Advanced Calculus (II)

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9.2: Limits Of Functions

Definition (9.13)

Let $n, m \in \mathbf{N}$ and $\mathbf{a} \in \mathbf{R}^n$, let V be an open set which contains $\mathbf{a}$, and suppose that $f : V \setminus \{\mathbf{a}\} \to \mathbf{R}^m$. Then $f(\mathbf{x})$ is said to *converge to* $\mathbf{L}$, as $\mathbf{x}$ *approaches* $\mathbf{a}$, if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ (which in general depends on ε, f, V, and $\mathbf{a}$) such that

$$0 < \|\mathbf{x} - \mathbf{a}\| \leq \delta \text{ implies } \|f(\mathbf{x}) - \mathbf{L}\| < \varepsilon.$$

In this case we write

$$\mathbf{L} = \lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x})$$

and call $\mathbf{L}$ the *limit* of f($\mathbf{x}$) as $\mathbf{x}$ approaches $\mathbf{a}$.

Theorem (9.14)

Let $n, m \in \mathbf{N}$, let $\mathbf{a} \in \mathbf{R}^n$, let V be an open ball which contains $\mathbf{a}$, and let $f, g : V \backslash \{\mathbf{a}\} \to \mathbf{R}^m$.

(i) If $f(\mathbf{x}) = g(\mathbf{x})$ for all $\mathbf{x} \in V \backslash \{\mathbf{a}\}$ and $f(\mathbf{x})$ has a limit as $\mathbf{x} \to \mathbf{a}$, then $g(\mathbf{x})$ also has a limit as $\mathbf{x} \to \mathbf{a}$, and

$$\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}).$$

(ii) [Sequential Characterization of Limits]
$\mathbf{L} = \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ exists if and only if $f(\mathbf{x}_k) \to \mathbf{L}$ as $k \to \infty$ for every sequence $\mathbf{x}_k \in V \backslash \{\mathbf{a}\}$ which converges to $\mathbf{a}$ as $k \to \infty$.

(iii) Suppose that $\alpha \in \mathbf{R}$. If $f(\mathbf{x})$ and $g(\mathbf{x})$ have limits, as $\mathbf{x}$ approaches $\mathbf{a}$, then so do $(f + g)(\mathbf{x})$, $(\alpha f)(\mathbf{x})$, $(f \cdot g)(\mathbf{x})$, and $\|f(\mathbf{x})\|$. In fact,

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$$\lim_{\mathbf{x}\to\mathbf{a}}(f+g)(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}),$$

$$\lim_{\mathbf{x}\to\mathbf{a}}(\alpha f)(\mathbf{x}) = \alpha \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}),$$

$$\lim_{\mathbf{x}\to\mathbf{a}}(f\cdot g)(\mathbf{x}) = \left(\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})\right)\cdot\left(\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})\right),$$

$$\left\|\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})\right\| = \lim_{\mathbf{x}\to\mathbf{a}} \|f(\mathbf{x})\|.$$

Moreover, when $m = 3$*,*

$$\lim_{\mathbf{x}\to\mathbf{a}}(f\times g)(\mathbf{x}) = \left(\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})\right)\times\left(\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})\right),$$

and when $m = 1$ *and the limit of* g *is nonzero,*

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})/g(\mathbf{x}) = \left(\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})\right)/\left(\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})\right).$$

Theorem (9.14)

(iv) [Squeeze Theorem for Functions] Suppose that $f, g, h : V \backslash \{\mathbf{a}\} \to \mathbf{R}$ *and* $g(\mathbf{x}) \leq h(\mathbf{x}) \leq f(\mathbf{x})$ *for all* $\mathbf{x} \in V \backslash \{\mathbf{a}\}$. *If*

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = \lim_{\mathbf{x} \to \mathbf{a}} g(\mathbf{x}) = L,$$

then the limit of h also exists, as $\mathbf{x} \to \mathbf{a}$, *and*

$$\lim_{\mathbf{x} \to \mathbf{a}} h(\mathbf{x}) = L.$$

(v) Suppose that U is open in $\mathbf{R}^m$, *that* $\mathbf{L} \in U$, *and* $h : U \backslash \{\mathbf{L}\} \to \mathbf{R}^p$ *for some* $p \in \mathbf{N}$. *If* $\mathbf{L} = \lim_{\mathbf{x} \to \mathbf{a}} g(\mathbf{x})$ *and* $\mathbf{M} = \lim_{\mathbf{y} \to \mathbf{L}} h(\mathbf{y})$, *then*

$$\lim_{\mathbf{x} \to \mathbf{a}} h \circ g(\mathbf{x}) = h(\mathbf{L}).$$

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(v) Suppose that U is open in $\mathbf{R}^m$, *that* $\mathbf{L} \in U$, *and* $h : U\backslash\{\mathbf{L}\} \to \mathbf{R}^p$ *for some* $p \in \mathbf{N}$. *If* $\mathbf{L} = \lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})$ *and* $\mathbf{M} = \lim_{\mathbf{y}\to\mathbf{L}} h(\mathbf{y})$, *then*

$$\lim_{\mathbf{x}\to\mathbf{a}} h \circ g(\mathbf{x}) = h(\mathbf{L}).$$

Theorem (9.15)

Let $\mathbf{a} \in \mathbf{R}^n$*, let* V *be an open ball that contains* $\mathbf{a}$*, let* $f = (f_1, \ldots, f_m) : V \backslash \{\mathbf{a}\} \to \mathbf{R}^m$*, and let* $\mathbf{L} = (L_1, L_2, \ldots, L_m) \in \mathbf{R}^m$*. Then*

$$\mathbf{L} = \lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) \tag{1}$$

exists if and only if

$$L_j = \lim_{\mathbf{x} \to \mathbf{a}} f_j(\mathbf{x}) \tag{2}$$

exists for each $j = 1, 2, \ldots, m$*.*

Example (9.17)

Prove that

$$f(x, y) = \frac{3x^2y}{x^2 + y^2}$$

converges as $(x, y) \to (0, 0)$.

Example (9.18)

Prove that the function

$$f(x, y) = \frac{2xy}{x^2 + y^2}$$

has no limit as $(x, y) \to (0, 0)$.

Example (9.20)

Evaluate the iterated limits of

$$f(x,y) = \frac{x^2}{x^2+y^2}$$

at (0,0)

Sol. (9.20)

For each $x \neq 0$, $\frac{x^2}{x^2+y^2} \to 1$ as $y \to 0$. Therefore,

$$\lim_{x\to 0}\lim_{y\to 0}\frac{x^2}{x^2+y^2} = \lim_{x\to 0}\frac{x^2}{x^2} = 1.$$

On the other hand,

$$\lim_{y\to 0}\lim_{x\to 0}\frac{x^2}{x^2+y^2} = \lim_{y\to 0}\frac{0^2}{y^2} = 0$$

Remark (9.21)

Suppose that I and J are open intervals, that $a \in I$ and $b \in J$, and that $f : (I \times J)\backslash\{(a, b)\} \to \mathbf{R}$. If

$$g(x) := \lim_{y \to b} f(x, y)$$

exists for each $x \in I\backslash\{a\}$, if $\lim_{x \to a} f(x, y)$ exists for each $y \in J\backslash\{b\}$, and if $f(x, y) \to L$ as $(x, y) \to (a, b)$ (in $\mathbf{R}^2$), then

$$L = \lim_{x \to a} \lim_{y \to b} f(x, y) = \lim_{y \to b} \lim_{x \to a} f(x, y).$$

Thank you.