Advanced Calculus (II)

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9.3: Continuous functions

Definition (9.22)

Let E be a nonempty subset of $\mathbf{R}^n$ and let $f : E \to \mathbf{R}^m$.
(i) f is said to be *continuous* at $\mathbf{a} \in E$ if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ (which in general depends on ε, f, and $\mathbf{a}$) such that

$$(3) \qquad \|\mathbf{x} - \mathbf{a}\| < \delta \text{ and } \mathbf{x} \in E \quad \text{imply} \quad \|f(\mathbf{x}) - f(\mathbf{a})\| < \varepsilon.$$

(ii) f is said to be *continuous* on E (notation: $f : E \to \mathbf{R}^m$ is continuous) if and only if f is continuous at every $\mathbf{x} \in E$.

Definition (9.23)

Let E be a nonempty subset of $\mathbf{R}^n$ and let $f : E \to \mathbf{R}^m$. Then f is said to be *uniformly continuous* on E (notation: $f : E \to \mathbf{R}^m$ is uniformly continuous) if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|\mathbf{x} - \mathbf{a}\| < \delta \quad \text{and} \quad \mathbf{x}, \mathbf{a} \in E \quad \text{imply} \quad \|f(\mathbf{x}) - f(\mathbf{a})\| < \varepsilon.$$

Theorem (9.24)

Let E be a nonempty compact subset of $\mathbf{R}^n$. If f is continuous on E, then f is uniformly continuous on E.

Proof.

Suppose that f is continuous on E.Given $\varepsilon > 0$ and $\mathbf{a} \in E$, choose $\delta(\mathbf{a}) > 0$ such that

$$\mathbf{x} \in B_{\delta(\mathbf{a})}(\mathbf{a}) \textit{ and } \mathbf{x} \in E \textit{ imply } \|f(\mathbf{x}) - f(\mathbf{a})\| < \frac{\varepsilon}{2}.$$

Since $\delta(\mathbf{a})/2$ is positive for all $\mathbf{a} \in E$, we can choose finitely many points $\mathbf{a}_j \in E$ and numbers $\delta_j := \delta(\mathbf{a}_j)/2$ such that

$$E \subset \bigcup_{j=1}^{N} B_{\delta_j}(\mathbf{a}_j). \tag{4}$$

Set $\delta := min\{\delta_1, \ldots, \delta_N\}$.

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Suppose that $\mathbf{x}, \mathbf{a} \in E$ and $\|\mathbf{x} - \mathbf{a}\| < \delta$. By (4), $\mathbf{x}$ belongs to $B_{\delta_j}(\mathbf{a}_j)$ for some $1 \le j \le N$. Hence, $\|\mathbf{a} - \mathbf{a}_j\| \le \|\mathbf{a} - \mathbf{x}\| + \|\mathbf{x} - \mathbf{a}_j\| < \delta_j + \delta_j = 2\delta_j = \delta(\mathbf{a}_j)$, i.e., $\mathbf{a}$ also belongs to $B_{\delta(\mathbf{a}_j)}(\mathbf{a}_j)$. It follows, therefore, from the choice of $\delta(\mathbf{a}_j)$ that

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$$\|f(\mathbf{x}) - f(\mathbf{a})\| \le \|f(\mathbf{x}) - f(\mathbf{a}_j)\| + \|f(\mathbf{a}_j) - f(\mathbf{a})\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

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Theorem (9.25)

Let $n, m \in N$ and $f : \mathbf{R}^n \to \mathbf{R}^m$. Then the following three conditions are equivalent.

(i) f is continuous on $\mathbf{R}^n$.

(ii) $f^{-1}(V)$ is open in $\mathbf{R}^n$ for every open subset V of $\mathbf{R}^m$.

(iii) $f^{-1}(E)$ is closed in $\mathbf{R}^n$ for every closed subset E of $\mathbf{R}^m$.

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Theorem (9.26)

Let $n, m \in \mathbf{N}$*, let* E *be open in* $\mathbf{R}^n$*, and suppose that* $f : E \to \mathbf{R}^m$*. Then* f *is continuous on* E *if and only if* $f^{-1}(V)$ *is open in* E *for every open set* V *in* $\mathbf{R}^m$*.*

Example (9.27)

(i) If $f(x) = \frac{1}{x^2+1}$ and $E = (0, 1]$, then f is continuous on **R** and E is bounded, but $f^{-1}(E) = (-\infty, \infty)$ is not bounded.

(ii) If $f(x) = x^2$ and $E = (1, 4)$, then f is continuous on **R** and E is connected, but $f^{-1}(E) = (-2, -1) \cup (1, 2)$ is not connected.

Theorem (9.29)

Let $n, m \in \mathbf{N}$. If H is compact in $\mathbf{R}^n$ and $f : H \to \mathbf{R}^m$ is continuous on H, then $f(H)$ is compact in $\mathbf{R}^m$.

Theorem (9.30)

Let $n, m \in \mathbf{N}$. If E is connected in $\mathbf{R}^n$ and $f : E \to \mathbf{R}^m$ is continuous on E, then $f(E)$ is connected in $\mathbf{R}^m$.

Remark (9.31)

The graph $y = f(x)$ of a continuous real function f on an interval $[a, b]$ is compact and connected.

Theorem (9.32 Extreme Value Theorem)

Suppose that H is a nonempty subset of $\mathbf{R}^n$ and $f : H \to \mathbf{R}$. If H is compact, and f is continuous on H, then

$$M := \sup\{f(\mathbf{x}) : \mathbf{x} \in H\} \quad \textit{and} \quad m := \inf\{f(\mathbf{x}) : \mathbf{x} \in H\}$$

are finite real numbers. Moreover, there exist points $\mathbf{x}_M, \mathbf{x}_m \in H$ such that $M = f(\mathbf{x}_M)$ and $m = f(\mathbf{x}_m)$.

Theorem (9.33)

Let $n, m \in \mathbf{N}$. If H is a compact subset of $\mathbf{R}^n$ and $f : H \to \mathbf{R}^m$ is 1-1 and continuous, then f^{-1} is continuous on $f(H)$.

Thank you.