Advanced Calculus (I)

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4.4 Monotone Function and The Inverse Function Theorem

Definition

Let E be a nonempty subset of $\mathbf{R}$ and $f : E \to \mathbf{R}$.

(i)
f is said to be *increasing* (respectively, *strictly increasing*) on E if and only if $x_1, x_2 \in E$ and $x_1 < x_2$ imply $f(x_1) \leq f(x_2)$ (respectively, $f(x_1) < f(x_2)$).

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Theorem

Suppose that $a, b \in \mathbf{R}$, with $a \neq b$, that f is continuous on [a,b], and that f is differentiable on (a,b).

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If $f'(x) > 0$ (respectively, f'(x)¡0) for all $x \in (a, b)$, then f is strictly increasing (respectively, strictly decreasing) on [a,b].

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If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on [a,b].

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Let $a \leq x_1 < x_2 \leq b$. By the Mean Value Theorem, there is a $c \in (a, b)$ such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Thus, $f(x_2) > f(x_1)$ when $f'(c) > 0$ and $f(x_2) < f(x_1)$ when $f'(c) < 0$. This proves part (i).

To prove part (ii), let $a \leq x \leq b$. By the Mean Value Theorem and hypothesis there is a $c \in (a, b)$ such that

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Suppose that f is increasing on [a,b].
(i) If $x_0 \in [a, b)$, then $f(x_0+)$ exists and $f(x_0) \leq f(x_0+)$.
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Fix $x_0 \in (a, b]$. By symmetry it suffices to show that $f(x_0-)$ exists and satisfies $f(x_0-) \leq f(x_0)$. Set $E = \{f(x) : a < x < x_0\}$ and $s = \sup E$. Since f is increasing, $f(x_0)$ is an upper bound of E. Hence, s is a finite real number that satisfies $s \leq f(x_0)$. Given $\epsilon > 0$, choose by the Approximation Property an $x_1 \in (a, x_0)$ such that $s - \epsilon < f(x_1) \leq s$. Since f is increasing,

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Fix $x_0 \in (a, b]$. By symmetry it suffices to show that $f(x_0-)$ exists and satisfies $f(x_0-) \leq f(x_0)$. Set $E = \{f(x) : a < x < x_0\}$ and $s = \sup E$. Since f is increasing, $f(x_0)$ is an upper bound of E. Hence, s is a finite real number that satisfies $s \leq f(x_0)$. Given $\epsilon > 0$, choose by the Approximation Property an $x_1 \in (a, x_0)$ such that $s - \epsilon < f(x_1) \leq s$. Since f is increasing,

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Thank you.