Advanced Calculus (I)

WEN-CHING LIEN

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Definition

Let f_k be a sequence of real functions defined on some set E and set

$$s_n(x) := \sum_{k=1}^n f_k(x), \quad x \in E, \quad n \in \mathbb{N}.$$

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The series $\sum_{k=1}^{\infty} f_k$ is said to *converges pointwise* on E if and only if the sequence $s_n(x)$ converges pointwise on E as $n \to \infty$.

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The series $\sum_{k=1}^{\infty} f_k$ is said to *converge uniformly* on E if and only if the sequence $s_n(x)$ converges uniformly on E as $n \to \infty$.

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The series $\sum_{k=1}^{\infty} f_k$ is said to *converge absolutely (pointwise)* on E if and only if $\sum_{k=1}^{\infty} |f_k(x)|$ converges for each $x \in E$.

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Let *E* be a nonempty subset of **R** and let $\{f_k\}$ be a sequence of real functions defined on *E*. (*i*) Suppose that $x_0 \in E$ and that each f_k is continuous at $x_0 \in E$. If $f = \sum_{k=1}^{\infty} f_k$ converges uniformly on *E*, then *f* is continuous at $x_0 \in E$.

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$$\int_a^b \sum_{k=1}^\infty f_k(x) dx = \sum_{k=1}^\infty \int_a^b f_k(x) dx.$$

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(iii)[Term-by-term differentation] Suppose that E is bounded, open interval and that each f_k is differentiable on E. If $\sum_{k=1}^{\infty} f_k$ converges at some $x_0 \in E$, and $\sum_{k=1}^{\infty} f'_k$ converges uniformly on E, then $f := \sum_{k=1}^{\infty} f_k$ converges uniformly on E, f is differentiable on E, and $\left(\sum_{k=1}^{\infty} f_k(x)\right)' = \sum_{k=1}^{\infty} f'_k(x)$

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Theorem (Weierstrass M-test)

Let *E* be a nonempty subset of **R**, let $f_k : E \to R$, $k \in \mathbf{N}$, and let $M_k \ge 0$ satisfy $\sum_{k=1}^{\infty} M_k < \infty$. If $|f_k(x)| \le M_k$ for $k \in \mathbf{N}$ and $x \in E$, then $\sum_{k=1}^{\infty} f_k$ converges absolutely and uniformly on *E*.

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Let *E* be a nonempty subset of **R** and suppose that $f_k, g_k : E \to R, k \in \mathbf{N}$. If

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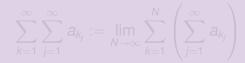
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 converges for each $k \in \mathbf{N}$

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Let $a_{k_i} \in \mathbf{R}$ for $k, j \in \mathbf{N}$ and suppose that

$$A_j = \sum_{k=1}^{\infty} |a_{k_j}| < \infty$$

for each $j \in \mathbf{N}$. If $\sum_{j=1}^{\infty}$ converges (i.e., the double sum converges absolutely), then

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Let $E = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$. For each $j \in \mathbb{N}$, define a function f_j on E by

$$f_j(0) = \sum_{k=1}^{\infty} a_{k_j}, \quad f_j\left(\frac{1}{n}\right) = \sum_{k=1}^{\infty} a_{k_j}, \quad n \in \mathbb{N}.$$

By hypothesis, $f_j(0)$ exists and by the definition of series convergence,

$$\lim_{n\to\infty}f_j\left(\frac{1}{n}\right)=f_j(0);$$

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i.e., f_j is continuous at $0 \in E$ for each $j \in \mathbb{N}$. Moreover, since $|f_j(x)| \le A_j$ for all $x \in E$ and $j \in \mathbb{N}$, The Weierstrass M-Test implies that



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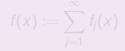


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$$f(x) := \sum_{j=1}^{\infty} f_j(x)$$

converges uniformly on E.

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{k_j} = \lim_{n \to \infty} \sum_{k=1}^{n} \sum_{j=1}^{\infty} a_{k_j}$$
$$= \lim_{n \to \infty} \sum_{j=1}^{\infty} \sum_{k=1}^{n} a_{k_j}$$
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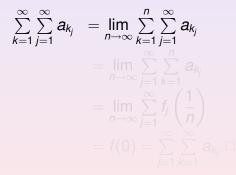
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Thank you.

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