Advanced Calculus (I)

WEN-CHING LIEN

Department of Mathematics
National Cheng Kung University

Definition

Let I be an interval and $f : I \to \mathbf{R}$
(i) f is said to be convex on I if and only if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

for all $0 \leq \alpha \leq 1$ and all $x, y \in I$.

(ii) f is said to be *concave* on I if and only if $-f$ is convex on I.

Definition

Let I be an interval and $f : I \to \mathbf{R}$

(i) f is said to be convex on I if and only if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

for all $0 \leq \alpha \leq 1$ and all $x, y \in I$.

(ii) f is said to be *concave* on I if and only if $-f$ is convex on I.

Definition

Let I be an interval and $f : I \to \mathbf{R}$
(i) f is said to be convex on I if and only if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

for all $0 \leq \alpha \leq 1$ and all $x, y \in I$.

(ii) f is said to be *concave* on I if and only if $-f$ is convex on I.

Definition

Let I be an interval and $f : I \to \mathbf{R}$
(i) f is said to be convex on I if and only if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

for all $0 \leq \alpha \leq 1$ and all $x, y \in I$.

(ii) f is said to be *concave* on I if and only if $-f$ is convex on I.

Remark:

Let I be an interval and $f : I \to \mathbf{R}$. Then f is convex on I if and only if given any [c,d]$\subseteq I$, the chord through the points $(c, f(c))$, $(d, f(d)$ lies on or above the graph of $y = f(x)$ for all $x \in [c, d]$. (See Figure 5.5.)

Remark:

Let I be an interval and $f : I \to \mathbf{R}$. Then f is convex on I if and only if given any [c,d]$\subseteq I$, the chord through the points $(c, f(c))$, $(d, f(d)$ lies on or above the graph of $y = f(x)$ for all $x \in [c, d]$. (See Figure 5.5.)

Proof:

Suppose that f is contex on I and $x_0 \in [c, d]$. Choose $0 \leq \alpha \leq 1$ such that $x_0 = \alpha c + (1 - \alpha)d$. The chord from $(c, f(c))$ to (d,f(d)) has slope $\dfrac{(f(d) - f(c))}{(d - c)}$. Hence, the point on this chord which has the form (x_0, y_0) must satisfy $y_0 = \alpha f(c) + (1 - \alpha)f(d)$. Since f is convex, it follows that $f(x_0) \leq y_0$; i.e., the point (x_0, y_0) lies on or above the point $(x_0, f(x_0))$. A similar argument establishes the reverse implication. $\square$

Proof:

Suppose that f is contex on I and $x_0 \in [c, d]$. Choose $0 \leq \alpha \leq 1$ such that $x_0 = \alpha c + (1 - \alpha)d$. The chord from $(c, f(c))$ to (d,f(d)) has slope $\frac{(f(d) - f(c))}{(d - c)}$. Hence, the point on this chord which has the form (x_0, y_0) must satisfy $y_0 = \alpha f(c) + (1 - \alpha)f(d)$. Since f is convex, it follows that $f(x_0) \leq y_0$; i.e., the point (x_0, y_0) lies on or above the point $(x_0, f(x_0))$. A similar argument establishes the reverse implication. □

Proof:

Suppose that f is contex on I and $x_0 \in [c, d]$. Choose $0 \leq \alpha \leq 1$ such that $x_0 = \alpha c + (1 - \alpha)d$. The chord from $(c, f(c))$ to (d,f(d)) has slope $\dfrac{(f(d) - f(c))}{(d - c)}$. Hence, the point on this chord which has the form (x_0, y_0) must satisfy $y_0 = \alpha f(c) + (1 - \alpha)f(d)$. Since f is convex, it follows that $f(x_0) \leq y_0$; i.e., the point (x_0, y_0) lies on or above the point $(x_0, f(x_0))$. A similar argument establishes the reverse implication. $\square$

Proof:

Suppose that f is contex on I and $x_0 \in [c, d]$. Choose $0 \leq \alpha \leq 1$ such that $x_0 = \alpha c + (1 - \alpha)d$. The chord from $(c, f(c))$ to (d,f(d)) has slope $\dfrac{(f(d) - f(c))}{(d - c)}$. Hence, the point on this chord which has the form (x_0, y_0) must satisfy $y_0 = \alpha f(c) + (1 - \alpha)f(d)$. Since f is convex, it follows that $f(x_0) \leq y_0$; i.e., the point (x_0, y_0) lies on or above the point $(x_0, f(x_0))$. A similar argument establishes the reverse implication. □

Proof:

Suppose that f is contex on I and $x_0 \in [c, d]$. Choose $0 \leq \alpha \leq 1$ such that $x_0 = \alpha c + (1 - \alpha)d$. The chord from $(c, f(c))$ to (d,f(d)) has slope $\dfrac{(f(d) - f(c))}{(d - c)}$. Hence, the point on this chord which has the form (x_0, y_0) must satisfy $y_0 = \alpha f(c) + (1 - \alpha) f(d)$. Since f is convex, it follows that $f(x_0) \leq y_0$; i.e., the point (x_0, y_0) lies on or above the point $(x_0, f(x_0))$. A similar argument establishes the reverse implication. □

Proof:

Suppose that f is contex on I and $x_0 \in [c, d]$. Choose $0 \le \alpha \le 1$ such that $x_0 = \alpha c + (1 - \alpha)d$. The chord from $(c, f(c))$ to (d,f(d)) has slope $\dfrac{(f(d) - f(c))}{(d - c)}$. Hence, the point on this chord which has the form (x_0, y_0) must satisfy $y_0 = \alpha f(c) + (1 - \alpha)f(d)$. Since f is convex, it follows that $f(x_0) \le y_0$; i.e., the point (x_0, y_0) lies on or above the point $(x_0, f(x_0))$. A similar argument establishes the reverse implication. □

Proof:

Suppose that f is contex on I and $x_0 \in [c, d]$. Choose $0 \leq \alpha \leq 1$ such that $x_0 = \alpha c + (1 - \alpha)d$. The chord from $(c, f(c))$ to (d,f(d)) has slope $\dfrac{(f(d) - f(c))}{(d - c)}$. Hence, the point on this chord which has the form (x_0, y_0) must satisfy $y_0 = \alpha f(c) + (1 - \alpha)f(d)$. Since f is convex, it follows that $f(x_0) \leq y_0$; i.e., the point (x_0, y_0) lies on or above the point $(x_0, f(x_0))$. A similar argument establishes the reverse implication. □

Proof:

Suppose that f is contex on I and $x_0 \in [c, d]$. Choose $0 \leq \alpha \leq 1$ such that $x_0 = \alpha c + (1 - \alpha)d$. The chord from $(c, f(c))$ to (d,f(d)) has slope $\dfrac{(f(d) - f(c))}{(d - c)}$. Hence, the point on this chord which has the form (x_0, y_0) must satisfy $y_0 = \alpha f(c) + (1 - \alpha)f(d)$. Since f is convex, it follows that $f(x_0) \leq y_0$; i.e., the point (x_0, y_0) lies on or above the point $(x_0, f(x_0))$. A similar argument establishes the reverse implication. □

Proof:

Suppose that f is contex on I and $x_0 \in [c, d]$. Choose $0 \leq \alpha \leq 1$ such that $x_0 = \alpha c + (1 - \alpha)d$. The chord from $(c, f(c))$ to (d,f(d)) has slope $\dfrac{(f(d) - f(c))}{(d - c)}$. Hence, the point on this chord which has the form (x_0, y_0) must satisfy $y_0 = \alpha f(c) + (1 - \alpha)f(d)$. Since f is convex, it follows that $f(x_0) \leq y_0$; i.e., the point (x_0, y_0) lies on or above the point $(x_0, f(x_0))$. A similar argument establishes the reverse implication. □

Proof:

Suppose that f is contex on I and $x_0 \in [c, d]$. Choose $0 \leq \alpha \leq 1$ such that $x_0 = \alpha c + (1 - \alpha)d$. The chord from $(c, f(c))$ to (d,f(d)) has slope $\dfrac{(f(d) - f(c))}{(d - c)}$. Hence, the point on this chord which has the form (x_0, y_0) must satisfy $y_0 = \alpha f(c) + (1 - \alpha)f(d)$. Since f is convex, it follows that $f(x_0) \leq y_0$; i.e., the point (x_0, y_0) lies on or above the point $(x_0, f(x_0))$. A similar argument establishes the reverse implication. □

Remark:

A function f is convex on a nonempty, open interval (a,b) if and only if the slope of the chord always increases on (a,b); i.e.,

$$a < c < x < d < b \ \ implies \ \ \frac{f(x) - f(c)}{x - c} \leq \frac{f(d) - f(x)}{d - x}.$$

Remark:

A function f is convex on a nonempty, open interval (a,b) if and only if the slope of the chord always increases on (a,b); i.e.,

$$a < c < x < d < b \ \ implies \ \ \frac{f(x) - f(c)}{x - c} \leq \frac{f(d) - f(x)}{d - x}.$$

Proof:

Fix $a < c < x < d < b$ and let $\lambda(x)$ be the equation of the chord to f through the points $(c, f(c))$ and $(d, f(d))$. If f is convex, then $f(x) \leq \lambda(x)$ (see Figure 5.6). Therefore,

$$\frac{f(x) - f(c)}{x - c} \leq \frac{\lambda(x) - \lambda(c)}{x - c} = \frac{\lambda(d) - \lambda(x)}{d - x} \leq \frac{f(d) - f(x)}{d - x}.$$

Conversely, if f is not convex, then $\lambda(x) < f(x)$ for some $x \in (c, d)$. It follows that

$$\frac{f(x) - f(c)}{x - c} > \frac{\lambda(x) - \lambda(c)}{x - c} = \frac{\lambda(d) - \lambda(x)}{d - x} > \frac{f(d) - f(x)}{d - x}.$$

Therefore, the slope of the chord does not increase on (a,b). □

Proof:

Fix $a < c < x < d < b$ and let $\lambda(x)$ be the equation of the chord to f through the points $(c, f(c))$ and $(d, f(d))$. If f is convex, then $f(x) \leq \lambda(x)$ (see Figure 5.6). Therefore,

$$\frac{f(x)-f(c)}{x-c} \leq \frac{\lambda(x)-\lambda(c)}{x-c} = \frac{\lambda(d)-\lambda(x)}{d-x} \leq \frac{f(d)-f(x)}{d-x}.$$

Conversely, if f is not convex, then $\lambda(x) < f(x)$ for some $x \in (c, d)$. It follows that

$$\frac{f(x)-f(c)}{x-c} > \frac{\lambda(x)-\lambda(c)}{x-c} = \frac{\lambda(d)-\lambda(x)}{d-x} > \frac{f(d)-f(x)}{d-x}.$$

Therefore, the slope of the chord does not increase on (a,b). □

Proof:

Fix $a < c < x < d < b$ and let $\lambda(x)$ be the equation of the chord to f through the points $(c, f(c))$ and $(d, f(d))$. If f is convex, then $f(x) \leq \lambda(x)$ (see Figure 5.6). Therefore,

$$\frac{f(x) - f(c)}{x - c} \leq \frac{\lambda(x) - \lambda(c)}{x - c} = \frac{\lambda(d) - \lambda(x)}{d - x} \leq \frac{f(d) - f(x)}{d - x}.$$

Conversely, if f is not convex, then $\lambda(x) < f(x)$ for some $x \in (c, d)$. It follows that

$$\frac{f(x) - f(c)}{x - c} > \frac{\lambda(x) - \lambda(c)}{x - c} = \frac{\lambda(d) - \lambda(x)}{d - x} > \frac{f(d) - f(x)}{d - x}.$$

Therefore, the slope of the chord does not increase on (a,b). □

Proof:

Fix $a < c < x < d < b$ and let $\lambda(x)$ be the equation of the chord to f through the points $(c, f(c))$ and $(d, f(d))$. If f is convex, then $f(x) \leq \lambda(x)$ (see Figure 5.6). Therefore,

$$\frac{f(x)-f(c)}{x-c} \leq \frac{\lambda(x)-\lambda(c)}{x-c} = \frac{\lambda(d)-\lambda(x)}{d-x} \leq \frac{f(d)-f(x)}{d-x}.$$

Conversely, if f is not convex, then $\lambda(x) < f(x)$ for some $x \in (c, d)$. It follows that

$$\frac{f(x)-f(c)}{x-c} > \frac{\lambda(x)-\lambda(c)}{x-c} = \frac{\lambda(d)-\lambda(x)}{d-x} > \frac{f(d)-f(x)}{d-x}.$$

Therefore, the slope of the chord does not increase on (a,b). □

Proof:

Fix $a < c < x < d < b$ and let $\lambda(x)$ be the equation of the chord to f through the points $(c, f(c))$ and $(d, f(d))$. If f is convex, then $f(x) \leq \lambda(x)$ (see Figure 5.6). Therefore,

$$\frac{f(x) - f(c)}{x - c} \leq \frac{\lambda(x) - \lambda(c)}{x - c} = \frac{\lambda(d) - \lambda(x)}{d - x} \leq \frac{f(d) - f(x)}{d - x}.$$

Conversely, if f is not convex, then $\lambda(x) < f(x)$ for some $x \in (c, d)$. It follows that

$$\frac{f(x) - f(c)}{x - c} > \frac{\lambda(x) - \lambda(c)}{x - c} = \frac{\lambda(d) - \lambda(x)}{d - x} > \frac{f(d) - f(x)}{d - x}.$$

Therefore, the slope of the chord does not increase on (a,b). □

Proof:

Fix $a < c < x < d < b$ and let $\lambda(x)$ be the equation of the chord to f through the points $(c, f(c))$ and $(d, f(d))$. If f is convex, then $f(x) \leq \lambda(x)$ (see Figure 5.6). Therefore,

$$\frac{f(x) - f(c)}{x - c} \leq \frac{\lambda(x) - \lambda(c)}{x - c} = \frac{\lambda(d) - \lambda(x)}{d - x} \leq \frac{f(d) - f(x)}{d - x}.$$

Conversely, if f is not convex, then $\lambda(x) < f(x)$ for some $x \in (c, d)$. It follows that

$$\frac{f(x) - f(c)}{x - c} > \frac{\lambda(x) - \lambda(c)}{x - c} = \frac{\lambda(d) - \lambda(x)}{d - x} > \frac{f(d) - f(x)}{d - x}.$$

Therefore, the slope of the chord does not increase on (a,b). □

Proof:

Fix $a < c < x < d < b$ and let $\lambda(x)$ be the equation of the chord to f through the points $(c, f(c))$ and $(d, f(d))$. If f is convex, then $f(x) \leq \lambda(x)$ (see Figure 5.6). Therefore,

$$\frac{f(x) - f(c)}{x - c} \leq \frac{\lambda(x) - \lambda(c)}{x - c} = \frac{\lambda(d) - \lambda(x)}{d - x} \leq \frac{f(d) - f(x)}{d - x}.$$

Conversely, if f is not convex, then $\lambda(x) < f(x)$ for some $x \in (c, d)$. It follows that

$$\frac{f(x) - f(c)}{x - c} > \frac{\lambda(x) - \lambda(c)}{x - c} = \frac{\lambda(d) - \lambda(x)}{d - x} > \frac{f(d) - f(x)}{d - x}.$$

Therefore, the slope of the chord does not increase on (a,b). □

Proof:

Fix $a < c < x < d < b$ and let $\lambda(x)$ be the equation of the chord to f through the points $(c, f(c))$ and $(d, f(d))$. If f is convex, then $f(x) \le \lambda(x)$ (see Figure 5.6). Therefore,

$$\frac{f(x) - f(c)}{x - c} \le \frac{\lambda(x) - \lambda(c)}{x - c} = \frac{\lambda(d) - \lambda(x)}{d - x} \le \frac{f(d) - f(x)}{d - x}.$$

Conversely, if f is not convex, then $\lambda(x) < f(x)$ for some $x \in (c, d)$. It follows that

$$\frac{f(x) - f(c)}{x - c} > \frac{\lambda(x) - \lambda(c)}{x - c} = \frac{\lambda(d) - \lambda(x)}{d - x} > \frac{f(d) - f(x)}{d - x}.$$

Therefore, the slope of the chord does not increase on (a,b). □

Proof:

Fix $a < c < x < d < b$ and let $\lambda(x)$ be the equation of the chord to f through the points $(c, f(c))$ and $(d, f(d))$. If f is convex, then $f(x) \leq \lambda(x)$ (see Figure 5.6). Therefore,

$$\frac{f(x) - f(c)}{x - c} \leq \frac{\lambda(x) - \lambda(c)}{x - c} = \frac{\lambda(d) - \lambda(x)}{d - x} \leq \frac{f(d) - f(x)}{d - x}.$$

Conversely, if f is not convex, then $\lambda(x) < f(x)$ for some $x \in (c, d)$. It follows that

$$\frac{f(x) - f(c)}{x - c} > \frac{\lambda(x) - \lambda(c)}{x - c} = \frac{\lambda(d) - \lambda(x)}{d - x} > \frac{f(d) - f(x)}{d - x}.$$

Therefore, the slope of the chord does not increase on (a,b). □

Proof:

Fix $a < c < x < d < b$ and let $\lambda(x)$ be the equation of the chord to f through the points $(c, f(c))$ and $(d, f(d))$. If f is convex, then $f(x) \leq \lambda(x)$ (see Figure 5.6). Therefore,

$$\frac{f(x) - f(c)}{x - c} \leq \frac{\lambda(x) - \lambda(c)}{x - c} = \frac{\lambda(d) - \lambda(x)}{d - x} \leq \frac{f(d) - f(x)}{d - x}.$$

Conversely, if f is not convex, then $\lambda(x) < f(x)$ for some $x \in (c, d)$. It follows that

$$\frac{f(x) - f(c)}{x - c} > \frac{\lambda(x) - \lambda(c)}{x - c} = \frac{\lambda(d) - \lambda(x)}{d - x} > \frac{f(d) - f(x)}{d - x}.$$

Therefore, the slope of the chord does not increase on (a,b). □

Proof:

Fix $a < c < x < d < b$ and let $\lambda(x)$ be the equation of the chord to f through the points $(c, f(c))$ and $(d, f(d))$. If f is convex, then $f(x) \leq \lambda(x)$ (see Figure 5.6). Therefore,

$$\frac{f(x) - f(c)}{x - c} \leq \frac{\lambda(x) - \lambda(c)}{x - c} = \frac{\lambda(d) - \lambda(x)}{d - x} \leq \frac{f(d) - f(x)}{d - x}.$$

Conversely, if f is not convex, then $\lambda(x) < f(x)$ for some $x \in (c, d)$. It follows that

$$\frac{f(x) - f(c)}{x - c} > \frac{\lambda(x) - \lambda(c)}{x - c} = \frac{\lambda(d) - \lambda(x)}{d - x} > \frac{f(d) - f(x)}{d - x}.$$

Therefore, the slope of the chord does not increase on (a,b). □

Proof:

Fix $a < c < x < d < b$ and let $\lambda(x)$ be the equation of the chord to f through the points $(c, f(c))$ and $(d, f(d))$. If f is convex, then $f(x) \leq \lambda(x)$ (see Figure 5.6). Therefore,

$$\frac{f(x) - f(c)}{x - c} \leq \frac{\lambda(x) - \lambda(c)}{x - c} = \frac{\lambda(d) - \lambda(x)}{d - x} \leq \frac{f(d) - f(x)}{d - x}.$$

Conversely, if f is not convex, then $\lambda(x) < f(x)$ for some $x \in (c, d)$. It follows that

$$\frac{f(x) - f(c)}{x - c} > \frac{\lambda(x) - \lambda(c)}{x - c} = \frac{\lambda(d) - \lambda(x)}{d - x} > \frac{f(d) - f(x)}{d - x}.$$

Therefore, the slope of the chord does not increase on (a,b). □

Theorem

Suppose that f is differentiable on a nonempty, open interval I. Then f is convex on I if and only if f' is increasing on I.

Theorem

Suppose that f is differentiable on a nonempty, open interval I. Then f is convex on I if and only if f' is increasing on I.

Proof:

Suppose that f is convex on $I =: (a, b)$ and that $c, d \in (a, b)$ satisfy $c < d$. Choose $h > 0$ so small that $c + h < d$ and $d + h < b$. Then by Remark 5.60,

$$\frac{f(c+h) - f(c)}{h} \leq \frac{f(d+h) - f(d)}{h}.$$

In particular, $f'(c) \leq f'(d)$.

Proof:

Suppose that f is convex on $I =: (a, b)$ and that $c, d \in (a, b)$ satisfy $c < d$. Choose $h > 0$ so small that $c + h < d$ and $d + h < b$. Then by Remark 5.60,

$$\frac{f(c+h) - f(c)}{h} \leq \frac{f(d+h) - f(d)}{h}.$$

In particular, $f'(c) \leq f'(d)$.

Proof:

Suppose that f is convex on $I =: (a, b)$ and that $c, d \in (a, b)$ satisfy $c < d$. Choose $h > 0$ so small that $c + h < d$ and $d + h < b$. Then by Remark 5.60,

$$\frac{f(c+h) - f(c)}{h} \leq \frac{f(d+h) - f(d)}{h}.$$

In particular, $f'(c) \leq f'(d)$.

Proof:

Suppose that f is convex on $I =: (a, b)$ and that $c, d \in (a, b)$ satisfy $c < d$. Choose $h > 0$ so small that $c + h < d$ and $d + h < b$. Then by Remark 5.60,

$$\frac{f(c+h) - f(c)}{h} \leq \frac{f(d+h) - f(d)}{h}.$$

In particular, $f'(c) \leq f'(d)$.

Proof:

Suppose that f is convex on $I =: (a, b)$ and that $c, d \in (a, b)$ satisfy $c < d$. Choose $h > 0$ so small that $c + h < d$ and $d + h < b$. Then by Remark 5.60,

$$\frac{f(c+h) - f(c)}{h} \leq \frac{f(d+h) - f(d)}{h}.$$

In particular, $f'(c) \leq f'(d)$.

Conversely, let f' be increasing on (a,b) and let $a < c < x < d < b$. Use the Mean value Theorem to choose x_0 (between c and x) and x_1 (between x and d) such that

$$\frac{f(x) - f(c)}{x - c} = f'(x_0) \text{ and } \frac{f(d) - f(x)}{d - x} = f'(x_1).$$

Since $x_0 < x_1$ it follows that $f'(x_0) \leq f'(x_1)$. In particular, we conclude by Remark 5.60 that f is convex on (a,b). □

Conversely, let f' be increasing on (a,b) and let $a < c < x < d < b$. Use the Mean value Theorem to choose x_0 (between c and x) and x_1 (between x and d) such that

$$\frac{f(x)-f(c)}{x-c} = f'(x_0) \quad and \quad \frac{f(d)-f(x)}{d-x} = f'(x_1).$$

Since $x_0 < x_1$ it follows that $f'(x_0) \le f'(x_1)$. In particular, we conclude by Remark 5.60 that f is convex on (a,b). □

Conversely, let f' be increasing on (a,b) and let $a < c < x < d < b$. Use the Mean value Theorem to choose x_0 (between c and x) and x_1 (between x and d) such that

$$\frac{f(x)-f(c)}{x-c} = f'(x_0) \ \ and \ \ \frac{f(d)-f(x)}{d-x} = f'(x_1).$$

Since $x_0 < x_1$ it follows that $f'(x_0) \le f'(x_1)$. In particular, we conclude by Remark 5.60 that f is convex on (a,b). □

Conversely, let f' be increasing on (a,b) and let $a < c < x < d < b$. Use the Mean value Theorem to choose x_0 (between c and x) and x_1 (between x and d) such that

$$\frac{f(x)-f(c)}{x-c} = f'(x_0) \;\; and \;\; \frac{f(d)-f(x)}{d-x} = f'(x_1).$$

Since $x_0 < x_1$ it follows that $f'(x_0) \le f'(x_1)$. In particular, we conclude by Remark 5.60 that f is convex on (a,b). □

Conversely, let f' be increasing on (a,b) and let $a < c < x < d < b$. Use the Mean value Theorem to choose x_0 (between c and x) and x_1 (between x and d) such that

$$\frac{f(x) - f(c)}{x - c} = f'(x_0) \ \ and \ \ \frac{f(d) - f(x)}{d - x} = f'(x_1).$$

Since $x_0 < x_1$ it follows that $f'(x_0) \leq f'(x_1)$. In particular, we conclude by Remark 5.60 that f is convex on (a,b). □

Conversely, let f' be increasing on (a,b) and let $a < c < x < d < b$. Use the Mean value Theorem to choose x_0 (between c and x) and x_1 (between x and d) such that

$$\frac{f(x)-f(c)}{x-c} = f'(x_0) \ \ and \ \ \frac{f(d)-f(x)}{d-x} = f'(x_1).$$

Since $x_0 < x_1$ it follows that $f'(x_0) \leq f'(x_1)$. In particular, we conclude by Remark 5.60 that f is convex on (a,b). □

Conversely, let f' be increasing on (a,b) and let $a < c < x < d < b$. Use the Mean value Theorem to choose x_0 (between c and x) and x_1 (between x and d) such that

$$\frac{f(x) - f(c)}{x - c} = f'(x_0) \;\; and \;\; \frac{f(d) - f(x)}{d - x} = f'(x_1).$$

Since $x_0 < x_1$ it follows that $f'(x_0) \leq f'(x_1)$. In particular, we conclude by Remark 5.60 that f is convex on (a,b). □

Conversely, let f' be increasing on (a,b) and let $a < c < x < d < b$. Use the Mean value Theorem to choose x_0 (between c and x) and x_1 (between x and d) such that

$$\frac{f(x) - f(c)}{x - c} = f'(x_0) \;\; and \;\; \frac{f(d) - f(x)}{d - x} = f'(x_1).$$

Since $x_0 < x_1$ it follows that $f'(x_0) \leq f'(x_1)$. In particular, we conclude by Remark 5.60 that f is convex on (a,b). □

Theorem

If f is convex on some nonempty, open interval I, then f is continuous on I.

Theorem

If f is convex on some nonempty, open interval I, then f is continuous on I.

Proof:

Let $x_0 \in I =: (a, b)$. By symmetry, it suffices to show that $f(x) \to f(x_0)$ as $x \to x_0+$. Let $a < c < x_0 < x < d < b$, $y = g(x)$ represent the euqation of the chord through $(x_0, f(x_0)), (d, f(d))$. Since f is convex, we have by Remark 5.59 that $f(x) \leq h(x)$. Since $f(x_0)$ lies on or below the chord from $(c, f(c))$ to $(x, f(x))$, we also have that $g(x) \leq f(x)$. Consequently,

$$g(x) \leq f(x) \leq h(x), \quad x \in (x_0, b).$$

Proof:

Let $x_0 \in I =: (a, b)$. By symmetry, it suffices to show that $f(x) \to f(x_0)$ as $x \to x_0+$. Let $a < c < x_0 < x < d < b$, $y = g(x)$ represent the euqation of the chord through $(x_0, f(x_0))$, $(d, f(d))$. Since f is convex, we have by Remark 5.59 that $f(x) \leq h(x)$. Since $f(x_0)$ lies on or below the chord from $(c, f(c))$ to $(x, f(x))$, we also have that $g(x) \leq f(x)$. Consequently,

$$g(x) \leq f(x) \leq h(x), \quad x \in (x_0, b).$$

Proof:

Let $x_0 \in I =: (a, b)$. By symmetry, it suffices to show that $f(x) \to f(x_0)$ as $x \to x_0+$. Let $a < c < x_0 < x < d < b$, $y = g(x)$ represent the euqation of the chord through $(x_0, f(x_0)), (d, f(d))$. Since f is convex, we have by Remark 5.59 that $f(x) \le h(x)$. Since $f(x_0)$ lies on or below the chord from $(c, f(c))$ to $(x, f(x))$, we also have that $g(x) \le f(x)$. Consequently,

$$g(x) \le f(x) \le h(x), \quad x \in (x_0, b).$$

Proof:

Let $x_0 \in I =: (a, b)$. By symmetry, it suffices to show that $f(x) \to f(x_0)$ as $x \to x_0+$. Let $a < c < x_0 < x < d < b$, $y = g(x)$ represent the euqation of the chord through $(x_0, f(x_0))$, $(d, f(d))$. Since f is convex, we have by Remark 5.59 that $f(x) \leq h(x)$. Since $f(x_0)$ lies on or below the chord from $(c, f(c))$ to $(x, f(x))$, we also have that $g(x) \leq f(x)$. Consequently,

$$g(x) \leq f(x) \leq h(x), \quad x \in (x_0, b).$$

Proof:

Let $x_0 \in I =: (a, b)$. By symmetry, it suffices to show that $f(x) \to f(x_0)$ as $x \to x_0+$. Let $a < c < x_0 < x < d < b$, $y = g(x)$ represent the euqation of the chord through $(x_0, f(x_0))$, $(d, f(d))$. Since f is convex, we have by Remark 5.59 that $f(x) \leq h(x)$. Since $f(x_0)$ lies on or below the chord from $(c, f(c))$ to $(x, f(x))$, we also have that $g(x) \leq f(x)$. Consequently,

$$g(x) \leq f(x) \leq h(x), \quad x \in (x_0, b).$$

Proof:

Let $x_0 \in I =: (a, b)$. By symmetry, it suffices to show that $f(x) \to f(x_0)$ as $x \to x_0+$. Let $a < c < x_0 < x < d < b$, $y = g(x)$ represent the euqation of the chord through $(x_0, f(x_0))$, $(d, f(d))$. Since f is convex, we have by Remark 5.59 that $f(x) \le h(x)$. Since $f(x_0)$ lies on or below the chord from $(c, f(c))$ to $(x, f(x))$, we also have that $g(x) \le f(x)$. Consequently,

$$g(x) \le f(x) \le h(x), \quad x \in (x_0, b).$$

Proof:

Let $x_0 \in I =: (a, b)$. By symmetry, it suffices to show that $f(x) \to f(x_0)$ as $x \to x_0+$. Let $a < c < x_0 < x < d < b$, $y = g(x)$ represent the euqation of the chord through $(x_0, f(x_0)), (d, f(d))$. Since f is convex, we have by Remark 5.59 that $f(x) \leq h(x)$. Since $f(x_0)$ lies on or below the chord from $(c, f(c))$ to $(x, f(x))$, we also have that $g(x) \leq f(x)$. Consequently,

$$g(x) \leq f(x) \leq h(x), \quad x \in (x_0, b).$$

Proof:

Let $x_0 \in I =: (a, b)$. By symmetry, it suffices to show that $f(x) \to f(x_0)$ as $x \to x_0+$. Let $a < c < x_0 < x < d < b$, $y = g(x)$ represent the euqation of the chord through $(x_0, f(x_0)), (d, f(d))$. Since f is convex, we have by Remark 5.59 that $f(x) \leq h(x)$. Since $f(x_0)$ lies on or below the chord from $(c, f(c))$ to $(x, f(x))$, we also have that $g(x) \leq f(x)$. Consequently,

$$g(x) \leq f(x) \leq h(x), \quad x \in (x_0, b).$$

Proof:

Let $x_0 \in I =: (a, b)$. By symmetry, it suffices to show that $f(x) \to f(x_0)$ as $x \to x_0+$. Let $a < c < x_0 < x < d < b$, $y = g(x)$ represent the euqation of the chord through $(x_0, f(x_0))$, $(d, f(d))$. Since f is convex, we have by Remark 5.59 that $f(x) \leq h(x)$. Since $f(x_0)$ lies on or below the chord from $(c, f(c))$ to $(x, f(x))$, we also have that $g(x) \leq f(x)$. Consequently,

$$g(x) \leq f(x) \leq h(x), \quad x \in (x_0, b).$$

Proof:

Let $x_0 \in I =: (a, b)$. By symmetry, it suffices to show that $f(x) \to f(x_0)$ as $x \to x_0+$. Let $a < c < x_0 < x < d < b$, $y = g(x)$ represent the euqation of the chord through $(x_0, f(x_0)), (d, f(d))$. Since f is convex, we have by Remark 5.59 that $f(x) \leq h(x)$. Since $f(x_0)$ lies on or below the chord from $(c, f(c))$ to $(x, f(x))$, we also have that $g(x) \leq f(x)$. Consequently,

$$g(x) \leq f(x) \leq h(x), \quad x \in (x_0, b).$$

Both chords $y = g(x)$ and $y = h(x)$ pass through the point $(x_0, f(x_0))$, so $g(x) \to f(x_0)$ and $h(x) \to f(x_0)$ as $x \to x_0+$. Hence, it follows from the Squeeze Theorem that $f(x) \to f(x_0)$ as $x \to x_0+$ $\square$

Both chords $y = g(x)$ and $y = h(x)$ pass through the point $(x_0, f(x_0))$, so $g(x) \to f(x_0)$ and $h(x) \to f(x_0)$ as $x \to x_0+$. Hence, it follows from the Squeeze Theorem that $f(x) \to f(x_0)$ as $x \to x_0+$ $\square$

Both chords $y = g(x)$ and $y = h(x)$ pass through the point $(x_0, f(x_0))$, so $g(x) \to f(x_0)$ and $h(x) \to f(x_0)$ as $x \to x_0+$. Hence, it follows from the Squeeze Theorem that $f(x) \to f(x_0)$ as $x \to x_0+$ $\Box$

Both chords $y = g(x)$ and $y = h(x)$ pass through the point $(x_0, f(x_0))$, so $g(x) \to f(x_0)$ and $h(x) \to f(x_0)$ as $x \to x_0+$. Hence, it follows from the Squeeze Theorem that $f(x) \to f(x_0)$ as $x \to x_0+$ $\square$

Both chords $y = g(x)$ and $y = h(x)$ pass through the point $(x_0, f(x_0))$, so $g(x) \to f(x_0)$ and $h(x) \to f(x_0)$ as $x \to x_0+$. Hence, it follows from the Squeeze Theorem that $f(x) \to f(x_0)$ as $x \to x_0+$ $\square$

Theorem (Jensen's Inequality)

Let ϕ be convex on a closed interval [a,b] and $f : [0,1] \to [a,b]$. If f and $\phi \circ f$ are integrable on [0,1], then

$$\phi\left(\int_0^1 f(x)dx\right) \le \left(\int_0^1 (\phi \circ f)(x)dx.\right)$$

Theorem (Jensen's Inequality)

Let ϕ be convex on a closed interval [a,b] and $f : [0,1] \to [a,b]$. If f and $\phi \circ f$ are integrable on [0,1], then

$$\phi\left(\int_0^1 f(x)dx\right) \leq \left(\int_0^1 (\phi \circ f)(x)dx.\right)$$

Proof:

Set

$$c = \int_0^1 f(x)dx$$

and observe that

$$(21) \qquad \phi\left(\int_0^1 f(x)dx\right) = \phi(c) + s\left(\int_0^1 f(x)dx - c\right)$$

for all $s \in \mathbf{R}$. (Note: Since $a \leq f(x) \leq b$ for each $x \in [0,1]$, c must belong to the interval [a,b] by the Comparison Theorem for Integrals. Thus $\phi(c)$ is defined.)

Proof:

Set

$$c = \int_0^1 f(x)dx$$

and observe that

$$\phi\left(\int_0^1 f(x)dx\right) = \phi(c) + s\left(\int_0^1 f(x)dx - c\right) \tag{21}$$

for all $s \in \mathbf{R}$. (Note: Since $a \leq f(x) \leq b$ for each $x \in [0, 1]$, c must belong to the interval [a,b] by the Comparison Theorem for Integrals. Thus $\phi(c)$ is defined.)

Proof:

Set

$$c = \int_0^1 f(x)dx$$

and observe that

$$\phi\left(\int_0^1 f(x)dx\right) = \phi(c) + s\left(\int_0^1 f(x)dx - c\right) \tag{21}$$

for all $s \in \mathbf{R}$. (Note: Since $a \leq f(x) \leq b$ for each $x \in [0,1]$, c must belong to the interval [a,b] by the Comparison Theorem for Integrals. Thus $\phi(c)$ is defined.)

Proof:

Set

$$c = \int_0^1 f(x)dx$$

and observe that

$$\phi\left(\int_0^1 f(x)dx\right) = \phi(c) + s\left(\int_0^1 f(x)dx - c\right) \tag{21}$$

for all $s \in \mathbf{R}$. (Note: Since $a \leq f(x) \leq b$ for each $x \in [0, 1]$, c must belong to the interval [a,b] by the Comparison Theorem for Integrals. Thus $\phi(c)$ is defined.)

Proof:

Set

$$c = \int_0^1 f(x)dx$$

and observe that

$$\phi\left(\int_0^1 f(x)dx\right) = \phi(c) + s\left(\int_0^1 f(x)dx - c\right) \tag{21}$$

for all $s \in \mathbf{R}$. (Note: Since $a \leq f(x) \leq b$ for each $x \in [0,1]$, c must belong to the interval [a,b] by the Comparison Theorem for Integrals. Thus $\phi(c)$ is defined.)

Proof:

Set

$$c = \int_0^1 f(x)dx$$

and observe that

$$\phi\left(\int_0^1 f(x)dx\right) = \phi(c) + s\left(\int_0^1 f(x)dx - c\right) \tag{21}$$

for all $s \in \mathbf{R}$. (Note: Since $a \leq f(x) \leq b$ for each $x \in [0, 1]$, c must belong to the interval [a,b] by the Comparison Theorem for Integrals. Thus $\phi(c)$ is defined.)

Let

$$s = \sup_{x \in [a,c)} \frac{\phi(c) - \phi(x)}{c - x}.$$

By Remark 5.60, $s \le (\phi(u) - \phi(c))/(u - c)$ for all $u \in (c, b]$; i.e.,

$$\phi(c) + s(u - c) \le \phi(u) \tag{22}$$

for all $u \in [c, b]$. On the other hand, if $u \in [a, c)$, we have by the definition of s that

$$s \le \frac{\phi(c) - \phi(u)}{c - u}.$$

Let

$$s = \sup_{x \in [a,c)} \frac{\phi(c) - \phi(x)}{c - x}.$$

By Remark 5.60, $s \leq (\phi(u) - \phi(c))/(u - c)$ for all $u \in (c, b]$; i.e.,

$$\phi(c) + s(u - c) \leq \phi(u) \tag{22}$$

for all $u \in [c, b]$. On the other hand, if $u \in [a, c)$, we have by the definition of s that

$$s \leq \frac{\phi(c) - \phi(u)}{c - u}.$$

Let

$$s = \sup_{x \in [a,c)} \frac{\phi(c) - \phi(x)}{c - x}.$$

By Remark 5.60, $s \leq (\phi(u) - \phi(c))/(u - c)$ for all $u \in (c, b]$; i.e.,

$$\phi(c) + s(u - c) \leq \phi(u) \tag{22}$$

for all $u \in [c, b]$. On the other hand, if $u \in [a, c)$, we have by the definition of s that

$$s \leq \frac{\phi(c) - \phi(u)}{c - u}.$$

Let

$$s = \sup_{x \in [a,c)} \frac{\phi(c) - \phi(x)}{c - x}.$$

By Remark 5.60, $s \leq (\phi(u) - \phi(c))/(u - c)$ for all $u \in (c, b]$; i.e.,

$$\phi(c) + s(u - c) \leq \phi(u) \tag{22}$$

for all $u \in [c, b]$. On the other hand, if $u \in [a, c)$, we have by the definition of s that

$$s \leq \frac{\phi(c) - \phi(u)}{c - u}.$$

Let

$$s = \sup_{x \in [a,c)} \frac{\phi(c) - \phi(x)}{c - x}.$$

By Remark 5.60, $s \le (\phi(u) - \phi(c))/(u - c)$ for all $u \in (c, b]$; i.e.,

$$\phi(c) + s(u - c) \le \phi(u) \tag{22}$$

for all $u \in [c, b]$. On the other hand, if $u \in [a, c)$, we have by the definition of s that

$$s \le \frac{\phi(c) - \phi(u)}{c - u}.$$

Let

$$s = \sup_{x \in [a,c)} \frac{\phi(c) - \phi(x)}{c - x}.$$

By Remark 5.60, $s \leq (\phi(u) - \phi(c))/(u - c)$ for all $u \in (c, b]$; i.e.,

$$\phi(c) + s(u - c) \leq \phi(u) \tag{22}$$

for all $u \in [c, b]$. On the other hand, if $u \in [a, c)$, we have by the definition of s that

$$s \leq \frac{\phi(c) - \phi(u)}{c - u}.$$

Let

$$s = \sup_{x \in [a,c)} \frac{\phi(c) - \phi(x)}{c - x}.$$

By Remark 5.60, $s \le (\phi(u) - \phi(c))/(u - c)$ for all $u \in (c, b]$; i.e.,

$$\phi(c) + s(u - c) \le \phi(u) \tag{22}$$

for all $u \in [c, b]$. On the other hand, if $u \in [a, c)$, we have by the definition of s that

$$s \le \frac{\phi(c) - \phi(u)}{c - u}.$$

Let

$$s = \sup_{x \in [a,c)} \frac{\phi(c) - \phi(x)}{c - x}.$$

By Remark 5.60, $s \le (\phi(u) - \phi(c))/(u - c)$ for all $u \in (c, b]$; i.e.,

$$\phi(c) + s(u - c) \le \phi(u) \tag{22}$$

for all $u \in [c, b]$. On the other hand, if $u \in [a, c)$, we have by the definition of s that

$$s \le \frac{\phi(c) - \phi(u)}{c - u}.$$

Thus (22) holds for all $u \in [a,b]$. Applying (22) to $u = f(x)$, we obtain

$$\phi(c) + s(f(x) - c) \le (\phi \circ f)(x).$$

Integrating this inequality as x runs from 0 to 1, we obtain

$$\phi(c) + s\left(\int_0^1 f(x)dx - c\right) \le \int_0^1 (\phi \circ f)(x)dx.$$

Combining this inequality with (21), we conclude that (20) holds. $\square$

Thus (22) holds for all $u \in [a, b]$. Applying (22) to $u = f(x)$, we obtain

$$\phi(c) + s(f(x) - c) \leq (\phi \circ f)(x).$$

Integrating this inequality as x runs from 0 to 1, we obtain

$$\phi(c) + s\left(\int_0^1 f(x)dx - c\right) \leq \int_0^1 (\phi \circ f)(x)dx.$$

Combining this inequality with (21), we conclude that (20) holds. $\square$

Thus (22) holds for all $u \in [a, b]$. Applying (22) to $u = f(x)$, we obtain

$$\phi(c) + s(f(x) - c) \le (\phi \circ f)(x).$$

Integrating this inequality as x runs from 0 to 1, we obtain

$$\phi(c) + s\left(\int_0^1 f(x)dx - c\right) \le \int_0^1 (\phi \circ f)(x)dx.$$

Combining this inequality with (21), we conclude that (20) holds. □

Thus (22) holds for all $u \in [a, b]$. Applying (22) to $u = f(x)$, we obtain

$$\phi(c) + s(f(x) - c) \leq (\phi \circ f)(x).$$

Integrating this inequality as x runs from 0 to 1, we obtain

$$\phi(c) + s\left(\int_0^1 f(x)dx - c\right) \leq \int_0^1 (\phi \circ f)(x)dx.$$

Combining this inequality with (21), we conclude that (20) holds. □

Thus (22) holds for all $u \in [a, b]$. Applying (22) to $u = f(x)$, we obtain

$$\phi(c) + s(f(x) - c) \leq (\phi \circ f)(x).$$

Integrating this inequality as x runs from 0 to 1, we obtain

$$\phi(c) + s\left(\int_0^1 f(x)dx - c\right) \leq \int_0^1 (\phi \circ f)(x)dx.$$

Combining this inequality with (21), we conclude that (20) holds. □

Thus (22) holds for all $u \in [a, b]$. Applying (22) to $u = f(x)$, we obtain

$$\phi(c) + s(f(x) - c) \leq (\phi \circ f)(x).$$

Integrating this inequality as x runs from 0 to 1, we obtain

$$\phi(c) + s\left(\int_0^1 f(x)dx - c\right) \leq \int_0^1 (\phi \circ f)(x)dx.$$

Combining this inequality with (21), we conclude that (20) holds. $\square$

Thus (22) holds for all $u \in [a, b]$. Applying (22) to $u = f(x)$, we obtain

$$\phi(c) + s(f(x) - c) \leq (\phi \circ f)(x).$$

Integrating this inequality as x runs from 0 to 1, we obtain

$$\phi(c) + s\left(\int_0^1 f(x)dx - c\right) \leq \int_0^1 (\phi \circ f)(x)dx.$$

Combining this inequality with (21), we conclude that (20) holds. □

Thus (22) holds for all $u \in [a, b]$. Applying (22) to $u = f(x)$, we obtain

$$\phi(c) + s(f(x) - c) \leq (\phi \circ f)(x).$$

Integrating this inequality as x runs from 0 to 1, we obtain

$$\phi(c) + s\left(\int_0^1 f(x)dx - c\right) \leq \int_0^1 (\phi \circ f)(x)dx.$$

Combining this inequality with (21), we conclude that (20) holds. $\square$

Thank you.