Advanced Calculus (I)

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5.5 Functions of bounded variation

Let $\phi : [a, b] \to \mathbf{R}$. To measure how much ϕ wiggles on an interval [a,b], set

$$V(\phi, P) = \sum_{j=1}^{n} |\phi(x_j) - \phi(x_{j-1})|$$

for each partition $P = \{x_0, x_1, \cdots, x_n\}$ of [a,b]. The *variation* of ϕ is defined by

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If $\phi \in \mathcal{C}^1[a, b]$, then ϕ is of bounded variation on [a,b]. However, there exist functions of bounded variation that are not continuously differentiable.

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Proof:

Let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of [a,b]. By the Extreme Value Theorem, there is an $M > 0$ such that $|\phi'(x)| \leq M$ for all $x \in [a, b]$. Therefore, it follows from the Mean Value Theorem that for each k between 1 and n, there is a point c_k between x_{k-1} and x_k such that

$$|\phi(x_k) - \phi(x_{k-1})| = |\phi'(c_k)|(x_k - x_{k-1}) \leq M(x_k - x_{k-1}).$$

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By telescoping, we obtain $V(\phi, P) \leq M(b-a)$ for any partition P of [a,b]. Therefore,

$$Var(\phi) \leq M(b-a).$$

On the other hand, $x^2 \sin(1/x)$ is of bounded variation on [0,1] (see Exercise 2) but does not belong to $\mathcal{C}^1[0,1]$ (see Example 4.8) □

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Let $x \in [a, b]$ and note by definition that

$$|\phi(x) - \phi(a)| \leq |\phi(x) - \phi(a)| + |\phi(b) - \phi(x)| \leq Var(\phi).$$

Hence, by the triangle inequality,

$$|\phi(x)| \leq |\phi(a)| + Var(\phi).$$

To find a bounded function that is not of bounded variation, consider

$$\phi(x) := \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

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Clearly, ϕ is bounded by 1. On the other hand, if

$$x_j = \begin{cases} 0 & j = 0 \\ \dfrac{2}{(n-j)\pi} & 0 < j < n, \end{cases}$$

then

$$\sum_{j=1}^{n} |\phi(x_j) - \phi(x_{j-1})| = 2n \to \infty$$

as $n \to \infty$. Thus ϕ is not of bounded variation on $[0, \frac{2}{\pi}]$. □

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If ϕ and ψ are of bounded variation on a closed interval [a,b], then so are $\phi + \psi$ and $\phi - \psi$.

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Let ϕ be of bounded variation on a closed interval [a,b]. The *total variation* of ϕ is the function Φ deifined on [a,b] by

$$\Phi(x) := \sup \left\{ \sum_{j=1}^{k} |\phi(x_j) - \phi(x_{j-1})| : \{x_0, x_1, \cdots, x_k\} \right.$$

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Let ϕ be of bounded variation on [a,b] and Φ be its total variation. Then

(i) $|\phi(y) - \phi(x)| \leq \Phi(y) - \Phi(x)$ for all $a \leq x < y \leq b$,

(ii) Φ and $\Phi - \phi$ are increasing on [a,b], and

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Let $x < y$ belong to [a,b] and $\{x_0, x_1, \cdots, x_k\}$ be a partition of [a,x]. Then $\{x_0, x_1, \cdots, x_k, y\}$ is a partition [a,y], and we have by Definition 5.55 that

$$\sum_{j=1}^{k} |\phi(x)-\phi(x_{j-1})| \leq \sum_{j=1}^{k} |\phi(x_j)-\phi(x_{j-1})|+|\phi(y)-\phi(x)| \leq \Phi(y).$$

Taking the supremum of this inequality over all partitions $\{x_0, x_1, \cdots, x_k\}$ of [a,x], we obtain

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By the Monotone Property of Suprema, Φ is increasing on [a,b]. To show that $\Phi - \phi$ is increasing, suppose that $a \leq x < y \leq b$. By part (i),

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Let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of [a,b]. By part (i) and Definition 5.50,

$$\sum_{j=1}^{n} |\phi(x_j) - \phi(x_{j-1}| \leq \sum_{j=1}^{n} |\Phi(x_j) - \Phi(x_{j-1})| \leq Var(\Phi).$$

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Taking the supremum of this inequality over all partitions P of [a,b], we obtain $Var(\phi) \leq Var(\Phi)$ $\square$

Corollary:

Let [a,b] be a closed interval. Then ϕ is of bounded variation on [a,b] if and only if there exist increasing functions f, g on [a,b] such that

$$\phi(x) = f(x) - g(x), \quad x \in [a, b].$$

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Proof:

Suppose that ϕ is of bounded variation, let Φ represent the total variation of ϕ, $f = \Phi$, and $g = \Phi - \phi$. By theorem 5.56, f and g are increasing , and by construction, $\phi = f - g$.

Conversely, suppose that $\phi = f - g$ for some increasing f,g on [a,b]. Then by Remark 5.52 and Theorem 5.54, ϕ is of bounded variation on [a,b]. □

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Thank you.