Advanced Calculus (II)

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9.1: Limits Of Sequences

Definition (9.1)

Let $\{\mathbf{x}_k\}$ be a sequence points in $\mathbf{R}^n$.
(i) $\{\mathbf{x}_k\}$ is said to *converge* to some point $\mathbf{a} \in \mathbf{R}^n$ (call the *limit* of $\mathbf{x}_k$) if and only if for every $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$k \geq N \quad \textit{implies} \quad \|\mathbf{x}_k - \mathbf{a}\| < \varepsilon.$$

(ii) $\{\mathbf{x}_k\}$ is said to be *bounded* if and only if there is an $M > 0$ such that $\|\mathbf{x}_k\| \leq M$ for all $k \in \mathbf{N}$.
(iii) $\mathbf{x}_k \in \mathbf{R}^n$ is said to be *Cauchy* if and only if for every $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$k, m \geq N \quad \textit{imply} \quad \|\mathbf{x}_k - \mathbf{x}_m\| < \varepsilon.$$

Theorem (9.2)

Let $\mathbf{a} := (a(1), \ldots, a(n))$ *and* $\mathbf{x}_k := (x_k(1), \ldots, x_k(n))$ *belong to* $\mathbf{R}^n$ *for* $k \in \mathbf{N}$. *Then* $\mathbf{x}_k \to \mathbf{a}$, *as* $k \to \infty$, *if and only if the component sequences* $x_k(j) \to a(j)$, *as* $k \to \infty$, *for all* $j = 1, 2, \ldots, n$.

Proof.

Fix $j \in \{1, \ldots, n\}$. By Remark 8.7,

$$\|x_k(j) - a(j)\| \leq \|\mathbf{x}_k - \mathbf{a}\| \leq \sqrt{n} \max_{1 \leq \ell \leq n} |x_k(\ell) - a(\ell)|.$$

Hence, by the Squeeze Theorem, $x_k(j) \to a(j)$ as $k \to \infty$ for all $1 \leq j \leq n$ if and only if the real sequence $\|\mathbf{x}_k - \mathbf{a}\| \to 0$ as $k \to \infty$. Since $\|\mathbf{x}_k - \mathbf{a}\| \to 0$ if and only if $\mathbf{x}_k \to \mathbf{a}$, as $k \to \infty$, the proof of the theorem is complete. □

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Theorem (9.3)

For each $\mathbf{a} \in \mathbf{R}^n$ *there is a sequence* $\mathbf{x}_k \in \mathbf{Q}^n$ *such that* $\mathbf{x}_k \to \mathbf{a}$ *as* $k \to \infty$.

Theorem (9.4)

Let $n \in \mathbf{N}$.
(i) A sequence in $\mathbf{R}^n$ can have at most one limit.

(ii) If $\{\mathbf{x}_k\}_{k\in\mathbf{N}}$ is a sequence in $\mathbf{R}^n$ that converges to $\mathbf{a}$ and $\{\mathbf{x}_{k_j}\}_{k\in\mathbf{N}}$ is any subsequence of $\{\mathbf{x}_k\}_{k\in\mathbf{N}}$, then $\mathbf{x}_{k_j}$ converges to $\mathbf{a}$ as $j \to \infty$.

(iii) Every convergent sequence in $\mathbf{R}^n$ is bounded, but not conversely.

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Theorem (9.4)

(v) If $\{\mathbf{x}_k\}$ *and* $\{\mathbf{y}_k\}$ *are convergent sequences in* $\mathbf{R}^n$ *and* $\alpha \in \mathbf{R}$, *then*

$$\lim_{k\to\infty}(\mathbf{x}_k + \mathbf{y}_k) = \lim_{k\to\infty}\mathbf{x}_k + \lim_{k\to\infty}\mathbf{y}_k,$$

$$\lim_{k\to\infty}(\alpha\mathbf{x}_k) = \alpha\lim_{k\to\infty}(\mathbf{x}_k),$$

and

$$\lim_{k\to\infty}(\mathbf{x}_k \cdot \mathbf{y}_k) = \left(\lim_{k\to\infty}(\mathbf{x}_k)\right)\cdot\left(\lim_{k\to\infty}(\mathbf{y}_k)\right).$$

Moreover, when $n = 3$,

$$\lim_{k\to\infty}(\mathbf{x}_k \times \mathbf{y}_k) = \left(\lim_{k\to\infty}(\mathbf{x}_k)\right)\times\left(\lim_{k\to\infty}(\mathbf{y}_k)\right).$$

Theorem (9.5 Bolzano-Weierstrass Theorem for $\mathbf{R}^n$)

Every bounded sequence in $\mathbf{R}^n$ has a convergent subsequence.

Theorem (9.6)

A sequence $\{\mathbf{x}_k\}$ *in* $\mathbf{R}^n$ *is Cauchy if and only if it converges.*

Theorem (9.7)

Let $\mathbf{x}_k \in \mathbf{R}^n$*. Then* $\mathbf{x}_k \to \mathbf{a}$ *as* $k \to \infty$ *if and only if for every open set* V *that contains* $\mathbf{a}$ *there is an* $N \in \mathbf{N}$ *such that* $k \geq N$ *implies* $\mathbf{x}_k \in V$*.*

Theorem (9.8)

Let $E \subseteq \mathbf{R}^n$. Then E is closed if and only if E contains all its limit points; i.e., $\mathbf{x}_k \in E$ and $\mathbf{x}_k \to \mathbf{x}$ imply $\mathbf{x} \in E$.

Definition (9.9)

Let E be a subset of $\mathbf{R}^n$.
(i) An *open covering* of E is a collection of sets $\{V_\alpha\}_{\alpha\in A}$ such that each V_α is open and

$$E \subseteq \bigcup_{\alpha\in A} V_\alpha.$$

(ii) The set E is said to be *compact* if and only if every open covering of E has a finite subcovering; i.e., if $\{V_\alpha\}_{\alpha\in A}$ is an open covering of E, then there is a finite subset A_0 of A such that

$$E \subseteq \bigcup_{\alpha\in A_0} V_\alpha.$$

Lemma (9.10 Borel Covering Lemma)

Let E be a closed, bounded subset of $\mathbf{R}^n$. If r is any function from E into $(0, \infty)$, then there exist finitely many points $\mathbf{y}_1, \ldots, \mathbf{y}_N \in E$ such that

$$E \subseteq \bigcup_{j=1}^{N} B_{r(\mathbf{y}_j)}(\mathbf{y}_j).$$

Theorem (9.11 Heine-Borel Theorem)

Let E be a subset of $\mathbf{R}^n$. Then E is compact if and only if E is closed and bounded.

Thank you.