Advanced Calculus (I)

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6.2 Series with nonnegative terms

Theorem

Suppose that $a_k \geq 0$ *for* $k \geq N$*. Then* $\sum_{k=1}^{\infty} a_k$ *converges if and only if its sequence of partial sums* $\{s_n\}$ *is bounded, i.e., if and only if there exists a finite number* $M > 0$ *such that*

$$\left| \sum_{k=1}^{n} a_k \right| \leq M \quad \text{for} \quad \text{all} \quad n \in \mathbf{N}.$$

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Proof:

Set $s_n = \sum_{k=1}^{n} a_k$ for $n \in \mathbf{N}$. If $\sum_{k=1}^{\infty} a_k$ converges, then s_n converges as $n \to \infty$. Since every convergent sequence is bounded (Theorem 2.8), $\sum_{k=1}^{\infty} a_k$ has bounded partial sums.

Conversely, suppose that $|s_n| \leq M$ for $n \in \mathbf{N}$. Since $a_k \geq 0$ for $k \geq N$, s_n is an increasing sequence when $n \geq N$. Hence by the Monotone Convergence Theorem (Theorem 2.19), s_n converges. □

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Theorem (Integral Test)

Suppose that $f : [1, \infty) \to \mathbf{R}$ is positive and decreasing on $[1, \infty)$. Then $\sum_{k=1}^{\infty} f(k)$ converges if and only if f is improperly integrable on $[1, \infty)$, i.e., if and only if

$$\int_1^{\infty} f(x)dx < \infty.$$

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Corollary: [p-Series Test]

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If $p = 1$ or $p \leq 0$, the series diverges. If $p > 0$ and $p \neq 1$, set $f(x) = x^{-p}$ and observe that $f'(x) = -px^{-p-1} < 0$ for all $x \in [1, \infty)$. Hence, f is nonnegative and decreasing on $[1, \infty)$. Since,

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{x \to \infty} \frac{x^{1-p}}{1-p}\bigg|_1^n = \lim_{n \to \infty} \frac{n^{1-p} - 1}{1-p}$$

has a finite limit if and only if $1 - p < 0$, it follows from the Integral Test that (4) converges if and only if $p > 1$. $\square$

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Theorem (Comparison Test)

Suppose that $0 \leq a_k \leq b_k$ *for large k.*

(i)If $\sum_{k=1}^{\infty} b_k < \infty$*, then* $\sum_{k=1}^{\infty} a_k < \infty$.

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Determine whether the following series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{3k}{k^2+k}\sqrt{\frac{\log k}{k}}$$

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Theorem (Limit Comparison Test)

Suppose that $a_k \geq 0$ and $b_k > 0$ for large k and $L := \lim_{n \to \infty} \frac{a_n}{b_n}$ exists as an extended real number.

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