Advanced Calculus (I)

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Definition

Let f_k be a sequence of real functions defined on some set E and set

$$s_n(x) := \sum_{k=1}^{n} f_k(x), \quad x \in E, \quad n \in \mathbf{N}.$$

(i)

The series $\sum_{k=1}^{\infty} f_k$ is said to *converges pointwise* on E if and only if the sequence $s_n(x)$ converges pointwise on E as $n \to \infty$.

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The series $\sum_{k=1}^{\infty} f_k$ is said to *converge absolutely (pointwise)* on E if and only if $\sum_{k=1}^{\infty} |f_k(x)|$ converges for each $x \in E$.

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Theorem

Let E be a nonempty subset of $\mathbf{R}$ and let $\{f_k\}$ be a sequence of real functions defined on E.

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Suppose that $x_0 \in E$ and that each f_k is continuous at $x_0 \in E$. If $f = \sum_{k=1}^{\infty} f_k$ converges uniformly on E, then f is continuous at $x_0 \in E$.

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Theorem

(ii)[Term-by-term integration]
Suppose that $E = [a, b]$ and that each f_k is integrable on [a,b]. If $f = \sum_{k=1}^{\infty} f_k$ converges uniformly on [a,b], then f is integrable on [a,b] and

$$\int_a^b \sum_{k=1}^{\infty} f_k(x)dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)dx.$$

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(iii)[Term-by-term differentation]
Suppose that E is bounded, open interval and that each f_k is differentiable on E. If $\sum_{k=1}^{\infty} f_k$ converges at some $x_0 \in E$, and $\sum_{k=1}^{\infty} f'_k$ converges uniformly on E, then $f := \sum_{k=1}^{\infty} f_k$ converges uniformly on E, f is differentiable on E, and

$$\left(\sum_{k=1}^{\infty} f_k(x)\right)' = \sum_{k=1}^{\infty} f'_k(x)$$

for $x \in E$.

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Theorem (Weierstrass M-test)

Let E be a nonempty subset of $\mathbf{R}$, let $f_k : E \to R$, $k \in \mathbf{N}$, and let $M_k \geq 0$ satisfy $\sum_{k=1}^{\infty} M_k < \infty$. If $|f_k(x)| \leq M_k$ for $k \in \mathbf{N}$ and $x \in E$, then $\sum_{k=1}^{\infty} f_k$ converges absolutely and uniformly on E.

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Theorem (Dirichlet's Test for uniform convergence)

Let E be a nonempty subset of $\mathbf{R}$ *and suppose that* $f_k, g_k : E \to R$, $k \in \mathbf{N}$. *If*

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for $n \in \mathbf{N}$ *and* $x \in E$, *and if* $g_k \downarrow 0$ *uniformly on E as* $k \to \infty$, *then* $\sum_{k=1}^{\infty} f_k g_k$ *converges uniformly on E.*

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Definition

A double series is convergent if and only if

$$\sum_{j=1}^{\infty} a_{k_j} \quad converges \quad for \quad each \quad k \in \mathbf{N}$$

and

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{k_j} := \lim_{N\to\infty}\sum_{k=1}^{N}\left(\sum_{j=1}^{\infty} a_{k_j}\right)$$

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Let $a_{k_j} \in \mathbf{R}$ for $k, j \in \mathbf{N}$ and suppose that

$$A_j = \sum_{k=1}^{\infty} |a_{k_j}| < \infty$$

for each $j \in \mathbf{N}$. If $\sum_{j=1}^{\infty}$ converges (i.e., the double sum converges absolutely), then

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{k_j} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{k_j}.$$

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Proof:

Let $E = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\}$. For each $j \in \mathbf{N}$, define a function f_j on E by

$$f_j(0) = \sum_{k=1}^{\infty} a_{k_j}, \quad f_j\left(\frac{1}{n}\right) = \sum_{k=1}^{\infty} a_{k_j}, \quad n \in \mathbf{N}.$$

By hypothesis, $f_j(0)$ exists and by the definition of series convergence,

$$\lim_{n\to\infty} f_j\left(\frac{1}{n}\right) = f_j(0);$$

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i.e., f_j is continuous at $0 \in E$ for each $j \in \mathbf{N}$. Moreover, since $|f_j(x)| \leq A_j$ for all $x \in E$ and $j \in \mathbf{N}$, The Weierstrass M-Test implies that

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Thus f is continuous at $0 \in E$ by theorem 7.9. It follows from the sequential characteristization of continuity (Theorem 3.21) that $f(\frac{1}{n}) \to f(0)$ as $n \to \infty$. Therefore,

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