Advanced Calculus (I)

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Definition

A real-valued function is said to be (real) *analytic* on a nonempty, open interval (a,b) if and only if given $x_0 \in (a, b)$ there is a power series centered at x_0 that converges to f near x_0; i.e., there exist coefficients $\{a_k\}_{k=0}^{\infty}$ and points $c, d \in (a, b)$ such that $c < x_0 < d$ and

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

for all $x \in (c, d)$

7.4 Analytic function

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Theorem (Uniqueness)

Let c,d be extended real numbers with $c < d$, let $x_0 \in (c, d)$ and suppose that $f : (c, d) \to \mathbf{R}$. If $f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$ for each $x \in (c, d)$, then $f \in \mathcal{C}^{\infty}(c, d)$ and

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Proof:

Clearly, $f(x_0) = a_0$. Fix $k \in \mathbf{N}$. By hypothesis, the radius of convergence R of the power series $\sum_{k=0}^{\infty} a_k(x - x_0)^k$ is positive and $(c, d) \subseteq (x_0 - R, x_0 + R)$. Hence, by Corollary 7.31, $f \in \mathcal{C}^{\infty}(c, d)$ and

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (x - x_0)^{n-k}$$

for $x \in (c, d)$. Apply, this to $x = x_0$. The terms on the right side of (10) are zero when $n > k$ and $k!a_k$ when $n = k$. Hence, $f^{(k)}(x_0) = k!a_k$ for each $k \in \mathbf{N}$. $\square$

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Definition

Let $f \in \mathcal{C}^\infty(a, b)$ and let $x_0 \in (a, b)$. The *Taylor expansion (or Taylor series) of f centered at x_0* is the series

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Remark: [Cauchy]

The function

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

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It is easy to see (Exercise 3, p.101) that $f \in \mathcal{C}^\infty(-\infty, \infty)$ and $f^{(k)}(0) = 0$ for all $k \in \mathbf{N}$. Thus the Taylor expansion of f about the point $x_0 = 0$ is identically zero, but $f(x) = 0$ only when $x = 0$. □

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Let $f \in C^\infty(a, b)$ and $x_0 \in (a, b)$. The *remainder term of order n* of the Taylor expansion of f centered at x_0 is the function

$$R_n(x) = R_n^{f,x_0}(x) := f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

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Theorem (Taylor's Formula)

Let $n \in \mathbf{N}$, let a,b be distinct extended real numbers, let $f : (a, b) \to \mathbf{R}$, and suppose that $f^{(n)}$ exists on (a,b). Then for each pair of points $x, x_0 \in (a, b)$ there is a number c between x and x_0 such that

$$R_n^{f,x_0}(x) = \frac{f^{(n)}(c)}{n!}(x - x_0)^n.$$

In particular,

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n)}(c)}{n!}(x - x_0)^n$$

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Without loss of generality, suppose that $x_0 < x$. Define

$$F(t) := \frac{(x-t)^n}{n!} \ \ and \ \ G(t) := R_n^{f,t}(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(t)}{k!}(x-t)^k$$

for each $t \in (a, b)$. In order to apply the Generalized Mean Value Theorem to F and G, we need to be sure the hypotheses of that result hold.

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Notice by the Chain rule that

$$F'(t) = -\frac{(x-t)^{n-1}}{(n-1)!} \tag{11}$$

for $t \in \mathbf{R}$. On the other hand, since

$$\frac{d}{dt}\left(\frac{f^{(k)(t)}}{k!}(x-t)^k\right) = \frac{f^{(k+1)}(t)}{k!}(x-t)^k - \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1}$$

for $t \in (a, b)$. and $k \in \mathbf{N}$, we can telescope to obtain

$$G'(t) = -\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} \tag{12}$$

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Thus, F and G are differentiable on (x_0, x) and continuous on $[x_0, x]$. By the Generalized Mean Value Theorem and the fact that $F(x) = G(x) = 0$, there is a number $c \in (x_0, x)$ such that

$$\begin{aligned} -F(x_0)G'(c) &= (F(x) - F(x_0))G'(c) \\ &= (G(x) - G(x_0))F'(c) \\ &= -G(x_0)F'(c) \\ &= -G(x_0)F'(c). \end{aligned}$$

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\end{aligned}
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Thus, F and G are differentiable on (x_0, x) and continuous on $[x_0, x]$. By the Generalized Mean Value Theorem and the fact that $F(x) = G(x) = 0$, there is a number $c \in (x_0, x)$ such that

$$\begin{aligned} -F(x_0)G'(c) &= (F(x) - F(x_0))G'(c) \\ &= (G(x) - G(x_0))F'(c) \\ &= -G(x_0)F'(c) \\ &= -G(x_0)F'(c). \end{aligned}$$

Hence, it follows from (11) and (12) that

$$\frac{(x-x_0)^n}{n!}\left(\frac{f^{(n)}(c)(x-c)^{n-1}}{(n-1)!}\right) = R_n{}^{f,x_0}(x) = \frac{(x-c)^{n-1}}{(n-1)!}.$$

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Theorem

Let $f \in \mathcal{C}^{\infty}(a,b)$. If there is an $M > 0$ such that

$$|f^{(n)}(x)| \leq M^n$$

for all $x \in (a,b)$ and $n \in \mathbf{N}$, then f is analytic on (a,b). In fact, for each $x_0 \in (a,b)$,

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Example:

Prove that $\sin x$ and $\cos x$ are analytic on $\mathbf{R}$ and have Maclaurin expansions

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

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Theorem

Suppose that I is an open interval centered at c and

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k, \quad x \in I.$$

If $x_0 \in I$ and $r > 0$ satisfy $(x_0 - r, x_0 + r) \subseteq I$, then

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Proof:

It suffices to prove the first statement. By making the change of variables $w = x - c$, we may suppose that $c = 0$ and $I = (-R, R)$ i.e., that $f(x) = \sum_{k=0}^{\infty} a_k x^k$, for all $x \in (-R, R)$. Suppose that $(x_0 - r, x_0 + r) \subseteq (-R, R)$ and fix $x \in (x_0 - r, x_0 + r)$. By hypothesis and the Binomial Formula,

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} a_k x^k \\ &= \sum_{k=0}^{\infty} a_k ((x - x_0) + x_0)^k \\ &= \sum_{k=0}^{\infty} a_k \sum_{j=0}^{k} \binom{k}{j} {x_0}^{k-J} (x - x_0)^j. \end{aligned} \tag{13}$$

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Since $\sum_{k=0}^{\infty} a_k y^k$ converges absolutely at $y := |x - x_0| + |x_0| < R$, we have

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Hence, by(13), Theorem 7.18, and Corollary 7.31,

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Example:

Prove that arctan x is analytic on (-1,1) and has Maclaurin expansion

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Theorem (Lagrange)

Let $n \in \mathbf{N}$. If $f \in \mathcal{C}^n(a,b)$, then

$$R_n(x) := R_n{}^{f,x_0}(x) = \frac{1}{(n-1)!}\int_{x_0}^{x}(x-t)^{n-1}f^{(n)}(t)dt$$

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The proof is by induction on n. If $n = 1$, the formula holds by the Fundamental Theorem of Calculus.
Suppose that the formula holds for some $n \in \mathbf{N}$. Since

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$$R_{n+1}(x) = -\frac{1}{(n-1)!}\int_{x_0}^{x}(x - t)^{n-1}\left(f^{(n)}(t) - f^{(n)}(x_0)\right)dt.$$

Proof:

The proof is by induction on n. If $n = 1$, the formula holds by the Fundamental Theorem of Calculus.
Suppose that the formula holds for some $n \in \mathbf{N}$. Since

$$R_{n+1}(x) = R_n(x) - \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

and

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Let $u = f^{(n)}(t) - f^{(n)}(x_0)$, $dv = (x-t)^{n-1}$ and integrate the right side of the identity above by parts. Since $u(x_0) = 0$ and $v(x) = 0$, we have

$$R_{n+1}(x) = -\frac{1}{(n-1)!}\int_{x_0}^{x} u'(t)v(t)dt = \frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)dt.$$

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Theorem (Bernsten)

If $f \in C^{\infty}(a,b)$ and $f^{(n)}(x) \geq 0$ for all $x \in (a,b)$ and $n \in \mathbf{N}$, then f is analytic on (a,b). In fact, if $x_0 \in (a,b)$ and $f^{(n)}(x) \geq 0$ for $x \in [x_0, b)$ and $n \in \mathbf{N}$, then

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Lemma:

Suppose that f,g are analytic on an open interval (c,d) and that $x_0 \in (c, d)$. If $f(x) = g(x)$ for $x \in (c, x_0)$, then there is a $\delta > 0$, such that $f(x) = g(x)$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

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Theorem (Analysis Continuation)

Suppose that I and J are open intervals, that f is analytic on I, that g is analytic on J, and that $a < b$ are points in $I \cap J$. If $f(x) = g(x)$ for $x \in (a, b)$, then $f(x) = g(x)$ for all $x \in I \cap J$.

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Thank you.