Advanced Calculus (I)

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Definition

Let $S = \sum_{k=1}^{\infty} a_k$ be an infinite series.

(i)S is said to *converge absolutely* if and only if $\sum_{k=1}^{\infty} |a_k| < \infty$.

(ii)S is said to converge conditionally if and only if S converges but not absolutely.

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Proof:

Suppose that $\sum_{k=1}^{\infty} a_k$ converges absolutely. Given $\epsilon > 0$, choose $N \in \mathbf{N}$ so that (6) holds. Then

$$\left| \sum_{k=n}^{m} a_k \right| \le \sum_{k=n}^{m} |a_k| < \epsilon$$

for $M > n \geq N$. Hence, by the Cauchy Criterion, $\sum_{k=1}^{\infty} a_k$ converges.

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We shall finish the proof by showing that $S := \sum_{k=1}^{\infty} (-1)^k/k$ converges conditionally. Since the harmonic series diverges, S does not converge absolutely. On the other hand, the tails of S look like

$$\sum_{j=k}^{\infty} \frac{(-1)^j}{j} = (-1)^k \left(\frac{1}{k} - \frac{1}{k+1} + \frac{1}{k+2} - \frac{1}{k+3} + \cdots \right).$$

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By grouping pairs of terms together,
it is easy to see that the sum inside the parentheses is greater than 0 but less than $1/k$,
i.e.,

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Hence $\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$ converges by Corlllary 6.9. □

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Let $x \in \mathbf{R}$ and $\{x_k\}$ be a real sequence.

(i)If $\limsup_{k\to\infty} x_k < x$, then $x_k < x$ for large k.

(ii)If $\limsup_{k\to\infty} x_k > x$, then $x_k > x$ for infinitely many k.

(iii)If $x_k \to x$ as $k \to \infty$, then $\limsup_{k\to\infty} x_k = x$.

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Proof:

(i)

Let $s := \limsup_{k \to \infty} x_k < x$ but suppose to the contrary that there exist natural numbers $k_1 < k_2 < \cdots$ such that $x_{k_j} \geq x$ for $j \in \mathbf{N}$. If $\{x_k\}$ is unbounded above, then ∞ is an adherent point of $\{x_k\}$ so $s = \infty$, a contradiction.

If $\{x_{k_j}\}$ is bounded above by (by C), then it is bounded (since $x \leq x_k \leq C$ for all $j \in \mathbf{N}$). Hence, by the Bolzano-Weierstrass Theorem and the fact that $x_{k_j} \geq x$, $\{x_k\}$ has an adherent point $\geq x$, i.e., $s \geq x$, another contradiction.

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If $s > x$, then choose an adherent point $a \in (x, s)$. By the Approximation Propert there is a subsequence $\{x_{k_j}\}$ which converges to a, i.e., $x_{k_j} > x$ for large j.

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If x_k converges to x, then any subsequence x_{k_j} also converges to x (see Theorem 2.6). □

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Theorem (Root Test)

Let $a_k \in \mathbf{R}$ *and* $r := \limsup_{k\to\infty} |a_k|^{1/k}$.

(i) If $r < 1$*, then* $\sum_{k=1}^{\infty} a_k$ *converges absolutely.*

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Let $a_k \in \mathbf{R}$ with $a_k \neq 0$ for large k and suppose that

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A series $\sum_{j=1}^{\infty} b_j$ is called a *rearrangment* of a series $\sum_{k=1}^{\infty} a_k$ if and only if there is a 1-1 function f from N onto N such that

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Proof:

Let $\epsilon > 0$. Set $S_n = \sum_{k=1}^{n} a_k$, $s = \sum_{k=1}^{\infty} a_k$, and $t_m = \sum_{j=1}^{m} b_j$, $n, m \in \mathbf{N}$. Since $\sum_{k=1}^{\infty} a_k$ converges absolutely, we can choose $N \in \mathbf{N}$ (see Corollary 6.9) such that

$$\sum_{k=N+1}^{\infty} |a_k| \leq \frac{\epsilon}{2}. \tag{7}$$

Thus

$$|s_N - s| = \left| \sum_{k=N+1}^{\infty} a_k \right| \leq \sum_{k=N+1}^{\infty} |a_k| < \frac{\epsilon}{2} \tag{8}$$

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Let f be a 1-1 function from **N** onto **N** that satisfies

$$b_{f(k)} = a_k, \quad k \in \mathbf{N}$$

and set $M = \max\{f(1), \cdots, f(N)\}$. Notice that

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Let $m \geq M$. Then $t_m - s_N$ contains only $a_k's$ whose indices satisfy $k > N$. Thus, it follows from (7) that

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Theorem

Suppose that $a_k \in \mathbf{R}$ for $k \in \mathbf{N}$.

(i) If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then so do $\sum_{k=1}^{\infty} a_k^+$ and $\sum_{k=1}^{\infty} a_k^-$. In fact,

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} a_k^+ + \sum_{k=1}^{\infty} a_k^- \text{ and } \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_k^+ - \sum_{k=1}^{\infty} a_k^-$$

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