Advanced Calculus (I)

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Remark:

If f is integrable on [a,b], then

$$\int_a^b f(x)dx = \lim_{c \to a+} \left(\lim_{d \to b-} \int_c^d f(x)dx \right).$$

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Proof:

By Theorem 5.26,

$$F(x) = \int_a^x f(t)dt$$

is continuous on [a,b]. Thus

$$\begin{aligned}\int_a^b f(x)dx &= F(b) - F(a) \\ &= \lim_{c \to a+} \left(\lim_{d \to b-} F(d) - F(c) \right) \\ &= \lim_{c \to a+} \left(\lim_{d \to b-} \int_c^d f(x)dx \right). \square\end{aligned}$$

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Definition

Let [a,b] be a nonempty, open (possibly unbounded) interval and $f : (a, b) \to \mathbf{R}$.

(i)

f is said to be *locally integrable* on (a,b) if and only if f is integrable on each closed subinterval [c,d] of (a,b).

(ii)

f is said to be *improperly integrable* on (a,b) if and only if f is locally integrable on (a,b) and

$$\int_a^b f(x)dx := \lim_{c \to a+} \left(\lim_{d \to b-} \int_c^d f(x)dx \right) \tag{18}$$

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The order of the limits in (18) does not matter. In particular, if the limits in (18) exists, then

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Proof:

Let $x_0 \in (a, b)$ be fixed. By Theorem 5.20 and 3.8

$$\begin{aligned}
&\lim_{c \to a+} \left(\lim_{d \to b-} \int_c^d f(x)dx \right) \\
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Theorem

If f,g are improperly integrable on (a,b) and $\alpha, \beta \in \mathbf{R}$, then $\alpha f + \beta g$is improperly integrable on (a,b) and

$$\int_a^b (\alpha f(x) + \beta g(x))dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx.$$

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Theorem (Comparison Theorem For Improper Integrals)

Suppose that f,g are locally integrable on (a,b). If $0 \le f(x) \le g(x)$ for $x \in (a, b)$, and g is improperly integrable on (a,b), then f is improperly integrable on (a,b) and

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Fix $c \in (a, b)$. Let $F(d) = \int_c^d f(x)dx$ and $G(d) = \int_c^d g(x)dx$ for $d \in [c, b)$. By the Comparison Theorem for Integrals, $F(d) \le G(d)$. Since $f \ge 0$, the function F is increasingon [c,d], hence $F(b-)$ exists. Thus, by definition, f is improper integrable on (c,b) and

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Prove that the function $\dfrac{\sin x}{x}$ is conditionally integrable on $[1, \infty)$.

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Integrating by parts, we have

$$\int_a^b \frac{\sin x}{x} dx = -\frac{\cos x}{x}\Big|_1^d - \int_1^d \frac{\cos x}{x^2} dx$$

$$= \cos(1) - \frac{\cos d}{d} - \int_1^d \frac{\cos x}{x^2} dx$$

Since $\frac{1}{x^2}$ is absolutely integrable on $[1, \infty)$, it follows from Remark 5.46 that $\frac{\cos x}{x^2}$ is absolutely integrable on $[1, \infty)$. Therefore, $\frac{\sin x}{x}$ is improperly integrable on $[1, \infty)$ and

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Since $\frac{1}{x^2}$ is absolutely integrable on $[1,\infty)$, it follows from Remark 5.46 that $\frac{\cos x}{x^2}$ is absolutely integrable on $[1,\infty)$.

Therefore, $\frac{\sin x}{x}$ is improperly integrable on $[1,\infty)$ and

$$\int_1^\infty \frac{\sin x}{x}dx = \cos(1) - \int_1^\infty \frac{\cos x}{x^2}dx$$

Proof:

Integrating by parts, we have

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To show that $\frac{\sin x}{x}$ is not absolutely integrable on $[1, \infty)$, notice that

$$\begin{aligned} \int_1^{n\pi} \frac{|\sin x|}{x} dx &\geq \sum_{k=2}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x} dx \\ &\geq \sum_{k=2}^{n} \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin x| dx \\ &= \sum_{k=2}^{n} \frac{2}{k\pi} = \frac{2}{\pi} \sum_{k=2}^{n} \frac{1}{k} \end{aligned}$$

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as $n \to \infty$, it follows from the Squeeze Theorem that

$$\lim_{n \to \infty} \int_{1}^{n\pi} \frac{\sin x}{x} dx = \infty.$$

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