Advanced Calculus (I)

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Theorem (Fundamental Theorem of Calculus)

Let [a,b] be nondegenerate and suppose that $f : [a, b] \to \mathbf{R}$.

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If f is continuous on [a,b] and $F(x) = \int_a^x f(t)dt$, *then* $F \in C^1[a, b]$ *and*

$$\frac{d}{dx}\int_a^x f(t)dt := F'(x) = f(x)$$

for each $x \in [a, b]$

5.3 Fundamental Theorem of Calculus

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If f is differentiable on [a,b] and f' is integrable on [a,b], then

$$\int_a^x f'(t)dt = f(x) - f(a)$$

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If f is differentiable on [a,b] and f′ is integrable on [a,b], then

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Proof:

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$$F(x) = \int_a^x f(t)dt, \quad x \in [a, b].$$

By symmetry, it suffices to show that if $f(x_0+) = f(x_0)$ for some $x_0 \in [a, b)$, then

$$(11) \qquad \lim_{h \to 0+} \frac{F(x_0 + h) - F(x_0)}{h} = f(x_0)$$

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(see Definition 4.6) Let $\epsilon > 0$ and choose a $\delta > 0$ such that $x_0 \le t < x_0 + \delta$ implies $|f(t) - f(x_0)| < \epsilon$. Fix $0 < h < \delta$. Notice that by Theorem 5.20,

$$F(x_0 + h) - F(x_0) = \int_{x_0}^{x_0+h} f(t)dt$$

and that by Theorem 5.16

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Therefore,

$$\frac{F(x_0+h)-F(x_0)}{h} - f(x_0) = \frac{1}{h}\int_{x_0}^{x_0+h} (f(t) - f(x_0))dt.$$

Since $0 < h < \delta$, it follows from Theorem 5.22 and the choice of δ that

$$\left|\frac{F(x_0+h)-F(x_0)}{h} - f(x_0)\right| \leq \frac{1}{h}\int_{x_0}^{x_0+h} |f(t) - f(x_0)|dt \leq \epsilon$$

This verifies (11) and the proof of part (i) is complete.

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(ii)

We may suppose that $x = b$. Let $\epsilon > 0$. Since f' is integrable, choose a partition $P = \{x_0, x_1, \ldots, x_n\}$ of [a,b] such that

$$\left| \sum_{j=1}^{n} f'(t_j)(x_j - x_{j-1}) - \int_a^b f'(t)dt \right| < \epsilon$$

for any choice of points $t_j \in [x_{j-1}, x_j]$.

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Use the Mean Value Theorem to choose points $t_j \in [x_{j-1}, x_j]$ such that $f(x_j) - f(x_{j-1}) = f'(t_j)(x_j - x_{j-1})$. It follows by telescoping that

$$\begin{aligned} & \left| f(b) - f(a) - \int_a^b f'(t)dt \right| \\ = & \left| \sum_{j=1}^{n} (f(x_j) - f(x_{j-1})) - \int_a^b f'(t)dt \right| \\ < & \ \epsilon. \end{aligned}$$

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Theorem (Integration By parts)

Suppose that f,g are differentiable on [a,b] with f', g' integrable on [a,b]. Then

$$\int_a^b f'(x)g(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)dx.$$

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Theorem (Change of Variables)

Let ϕ be continuously differentiable on a closed, nondegenerate interval [a,b]. If

$$f \ \ is \ \ continuous \ \ on \ \ \phi([a,b]),$$

or if ϕ is strictly increasing on [a,b] and f is integrable on $[\phi(a), \phi(b)]$, then

$$\int_{\phi(a)}^{\phi(b)} f(t)dt = \int_a^b f(\phi(x))\phi'(x)dx.$$

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