Advanced Calculus (I)

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3.4 Uniform continuity

Definition

Let E be a nonempty subset of **R** and $f : E \to \mathbf{R}$. Then f is said to be *uniformly* continuous on E (notation: $f : E \to \mathbf{R}$ is uniformly continuous) if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|x - a| < \delta \quad \text{and} \quad x, a \in E \quad \text{imply} \quad |f(x) - f(a)| < \epsilon.$$

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Suppose to the contrary that f is continuous but not uniformly continuous on I Then there is an $\epsilon_0 > 0$ and points $x_n, y_n \in I$ such that $|x_n - y_n| < 1/n$ and

$$|f(x_n) - f(y_n)| \geq \epsilon_0, \quad n \in \mathbf{N}. \tag{10}$$

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By the Bolzano-Weierstrass Theorem and the Comparison Theorem, the sequence $\{x_n\}$ has a subsequence, say x_{n_k}, that converges, as $k \to \infty$, to some $x \in I$. Similarly, the sequence $\{y_{n_k}\}_{k \in \mathbf{N}}$ has a convergent subsequence, say $y_{n_{k_j}}$, that converges, as $j \to \infty$, to some $y \in I$. Since $x_{n_{k_j}} \to x$ as $j \to \infty$ and f is continuous, it follows from (10) that $|f(x) - f(y)| \geq \epsilon_0$; i.e., $f(x) \neq f(y)$. But $|x_n - y_n| < 1/n$ for all $n \in \mathbf{N}$, so Theorem 2.9 (the Squeeze Theorem) implies that $x = y$. Therefore, $f(x) = f(y)$, a contradiction. $\square$

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Theorem

Let (a, b) be a bounded, open, nonempty interval and $f : (a, b) \to \mathbf{R}$. Then f is uniformly continuous on (a, b) if and only if f can be extended continuously to $[a, b]$, i.e., if and only if there is a continuous function $g : [a, b] \to \mathbf{R}$ that satifies

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Suppose that f is uniformly continuous on (a, b). Let $x_n \in (a, b)$ converge to b as $n \to \infty$. Then $\{x_n\}$ is Cauchy; hence, by Lemma 3.38, so is $\{f(x_n)\}$. In particular,

$$g(b) := \lim_{n\to\infty} f(x_n)$$

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Indeed, let $y_n \in (a, b)$ be another sequence that converges to b as $n \to \infty$. Given $\epsilon > 0$, choose $\delta > 0$ such that (9) holds for $E = (a, b)$. Since $x_n - y_n \to 0$, choose $N \in \mathbf{N}$ so that $n \geq N$ implies $|x_n - y_n| < \delta$. By (9), then, $|f(x_n) - f(y_n)| < \epsilon$ for all $n \geq N$. Taking the limit of this inequality as $n \to \infty$, we obtain

$$|\lim_{n\to\infty} f(x_n) - \lim_{n\to\infty} f(y_n)| \leq \epsilon$$

for all $\epsilon > 0$. It follows from Theorem 1.9 that

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Thus, $g(b)$ is well defined. A similar argument defines $g(a)$. Set $g(x) = f(x)$ for $x \in (a, b)$. Then g is defined on $[a, b]$, satifies (11), and is continuous on $[a, b]$ by the Sequential Characterization of Limits. Thus, f can be "continuously extended" to g as required.

Conversely, suppose that there is a function g continuouos on [a,b] that satifies (11). By theorem 3.39, g is uniformly continuous on [a,b]; hence, g is uniformly continuous on (a,b). We conclude that f is uniformly continuous on (a,b).□

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Example:

Prove that $f(x) = \dfrac{(x-1)}{\log x}$ is uniformly continuous on (0,1).

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Thank you.