Advanced Calculus (I)

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Definition

Let E be a nonempty subset of **R** and $f : E \to \mathbf{R}$.
(i)

f is said to be *continuous at a point* $a \in E$ if and only if given $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, and a) such that

$$|x - a| < \delta \quad and \quad x \in E \quad imply \quad |f(x) - f(a)| < \epsilon.$$

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Remark:

Let I be an open interval that contains a point a and $f : I \to R$. Then f is continuous at $a \in I$ if and only if

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Example:

1. $f(x) = x^2, x \in [0, 1]$.
2. $f(x) = x^2, x \in R$.
3. $f(x) = \dfrac{1}{x}, x \in (0, 1]$.
4. $f(x) = \sqrt{x}, x \in R^+$.

Theorem

Suppose that E is a nonempty subset of $\mathbf{R}$, $a \in E$, and $f : E \to \mathbf{R}$. Then the following statements are equivalent:

(i) f is continuous at $a \in E$

(ii) If x_n converges to a and $x_n \in E$, then $f(x_n) \to f(a)$ as $n \to \infty$.

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Suppose that E is a nonempty subset of $\mathbf{R}$, $a \in E$, and $f : E \longrightarrow \mathbf{R}$. Then the following statements are equivalent:
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Suppose that A and B are subsets of $\mathbf{R}$ and that $f : A \longrightarrow \mathbf{R}$ and $g : B \longrightarrow R$ with $f(A) \subseteq B$.

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If $A := I \setminus \{a\}$, where I is a nondegenerate interval that either contains a or has a as one of its endpoint if

$$L := \lim_{\substack{x \to a \\ x \in I}} f(x)$$

exists and belongs to B, and if g is continuous at $L \in B$, then

$$\lim_{\substack{x \to a \\ x \in I}} (g \circ f)(x) = g\left(\lim_{\substack{x \to a \\ x \in I}} f(x) \right).$$

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Proof:

Suppose that $x_n \in I \setminus \{a\}$ and $x_n \to a$ as $n \to \infty$. Since $f(A) \subseteq B$, $f(x_n) \in B$. Also, by the Sequential Characterization of Limits (Theorem 3.17), $f(x_n) \to L$ as $n \to \infty$. Since g is continuous at $L \in B$, it follows from Theorem 3.21 that $g \circ f(x_n) := g(f(x_n)) \to g(L)$ as $n \to \infty$. Hence by Theorem 3.17, $g \circ f(x) \to g(L)$ as $x \to a$ in I. This prove (i). A similar proof establishes part (ii). $\square$

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Theorem (Extreme Value Theorem)

If I is a closed, bounded interval and $f : I \to \mathbf{R}$ is continuous on I, then f is bounded on I. Moreover, if

$$M = \sup_{x \in I} f(x) \quad and \quad m = \inf_{x \in I} f(x),$$

then there exist points $x_m, x_M \in I$ such that

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$$f(x_M) = M \quad and \quad f(x_m) = m.$$

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Suppose first that f is not bounded on I. Then there exist $x_n \in I$ such that

$$(7) \qquad |f(x_n)| > n, \quad n \in \mathbf{N}.$$

Since I is bounded, we know (by the Bolzano-Weierstrass Theorem) that $\{x_n\}$ has a convergent subsequence, say $x_{n_k} \to a$ as $k \to \infty$. Since I is closed, we also know (by the Comparison Theorem) that $a \in I$. In particular, $f(a) \in R$. On the other hand, substituting n_k for n in (7) and taking the limit of this inequility as $k \to \infty$, we have $|f(a)| = \infty$, a contradiction. Hence, the function f is bounded on I.

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We have proved that both M and m are finite real numbers. To show that there is an $x_M \in I$ such that $f(x_M) = M$, suppose to the contrary that $f(x) < M$ for all $x \in I$. Then the function

$$g(x) = \frac{1}{M - f(x)}$$

is continuous, hence, bounded on I.

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$$g(x) = \frac{1}{M - f(x)}$$

is continuous, hence, bounded on I.

In particular, there is a $C > 0$ such that $|g(x)| = g(x) \leq C$. It follows that

$$(8) \qquad f(x) \leq M - \frac{1}{c}$$

for all $x \in I$. Taking the supremum of (8) over all $x \in I$, we obtain $M \leq M - 1/C < M$, a contradiction. Hence, there is an $x_M \in I$, such that $f(x_M) = M$. A similar argument proves that there is an $x_m \in I$ such that $f(x_m) = m$. □

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Lemma: [Sign-Preserving Property].

Let $f : I \to \mathbf{R}$ where I is an open, nondegenerate interval. If f is continuous at a point $x_0 \in I$ and $f(x_0) > 0$, then there are positive number ϵ and δ such that $|x - x_0| < \delta$ implies $f(x) > \epsilon$.

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Let I be a nondegenerate interval and $f : I \to \mathbf{R}$ be continuous. If $a, b \in I$ with $a < b$, and if y_0 lies between $f(a)$ and $f(b)$, then there is an $x_0 \in (a, b)$ such that $f(x_0) = y_0$.

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Example:

Assuming that $\sin x$ is continuous on **R**, prove that

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is continuous on $(-\infty, 0)$ and $(0, \infty)$, discontinuous at 0, and neither $f(0+)$ nor $f(0-)$ exists. (see Figure 3.1 on p.61.)

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Thank you.