Advanced Calculus (II)

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2009

9.4: Compact Sets

Definition (9.35)

Let $\mathcal{V} = \{V_\alpha\}_{\alpha\in A}$ be a collection of subsets of $\mathbf{R}^n$, and suppose that $E \subseteq \mathbf{R}^n$.

(i) $\mathcal{V}$ is said to *cover* E (or be a *covering* of E) if and only if

$$E \subseteq \bigcup_{\alpha\in A} V_\alpha.$$

(ii) $\mathcal{V}$ is said to be an *open covering* of E if and only if $\mathcal{V}$ covers E and each V_α is open.

(iii) Let $\mathcal{V}$ be a covering of E. $\mathcal{V}$ is said to have a *finite* (respectively, *countable*) *subcovering* if and only if there is a finite (respectively, an at most countable) subset A_0 of A such that $\{V_\alpha\}_{\alpha\in A_0}$ covers E.

Theorem (9.39 Lindelöf)

Let $n \in \mathbf{N}$ and let E be a subst of $\mathbf{R}^n$. If $\{V_\alpha\}_{\alpha \in A}$ is a collection of open sets and $E \subseteq \cup_{\alpha \in A} V_\alpha$, then there is an at most countable subset A_0 of A such that

$$E \subseteq \bigcup_{\alpha \in A_0} V_\alpha .$$

Proof.

Let $\mathcal{T}$ be the collection of open balls with rational radii and rational centers, i.e., centers that belong to $\mathbf{Q}^n$. This collection is countable. Moreover, by the proof of the Borel Covering Lemma, $\mathcal{T}$ "approximates" the collection of open balls in the following sense: Given any open ball $B_r(\mathbf{x}) \subseteq \mathbf{R}^n$, there is a ball $B_\rho(\mathbf{a}) \in \mathcal{T}$ such that $\mathbf{x} \in B_\rho(\mathbf{a})$ and $B_\rho(\mathbf{a}) \subseteq B_r(\mathbf{x})$.

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To prove the theorem, let $\mathbf{x} \in E$. By hypothesis, $\mathbf{x} \in V_\alpha$ for some $\alpha \in A$. Since V_α is open, there is a $r > 0$ such that $B_r(\mathbf{x}) \subset \mathbf{V}_\alpha$. Since $\mathcal{T}$ approximates open balls, we can choose a ball $B_\mathbf{x} \in \mathcal{T}$ such that $\mathbf{x} \in B_\mathbf{x} \subseteq V_\alpha$. The collection $\mathcal{T}$ is countable, hence so is the subcollection

$$\{U_1, U_2, \ldots\} := \{B_\mathbf{x} : \mathbf{x} \in E\}.$$

By the choice of the balls $B_\mathbf{x}$, for each $k \in \mathbf{N}$ there is at least one $\alpha_k \in A$ such that $U_k \subseteq V_{\alpha_k}$.
Hence, by construction,

$$E \subseteq \bigcup_{\mathbf{x} \in E} B_\mathbf{x} = \bigcup_{k \in \mathbf{N}} U_k \subseteq \bigcup_{k \in \mathbf{N}} V_{\alpha k}.$$

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