Advanced Calculus (I)

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3.2 One-sided Limits And Limits At Infinty

Definition (1)

Let $a \in \mathbf{R}$.

(i)

A real function is said to *converge to L as x approaches a from the right* if and only if f is defined on some open interval I with left endpoint a and for every $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, I, and a) such that $a + \delta \in I$ and

$$a < x < a + \delta \ \textit{implies} \ |f(x) - L| < \epsilon.$$

In this case we call L the *right-hand limit* of f at a, and denote it by

$$f(a+) := L =: \lim_{x \to a+} f(x).$$

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Let f be a real function. Then the limit

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Definition (2)

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$f(x) \to L$ as $x \to \infty$ if and only if for any given $\epsilon > 0$, there is an $M \in \mathbf{R}$ such that for $x > M$,

$$|f(x) - L| < \epsilon.$$

In this case, we write

$$\lim_{x \to \infty} f(x) = L$$

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$f(x) \to +\infty$ as $x \to a$ if and only if for any given $M \in \mathbf{R}$, there is a $\delta > 0$ such that

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Theorem

Let a be an extended real number, and I be a nondegenerate open interval which either contains a or has a as one of its endpoints. Suppose further that f is a real function defined on I except possibly at a. Then

$$\lim_{\substack{x \to a \\ x \in I}} f(x)$$

exists and equals L if and only if $f(x_n) \to L$ *for all sequence* $x_n \in I$ *that satisfy* $x_n \neq a$ *and* $x_n \to a$ *as* $n \to \infty$.

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Example:

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Proof:

Since the limit of a product is the product of the limits, we have by Example 3.15 that $1/x^m \to 0$ as $x \to \infty$ for any $m \in \mathbf{N}$. Multiplying numerator and denominator of the expression above by $\frac{1}{x^2}$ we have

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x^2-1}{1-x^2} &= \lim_{x\to\infty} \frac{2-1/x^2}{-1+1/x^2} \\
&= \frac{\lim_{x\to\infty}(2-1/x^2)}{\lim_{x\to\infty}(-1+1/x^2)} \\
&= \frac{2}{-1} = -2. \qquad \square
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Thank you.