Advanced Calculus (I)

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7.3 Power series

A power series is a series of the form

$$S(x) := \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

Definition

An extended real number R is said to be the *radius of convergence* of a power series $S(x) := \sum_{k=0}^{\infty} a_k(x - x_0)^k$ if and only if $S(x)$ converges absolutely for $|x - x_0| < R$ and $S(x)$ diverges for $|x - x_0| > R$.

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Theorem

Let $S(x) = \sum_{k=0}^{\infty} a_k(x - x_0)^k$ *be a power series centered at* x_0*, If* $R = \dfrac{1}{\limsup_{k\to\infty} |a_k|^{1/k}}$*, with the convention that* $\dfrac{1}{\infty} = 0$ *and* $\dfrac{1}{0} = \infty$*, then R is the radius of convergence of S.*

In fact,

(i) S(x) converges absolutely for each $x \in (x_0 - R, x_0 + R)$*,*

(ii) S(x) converges uniformly on any closed interval $[a, b] \subseteq (x_0 - R, x_0 + R)$*,*

(iii) and (when R is finite), S(x) diverges for each $x \notin [x_0 - R, x_0 + R]$*.*

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Proof:

Fix $x \in \mathbf{R}$, $x \neq x_0$, and set $\rho := \dfrac{1}{\limsup_{k\to\infty} |a_k|^{1/k}}$, with the convention that $\dfrac{1}{\infty} = 0$ and $\dfrac{1}{0} = \infty$. To apply the Root Test to $S(x)$, consider

$$r(x) := \limsup_{n\to\infty} |a_k(x - x_0)^k|^{1/k} = |x - x_0| \cdot \limsup_{k\to\infty} |a_k|^{1/k}.$$

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Case 1:

$\rho = 0$. By our convention, $\rho = 0$ implies that $r(x) = \infty > 1$, so by the Root Test, S(x) does not converge for any $x \neq x_0$. Hence, the radius of convergence of S is $R = 0 = \rho$.

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Case 2:

$\rho = \infty$. Then $r(x) = 0 < 1$, so by the Root Test, S(x) converges absolutely for all $x \in \mathbf{R}$. Hence, the radius of convergence of S is $R = \infty = \rho$.

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Case 3:

$\rho \in (0, \infty)$. Then $r(x) = \dfrac{|x - x_0|}{\rho}$. Since $r(x) < 1$ if and only if $|x - x_0| < \rho$, it follows from the Root Test that $S(x)$ converges absolutely when $x \in (x_0 - \rho, x_0 + \rho)$. Similarly, since $r(x) > 1$ if and only if $|x - x_0| > \rho$, we also have that S(x) diverges when $x \notin [x_0 - \rho, x_0 + \rho]$. This proves that ρ is the radius of convergence of S, and that parts (i) and (iii) hold.

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If the limit

$$R = \lim_{k\to\infty} \frac{a_k}{a_{k+1}}$$

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Definition

The *interval of convergence* of a power series S(x) is the largest interval on which S(x) converges.

By Theorem 7.21, for a given power series $S = \sum_{k=0}^{\infty} a_k(x - x_0)^k$, there are only three possibilities:

(i) $R = \infty$, in which case the interval of convergence of S is $(-\infty, \infty)$,

(ii) $R = 0$, in which case the interval of convergence of S is $\{x_0\}$, and

(iii) $0 < R < \infty$, in which case the interval of convergence of S is $(x_0 - R, x_0 + R)$, $[x_0 - R, x_0 + R)$, $(x_0 - R, x_0 + R]$, or $[x_0 - R, x_0 + R]$.

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If a power series $S(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$ converges at some $x_1 > x_0$, then S(x) converges uniformly on $[x_0, x_1]$ and absolutely on $[x_0, x_1)$. It might not converge absolutely at $x = x_1$.

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Lemma:

If $a_n \in \mathbf{R}$ for $n \in \mathbf{N}$, then

$$x := \limsup_{n\to\infty}(n|a_n|^{1/(n-1)}) = y := \limsup_{n\to\infty}|a_n|^{1/n}$$

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Proof:

Let $\epsilon > 0$. Since $n^{1/(n-1)} \to 1$ as $n \to \infty$, choose $N \in \mathbf{N}$ so that $n \geq N$ implies $1 - \epsilon < n^{1/(n-1)} < 1 + \epsilon$, i.e.,

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Suppose that z is an adherent point of $c_n := |a_n|^{1/n}$, i.e., $z_k := |a_{n_k}|^{1/n_k} \to z$ as $k \to \infty$. If $n_k > n \geq N$, then $|a_{n_k}|^{1/(n_k-1)} < z_k^{n/(n-1)} \to z^{n/(n-1)}$ as $k \to \infty$. Since y is the supremum of the set of adherent points of $|a_n|^{1/n}$, it follows from the right-most inequality above that $x \leq (1+\epsilon)y^{n/(n-1)}$. Let $n \to \infty$ and then let $\epsilon \to 0$. We obtain $x \leq y$. Similarly, the left-most inequality above implies $y \leq x$. $\square$

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If $f(x) = \sum_{k=0}^{\infty} a_k(x - x_0)^k$ *is a power series with positive radius of convergence* R*, then* $f'(x) = \sum_{k=1}^{\infty} k a_k(x - x_0)^{k-1}$ *for* $x \in (x_0 - R, x_0 + R)$*.*

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Let $f(x) = \sum_{k=0}^{\infty} a_k(x - x_0)^k$ be a power series and let $a, b \in \mathbf{R}$ with $a < b$.

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Let $a < t < b$ and set $A = \sum_{k=0}^{\infty} \frac{a_k(a - X_0)^{k+1}}{(k+1)}$. By part (i),

$$\int_a^t f(x)dx = \sum_{k=0}^{\infty} a_k \int_a^t (x-x_0)^k dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(t-x_0)^{k+1} - A.$$

The leftmost term of this last difference is a power series which by hypothesis converges at $t = b$.

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Thus, by the definition of improper integration and Abel's Theorem,

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&\int_a^b f(x)dx \\
&= \lim_{t\to b-}\int_a^t f(x)dx \\
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Theorem

If $f(x) = \sum_{k=0}^{\infty} a_k x^k$ *and* $g(x) = \sum_{k=0}^{\infty} b_k x^k$ *converge on* $(-r, r)$ *and*

$$c_k = \sum_{j=0}^{k} a_j b_{k-j}, \quad k = 0, 1, \cdots,$$

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Corollary:

Suppose that $a_k, b_k \in \mathbf{R}$ and $c_k := \sum_{j=0}^{k} a_j b_{k-j}$ for $k = 0, 1, \cdots$, If either

(i) $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ both converge, and at least one of them converges absolutely, or

(ii) if $\sum_{k=0}^{\infty} a_k$, $\sum_{k=0}^{\infty} b_k$, and $\sum_{k=0}^{\infty} c_k$ all converge,

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$$\sum_{k=0}^{\infty} c_k x^k = \left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{k=0}^{\infty} b_k x^k\right).$$

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Thank you.