering $\pm x_n$ we may suppose that $x = \infty$. Thus given $M \in \mathbf{R}$ there is an $N \in \mathbf{N}$ such that $\inf_{k \geq N} x_k > M$. It follows that $x_n > M$ for all $n \geq N$; i.e., $x_n \to \infty$ as $n \to \infty$.

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Case 2.

$-\infty < x < \infty$. Let $\epsilon > 0$. Choose $N \in \mathbf{N}$ such that

$$\sup_{k \geq N} x_k - x < \epsilon \;\; and \;\; x - \inf_{k \geq N} x_k < \epsilon.$$

Thus, for any $n \geq N$,

$$x_n - x \leq \sup_{k \geq N} x_k - x < \epsilon \;\; and \;\; x - x_n \leq x - \inf_{k \geq N} x_k < \epsilon.$$

That is, for any $n \geq N$,

$$|x - x_n| < \epsilon.$$

We conclude that $x_n \to x$ as $n \to \infty$. □

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We conclude that $x_n \to x$ as $n \to \infty$. □

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$-\infty < x < \infty$. Let $\epsilon > 0$. Choose $N \in \mathbf{N}$ such that

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Theorem

Let $\{x_n\}$ *be a real sequence of real numbers. Then* $\limsup_{n\to\infty} x_n$ *(respectively,* $\liminf_{n\to\infty} x_n$*) is the largest value (respectively, the smallest value) to which some subsequence of* $\{x_n\}$ *converges. Namely, if* $x_{n_k} \to x$ *as* $k \to \infty$*, then*

$$\liminf_{n\to\infty} x_n \leq x \leq \limsup_{n\to\infty} x_n. \tag{7}$$

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$$\liminf_{n\to\infty} x_n \leq x \leq \limsup_{n\to\infty} x_n. \tag{7}$$

Proof:

Suppose that $x_{n_k} \to x$ as $k \to \infty$. Fix $N \in \mathbf{N}$ and choose K so large that $k \geq K$ implies $n_k \geq N$. Clearly,

$$\inf_{j \geq N} x_j \leq x_{n_k} \leq \sup_{j \geq N} x_j$$

for all $k \geq K$. Taking the limit of this inequality as $k \to \infty$, we obtain

$$\inf_{j \geq N} x_j \leq x \leq \sup_{j \geq N} x_j$$

Taking the limit of this last inequality as $N \to \infty$ and applying Definition 2.32, we obtain(7). □

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If $\{x_n\}$ is any sequence of real numbers, then

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Thank you.