Advanced Calculus (II)

WEN-CHING LIEN

Department of Mathematics
National Cheng Kung University

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9.5: Applications

Theorem (9.40 Dini)

Suppose that H is a compact subset of $\mathbf{R}^n$ and $f_k : H \to \mathbf{R}$ is a pointwise monotone sequence of continuous functions. If $f_k \to f$ pointwise on H as $k \to \infty$ and f is continuous on H, then $f_k \to f$ uniformly on H. In particular, if ϕ_k is a pointwise monotone sequence of functions continuous on an interval [a,b] that converges pointwise to a continuous function, then

$$\lim_{k\to\infty} \int_a^b \phi_k(t)dt = \int_a^b \left(\lim_{k\to\infty} \phi_k(t) \right) dt.$$

Definition (9.41)

(i) A set $E \subseteq \mathbf{R}$ is said to be of *measure zero* if and only if for every $\varepsilon > 0$ there is a countable collection of intervals $\{I_j\}_{j\in\mathbf{N}}$ that covers E such that

$$\sum_{j=1}^{\infty} |I_j| \leq \varepsilon.$$

(ii) A function $f : [a, b] \to \mathbf{R}$ is said to be *almost everywhere continuous* on $[a, b]$ if and only if the set of points $x \in [a, b]$ where f is discontinuous is a set of measure zero.

Remark (9.43)

If $E_1, E_2, \ldots$ is a sequence of sets of measure zero, then

$$E = \bigcup_{k=1}^{\infty} E_k$$

is also a set of measure zero.

Proof.

Let $\varepsilon > 0$. By hypothesis, given $k \in \mathbf{N}$ we can choose a collection of intervals $\{I_j^{(k)}\}_{j\in\mathbf{N}}$ that covers E_k such that

$$\sum_{j=1}^{\infty} |I_j^{(k)}| < \frac{\varepsilon}{2^k}.$$

then the collection $\{I_j^{(k)}\}_{k,j\in\mathbf{N}}$ is countable, covers E, and

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |I_j^{(k)}| \le \sum_{j=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

Consequently, E is of measure zero.

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Theorem (9.51 Closed Graph Theorem)

Let I be a closed interval and $f : I \to \mathbf{R}$. Then f is continuous on I if and only if the graph of f is closed and connected in $\mathbf{R}^2$.

Proof.

For any interval $J \subseteq I$, let $\mathcal{G}(J)$ represent the graph of $y = f(x)$ for $x \in J$. Suppose that f is continuous on I. The function $x \mapsto (x, f(x))$ is continuous from I into $\mathbf{R}^2$, and I is connected in $\mathbf{R}$. Thus $\mathcal{G}(I)$ is connected in $\mathbf{R}^2$ by Theorem 9.30. To prove that $\mathcal{G}(I)$ is closed, we shall use Theorem 9.8. Let $x_k \in I$ and $(x_k, f(x_k)) \to (x, y)$ as $k \to \infty$. Then $x_k \to x$ and $f(x_k) \to y$, as $k \to \infty$. Hence, $x \in I$ and since f is continuous, $f(x_k) \to f(x)$. In particular, the graph of f is closed.

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Conversely, suppose that the graph of f is closed and connected in $\mathbf{R}^2$. We first show that f satisfies the Intermediate Value Theorem on I. Indeed, suppose to the contrary that there exist $x_1 < x_2$ in I with $f(x_1) \neq f(x_2)$ and a value y_0 between $f(x_1)$ and $f(x_2)$ such that $f(t) \neq y_0$ for all $t \in [x_1, x_2]$. Suppose for simplicity that $f(x_1) < f(x_2)$. Since $f(t) \neq y_0$ for any $t \in [x_1, x_2]$, the open sets

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$$f(x_{k_j}) > f(x_0) + \varepsilon_0 > f(x_0), \qquad j \in \mathbf{N}.$$

By the Intermediate Value Theorem, choose c_j between x_{k_j} and x_0 such that $f(c_j) = f(x_0) + \varepsilon_0$. By construction, $(c_j, f(c_j)) \to (x_0, f(x_0) + \varepsilon_0)$ and $c_j \to x_0$ as $j \to \infty$. Hence, the graph of f on I is not closed. □

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If f is not continuous on I, then there exist numbers $x_0 \in I, \varepsilon_0 > 0$, and $x_k \in I$ such that $x_k \to x_0$ and $|f(x_k) - f(x_0)| > \varepsilon_0$. By symmetry, we may suppose that $f(x_k) > f(x_0) + \varepsilon_0$ for infinitely many k's, say

$$f(x_{k_j}) > f(x_0) + \varepsilon_0 > f(x_0), \qquad j \in \mathbf{N}.$$

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Thank you.