Advanced Calculus (II)

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2009

8.2: Planes and Linear Transformations

Definition (8.12)

A function $T : \mathbf{R}^n \to \mathbf{R}^m$ is said to be *linear* (notation: $T \in \mathcal{L}(\mathbf{R}^n; \mathbf{R}^m)$) if and only if it satisfies

$$T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}) \quad \textit{and} \quad T(\alpha\mathbf{x}) = \alpha T(\mathbf{x})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and all scalars α.

Notations: $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$.

$[\mathbf{x}] = [x_1 \ x_2 \ \cdots \ x_n]$

Remark (8.13)

If $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and α is a scalar, then

$$[\mathbf{x} + \mathbf{y}] = [\mathbf{x}] + [\mathbf{y}], \quad [\mathbf{x} \cdot \mathbf{y}] = [\mathbf{x}][\mathbf{y}]^T, \quad \textit{and} \quad [\alpha \mathbf{x}] = \alpha[\mathbf{x}].$$

Remark (8.14)

Let $B = [b_{ij}]$ be an $m \times n$ matrix whose entries and real numbers, and let $\mathbf{e_1}, \ldots, \mathbf{e_n}$ represent the usual basis of $\mathbf{R}^n$. If

$$(6) \qquad T(\mathbf{x}) = B\mathbf{x}, \quad \mathbf{x} \in \mathbf{R}^n,$$

then T is a linear function from $\mathbf{R}^n$ to $\mathbf{R}^m$ and

$$(7) \qquad T(\mathbf{e_j}) = (b_{1j}, b_{2j}, \ldots, b_{mj}), \quad j = 1, 2, \ldots, n.$$

Theorem (8.15)

For each $T \in \mathcal{L}(\mathbf{R}^n; \mathbf{R}^m)$ there is a matrix $B = [b_{ij}]_{m \times n}$ such that (6) holds. Moreover, the matrix B is unique. Specifically, for each fixed T there is only one B that satisfies (6), and the entries of that B are defined by (7).

Proof.

Uniqueness has been established in Remark 8.14. To prove existence, suppose that $T \in \mathcal{L}(\mathbf{R}^n; \mathbf{R}^m)$. Define B by (7). Then

$$T(\mathbf{x}) = T\left(\sum_{j=1}^{n} x_j \mathbf{e}_j\right) = \sum_{j=1}^{n} x_j T(\mathbf{e}_j) = \sum_{j=1}^{n} x_j (b_{1j}, b_{2j}, \ldots, b_{mj})$$

$$= \left(\sum_{j=1}^{n} x_j b_{1j}, \sum_{j=1}^{n} x_j b_{2j}, \ldots, \sum_{j=1}^{n} x_j b_{mj}\right) = B\mathbf{x}.$$

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Definition (8.16)

Let $T \in \mathcal{L}(\mathbf{R}^n; \mathbf{R}^m)$. The *operator norm* of T is the extended real number

$$\|T\| := inf\{C > 0 : \|T(\mathbf{x})\| \leq C\,\|\mathbf{x}\| \;\; for\ all\ \mathbf{x} \in \mathbf{R}^n\}$$

one interesting corollary of Theorem 8.15 is that the operator norm of a linear function is always finite.

Theorem (8.17)

Let $T \in \mathcal{L}(\mathbf{R}^n; \mathbf{R}^m)$. *Then the operator norm of* T *is finite, and satisfies*

$$\|T(\mathbf{x})\| \leq \|T\| \, \|\mathbf{x}\| \tag{8}$$

for all $\mathbf{x} \in \mathbf{R}^n$.

Proof.

Let B be the $m \times n$ matrix that represents T, and suppose that the rows of T are given by $\mathbf{b}_1, \ldots, \mathbf{b}_m$. By the definition of matrix multiplication and our identification of $\mathbf{R}^m$ with $m \times 1$ matrices,

$$T(\mathbf{x}) = (\mathbf{b_1} \cdot \mathbf{x}, \ldots, \mathbf{b_m} \cdot \mathbf{x}).$$

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Proof.

If $B = O$, then $\|T\| = 0$ and (8) is an equality.If $B \neq O$, then by the Cauchy-Schwarz Inequality, the square of the Euclidean norm of $T(\mathbf{x})$ satisfies

$$\begin{aligned}\|T(\mathbf{x})\|^2 &= (\mathbf{b}_1 \cdot \mathbf{x})^2 + \ldots + (\mathbf{b}_m \cdot \mathbf{x})^2 \\ &\leq (\|\mathbf{b}_1\| \, \|\mathbf{x}\|)^2 + \ldots + (\|\mathbf{b}_m\| \, \|\mathbf{x}\|)^2 \\ &\leq m \cdot \max\{\|\mathbf{b}_j\|^2 : 1 \leq j \leq m\} \, \|\mathbf{x}\|^2 =: C \, \|\mathbf{x}\|^2\end{aligned}$$

and $C > 0$.Thus the set defining $\|T\|$ is nonempty. Since it is bounded below (by 0), it follows from the Completeness Axiom that $\|T\|$ exists and is finite.In particular, there are $C_k > 0$ such that $C_k \downarrow \|T\|$ and $\|T(\mathbf{x})\| \leq C_k \, \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbf{R}^n$. Taking the limit of this last inequality as $k \to \infty$, we obtain (8). □

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and $C > 0$.Thus the set defining $\|T\|$ is nonempty. Since it is bounded below (by 0), it follows from the Completeness Axiom that $\|T\|$ exists and is finite.In particular, there are $C_k > 0$ such that $C_k \downarrow \|T\|$ and $\|T(\mathbf{x})\| \leq C_k \, \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbf{R}^n$. Taking the limit of this last inequality as $k \to \infty$, we obtain (8). □

Proof.

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Thank you.