Advanced Calculus (I)

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3.1 Two-Sided Limits

Definition

Let $a \in \mathbf{R}$, Let I be an open interval that contains a, and let f be a real function defined everywhere on I except possibly at a. Then $f(x)$ is said to *converge* to L, *as x approaches a*, if and only if for every $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, I and a) such that

$$0 < |x - a| < \delta \ \ \textit{implies} \ \ |f(x) - L| < \epsilon.$$

In this case we write

$$L = \lim_{x \to a} f(x)$$

and call L the *limit* of $f(x)$ as x approaches a.

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Example:

1. $f(x) = 3$, $\lim\limits_{x \to 1} f(x) = ?$
2. $f(x) = 3x$, $\lim\limits_{x \to 1} f(x) = ?$
3. $f(x) = x^2$, $\lim\limits_{x \to 1} f(x) = ?$
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Remark:

Let $a \in \mathbf{R}$, let I be an open interval that contains a, and let f,g be real functions defined everywhere on I except possibly at a. If $f(x) = g(x)$ for all $x \in I \setminus \{a\}$ and $f(x) \to L$ as $x \to a$, then $g(x)$ also has a limit as $x \to a$, and

$$\lim_{x \to a} g(x) = \lim_{x \to a} f(x).$$

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$$g(x) = \frac{x^3 + x^2 - x - 1}{x^2 - 1}, \lim_{x \to 1} g(x) = ?$$

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Theorem (Sequential Characterization of Limits)

Let $a \in \mathbf{R}$, let I be an open interval that contains a, and let f be a real function defined everywhere on I except possibly at a. Then

$$L = \lim_{x \to a} f(x)$$

exists if and only if $f(x_n) \to L$ as $n \to \infty$ for every sequence $x_n \in I \setminus \{a\}$ that converges to a as $n \to \infty$.

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Example:

Prove that

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

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Proof:

By examing the graph of $y = f(x)$ (see Figure 3.1), we are led to consider two extremes:

$$a_n := \frac{2}{(4n+1)\pi} \ \ and \ \ b_n := \frac{2}{(4n+3)\pi}, \quad n \in \mathbf{N}.$$

Clearly, both a_n and b_n converge to 0 as $n \to \infty$. On the other hand, Since $f(a_n) = 1$ and $f(b_n) = -1$ for all $n \in \mathbf{N}$, $f(a_n) \to 1$ and $f(b_n) \to -1$ as $n \to \infty$. Thus by Theorem 3.6, the limit of f(x), as $x \to 0$, cannot exist. $\square$

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Theorem

Suppose that a $\in \mathbf{R}$, that I is an open interval that contains a, and that f,g are real functions defined everywhere on I except possibly at a. If f(x) and g(x) converge as x approaches a, then so do $(f+g)(x)$, $(fg)(x)$, $(\alpha f)(x)$, and $(f/g)(x)$ (when the limit of g(x) is nonzero).

In fact,

$$\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x),$$

$$\lim_{x\to a}(\alpha f)(x) = \alpha \lim_{x\to a} f(x),$$

$$\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x),$$

and (when the limit of g(x) is nonzero)

$$\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

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Theorem (Squeeze Theorem For Functions)

Suppose that $a \in \mathbf{R}$, that I is an open interval that contains a, and that f,g,h are real functions defined everywhere on I except possibly at a.

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If $g(x) \le h(x) \le f(x)$ for all $x \in I \setminus \{a\}$, and

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L,$$

then the limit of h(x) exists, as $x \to a$, and

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(ii)

If $|g(x)| \leq M$ *for all* $x \in I \setminus \{a\}$ *and* $f(x) \to 0$ *as* $x \to a$, *then*

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Theorem (Comparison Theorem For Functions)

Suppose that $a \in \mathbf{R}$, that I is an open interval that contains a, and that f,g are real functions defined everywhere on I except possibly at a. If f and g have a limit as x approaches a and

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Example:

For each function f define the *positive part* of f by

$$f^+(x) = \frac{|f(x)| + f(x)}{2}, \quad x \in Dom(f),$$

and the *negative part* by

$$f^-(x) = \frac{|f(x)| - f(x)}{2}, \quad x \in Dom(f).$$

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(a)

Prove that $f^+(x) \geq 0$, $f^-(x) \geq 0$, $f(x) = f^+(x) - f^-(x)$, and $|f(x)| = f^+(x) + f^-(x)$ hold for all $x \in Dom(f)$. (Compare with Exercise 1,p.11.)

(b)

Prove that if

$$L = \lim_{x \to a} f(x)$$

exists, then $f^+(x) \to L^+$ and $f^-(x) \to L^-$ as $x \to a$.

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Example:

Let f,g be real functions, and for each $x \in Dom(f) \cap Dom(g)$ define $(f \vee g)(x) := \max\{f(x), g(x)\}$ and $(f \vee g)(x) := \min\{f(x), g(x)\}$.

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Prove that

$$(f \vee g)(x) = \frac{(f+g)(x) + |(f-g)(x)|}{2}$$

and

$$(f \wedge g)(x) = \frac{(f+g)(x) - |(f-g)(x)|}{2}$$

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(b)

Prove that if

$$L = \lim_{x \to a} f(x) \quad and \quad M = \lim_{x \to a} g(x)$$

exist, then $(f \vee g)(x) \to L \vee M$ and $(f \wedge g)(x) \to L \wedge M$ as $x \to a$.

(b)

Prove that if

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exist, then $(f \vee g)(x) \to L \vee M$ and $(f \wedge g)(x) \to L \wedge M$ as $x \to a$.

Thank you.