Advanced Calculus (I)

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Definition

Let $f : [a, b] \to \mathbf{R}$.

(i)

A *Riemann sum* of f with respect to a partition $P = \{x_0, \ldots, x_n\}$ of [a,b] is a sum of the form

$$\sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}),$$

where the choice of $t_j \in [x_{j-1}, x_j]$ is arbitrary.

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5.2 Riemann Sums

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The Riemann sums of f are said to *converge* to I(f) as $||P|| \to 0$ if and only if given $\epsilon > 0$ there is a partition P_ϵ of [a,b] such that

$$P = \{x_0, \ldots, x_n\} \supseteq P_\epsilon \ \ implies \ \ \left| \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}) - I(f) \right| < \epsilon$$

for all choices of $t_j \in [x_{j-1}, x_j]$, $j = 1, 2, \ldots, n$. In this case we shall use the notation

$$I(f) = \lim_{||P|| \to 0} \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}).$$

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Theorem

Let $a, b \in \mathbf{R}$ with $a < b$, and suppose that $f : [a, b] \to \mathbf{R}$ is bounded. Then f is Riemann integrable on [a,b] if and only if

$$I(f) = \lim_{||P|| \to 0} = \sum_{j=1}^{n} f(t_j)(x_j - x_{j=1})$$

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Theorem (Linear Property)

If f,g are integrable on [a,b] and $\alpha \in \mathbf{R}$, then $f+g$ and αf are integrable on [a,b]. In fact,

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

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If f is integrable on [a,b], then f is integrable on each subinterval [c,d] of [a,b]. Moreover,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

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If f is (Riemann) integrable on [a,b], then $|f|$ is integrable on [a,b] and

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If f and g are (Riemann) integrable on [a,b], then so is fg.

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Theorem (First Mean Value Theorem For Integrals)

Suppose that f and g are integrable on [a,b] with $g(x) \geq 0$ *for all* $x \in [a, b]$. *If*

$$m = \inf_{x \in [a,b]} f(x) \ \ and \ \ M = \sup_{x \in [a,b]} f(x),$$

then there is a number $c \in [m, M]$ *such that*

$$\int_a^b f(x)g(x)dx = c \int_a^b g(x)dx.$$

In particular, if f is continuous on [a,b], then there is an $x_0 \in [a, b]$ *that satisfies*

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Theorem (Second Mean Value Theorem For Integrals)

Suppose that f,g are integrable on [a,b], that g is nonnegative on [a,b], and that m,M are real numbers that satisfy $m \leq \inf f([a,b])$ *and* $M \geq \sup f([a,b])$. *Then there is an* $x_0 \in [a,b]$ *such that*

$$\int_a^b f(x)g(x)dx = m\int_a^{x_0} g(x)dx + M\int_{x_0}^b g(x)dx.$$

In particular, if f is also nonnegative on [a,b], then there is an $x_0 \in [a,b]$ *that satisfies*

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