Advanced Calculus (I)

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Lemma: [Rolle's Theorem]

Suppose that $a, b \in \mathbf{R}$ with $a \neq b$. If f is continuous on [a,b], differentiable on (a, b), and if f(a)=f(b), then $f'(c) = 0$ for some $c \in (a, b)$.

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Proof:

By the Extreme Value Theorem, f has a finite maximum M and a finite minimum m on [a,b]. If $M = m$, then f is constant on (a,b) and f'(x)=0 for all $x \in (a, b)$. Suppose that $M \neq m$. Since $f(a) = f(b)$, f must assume one of the values M or m at some point $c \in (a, b)$. By symmetry, we may suppose that $f(c) = M$. (That is, if we can prove the theorem when $f(c) = M$, then a similar proof establishes the theorem when $f(c) = m$.)

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Since M is the maximum of f on $[a, b]$, we have

$$f(c+h) - f(c) \leq 0$$

for all h that satisfy $c + h \in (a, b)$. In this case $h > 0$ this implies that

$$f'(c) = \lim_{h \to 0+} \frac{f(c+h) - f(c)}{h} \leq 0,$$

and in the case $h < 0$ this implies that

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Suppose $a, b \in \mathbf{R}$ with $a \neq b$.
(i)[Generalized Mean Value Theorem]
If f,g are continuous on [a,b] and differentiable on (a,b), then there is a $c \in (a, b)$ such that

$$g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a)).$$

(ii)[Mean Value Theorem]
If f is continuous on [a,b] and differentiable on (a,b), then there is a $c \in (a, b)$ such that

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Set $h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$. Since $h'(x) = f'(x)(g(b) - g(a)) - g'(x)(f(b) - f(a))$, it is clear that h is continuous on [a,b], differentiable on (a,b), and $h(a) = h(b)$. Thus, by Rolle's Theorem, $h'(c) = 0$ for some $c \in (a, b)$.

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Set $g(x) = x$ and apply part (i). (For a geometric interpretation of this result, see the opening paragraph of this section and Figure 4.3 above.) □

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Theorem (Bernoulli's Inequality)

Let α be a positive real number and $\delta \geq -1$. If $0 < \alpha \leq 1$, then

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The proof of these inequalities are similar. We present the details only for the case $0 < \alpha \leq 1$. Let $f(x) = x^{\alpha}$. By the Mean Value Theorem,

$$f(1+\delta) = f(1) + \alpha\delta c^{\alpha-1}$$

for some c between 1 and $1+\delta$. If $\delta > 0$, then $c > 1$. Since $0 < \alpha \leq 1$, it follows that $c^{\alpha-1} \leq 1$(see Exercise 5, p. 134); hence, $\delta c^{\alpha-1} \leq \delta$. On the other hand, if $-1 \leq \delta \leq 0$, then $c^{\alpha-1} \geq 1$ and again $\delta c^{\alpha-1} \leq \delta$. Therefore,

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Theorem (L'Hopital's Rule)

Let a be an extended real number and I be an open interval that either contains a or has a as an endpoint. Suppose that f and g are differentiable on $I \setminus \{a\}$*, and* $g(x) \neq 0 \neq g'(x)$ *for all* $x \in I \setminus \{a\}$*. Suppose futher that*

$$A := \lim_{\substack{x \to a \\ x \in I}} f(x) = \lim_{\substack{x \to a \\ x \in I}} g(x)$$

is either 0 or ∞*. If*

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Thank you.