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Time-Asymptotic Interactions of Boltzmann Shock Layers in the Presence of Boundary

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ABSTRACT. We study the time-asymptotic behavior of the Boltzmann shock layers with a given physical boundary in a half-space. As boundary conditions, we prescribe a Maxwellian at the far field and require a specular reflection at the wall $x = 0$. When the macroscopic velocity at the far field is negative, we prove that if the initial data are suitably chosen, then a solution exists globally in time and tends toward the corresponding outgoing Boltzmann shock profile as time goes to infinity. The proof is essentially based on the macro-micro decomposition of solutions and the elementary energy methods.

CONTENTS

1. INTRODUCTION

Consider the Boltzmann equation with a given physical boundary in a half space. We study the plane wave propagation on the half space $R^+ = [0, \infty)$ with the following initial and boundary conditions:

$$F_t + \xi_1 F_x = Q(F, F), \quad (x, t, \xi) \in R^+ \times R^+ \times R^3, \tag{1.1}$$

$$\lim_{x \to \infty} F(x, 0, \xi) = \rho_+ \frac{e^{-|\xi - u_+|^2/(2T_+)}}{\sqrt{(2\pi T_+)^3}}, \tag{1.2}$$

$$F(0, t, \xi_1, \xi_2, \xi_3) = F(0, t, -\xi_1, \xi_2, \xi_3), \tag{1.3}$$

where $F(x, t, \xi)$ is the density function of the gas at time $t \geq 0$, position $x \geq 0$, and with the velocity $\xi = (\xi_1, \xi_2, \xi_3)$. ρ_+, $u_+ = (u_{+1}, 0, 0)$, and T_+ are the macroscopic density, the macroscopic velocity, and the temperature respectively in the thermal equilibrium at the far field. We assume that $u_{+1} < 0$. Here the specific gas constant is assumed to be one.

We also assume the hard sphere collision:

$$Q(g, h) \equiv \int_{R^3} \int_{S^2} \Big[g(\xi') h(\xi'_\star) - g(\xi) h(\xi_\star) \Big] C(\Omega, \xi - \xi_\star) \, \mathrm{d}\Omega \, \mathrm{d}\xi_\star,$$

where $C(\Omega, \xi - \xi_\star) \equiv |\Omega \cdot (\xi - \xi_\star)|$, and

$$\xi' = \xi + (\Omega \cdot (\xi_\star - \xi))\Omega, \ \xi'_\star = \xi_\star - (\Omega \cdot (\xi_\star - \xi))\Omega, \quad \Omega \in S^2.$$

In this paper we are interested in the time-asymptotic behavior of the Boltzmann shock layers. According to condition (1.2), the incoming flow travels along the positive x-axis and strikes the boundary $x = 0$. Due to condition (1.3), the incoming flow will be reflected as a wave of some form. We expect that the outgoing flow will tend toward a Boltzmann shock profile as time goes to infinity, which is a solution of the equation

$$-s\psi' + \xi_1 \psi' = Q(\psi, \psi).$$

Here s is the speed of the shock wave, and the space variable is one dimensional, $x \in R$. Motivated by the boundary condition, we first construct the approximate solution φ of (1.1)–(1.3) by superposing two travelling shock wave solutions moving in the opposite directions with the same speed. By writing the solution of (1.1) as $F(x,t,\xi) = \varphi(x,t,\xi) + J(x,t,\xi)$, we obtain the equation for J as follows:

$$J_t + \xi_1 J_x = Q(\varphi + J, \varphi + J) - Q(\varphi, \varphi) - (\varphi_t + \xi_1 \varphi_x - Q(\varphi, \varphi)). \tag{1.4}$$

Our goal is to prove the time-asymptotic convergence of the solution to (1.1)–(1.3) toward the outgoing Boltzmann shock profile; that is, $J(x,t,\xi)$ decays to zero in a suitable norm as $t \to \infty$.

There is extensive literature on the initial boundary value problems for the Boltzmann equation initiated by Cercignani. Existence, uniqueness and properties of asymptotic behavior are proved for solutions of the Milne and Kramers problems, which are to solve the linearized Boltzmann equation in a half space $x > 0$. Such studies on stationary solutions have been pursued analytically by [2], [8], [9], [27]. The Milne problem has also been studied by asymptotic expansions for the condensation and evaporation by Sone et al [1], [23].

The shock profiles of the Boltzmann equation are first constructed by [4], where the existence and uniqueness of weak plane shocks are obtained by using a projection method similar to Lyapunov-Schmidt method; however, the positivity property of the Boltzmann shock profile cannot be concluded from this approach. The time-asymptotic stability of the Maxwellian states has been shown by energy methods based on the Fourier transform and spectral analysis [15, 22, 24, 25]. Furthermore, the time-asymptotic stability and positivity of shock profiles are obtained in [18]. An elementary energy method is introduced in [18] based on a macro-micro decomposition of the equation into macroscopic and microscopic components to analyze the time-asymptotic stability of nonlinear waves. The decomposition effectively describes the Boltzmann dynamics so that the methods of analyzing viscous conservation laws can be implemented with small modifications. The positivity of a Boltzmann shock profile is thus shown by the time-asymptotic approach and the maximal principle for the collision operator.

Applying the macro-micro decomposition introduced in [18], we can regard the present problem as a time-asymptotic stability problem, of which the solution tends toward the nonlinear wave pattern, a superposition of two Boltzmann shock layers. We briefly describe the key steps for solving this problem in the following.

Step 1. We first construct the approximate solution φ of (1.1)–(1.3). (See Sections 2.2 and 3.1).

The state $(\rho_+, m_+, \mathfrak{E}_+)$ is given by (1.2) at $x = \infty$. Since (1.3) implies that the macroscopic velocity and momentum are zero at $x = 0$, we can construct a Boltzmann shock profile with these two given conditions. By solving the Rankine-Hugoniot condition (2.4) and equation (2.3), there exist ρ_0, $\mathfrak{E}_0$ and a wave speed $s > 0$ such that $(\rho_+, m_+, \mathfrak{E}_+)$ and $(\rho_0, 0, \mathfrak{E}_0)$ can be connected by a travelling wave solution $\varphi_+(x - st, \xi)$ on the whole space $R = (-\infty, \infty)$ satisfying

$$-s\varphi_+' + \xi_1\varphi_+' = Q(\varphi_+, \varphi_+).$$

Let $(\rho_+(x-st), m_+(x-st), \mathfrak{E}_+(x-st))$ denote the corresponding macroscopic variables of $\varphi_+(x - st, \xi)$. We then construct $\varphi_-(x, \xi_1, \xi_2, \xi_3) \equiv \varphi_+(-x, -\xi_1, \xi_2, \xi_3)$. We now choose the approximate solution $\varphi(x, t, \xi)$ to be

$$\varphi(x, t, \xi) \equiv \varphi_+(x - s(t + t_0), \xi) + \varphi_-(x + s(t + t_0), \xi) - \rho_0\omega(\xi; 0, T_0),$$

where

$$\omega(\xi; 0, T_0) = \frac{e^{-|\xi|^2/(2T_0)}}{\sqrt{(2\pi T_0)^3}}.$$

Here $\varepsilon \equiv |\rho_+ - \rho_0| \ll 1$ and T_0 is the temperature at $x = 0$. We choose $t_0 \equiv \varepsilon^{-3}$ large enough to study the time-asymptotic state. It follows from the above construction that

$$\varphi(x, t, \xi_1, \xi_2, \xi_3) = \varphi(-x, t, -\xi_1, \xi_2, \xi_3).$$

Step 2. Focus on the following equation and choose the suitable initial state $J(x, 0, \xi)$. (See Sections 3.1 and 3.3).

$$J_t + \xi_1 J_x = Q(\varphi + J, \varphi + J) - Q(\varphi, \varphi) - (\varphi_t + \xi_1\varphi_x - Q(\varphi, \varphi)).$$

Let F denote a solution of (1.1)–(1.3), extended to the whole space R by setting

$$F(x, t, \xi_1, \xi_2, \xi_3) = F(-x, t, -\xi_1, \xi_2, \xi_3), \quad \text{for } x < 0.$$

We treat F as a perturbation of the approximate solution φ. Thus, we write

$$F(x, t, \xi) = \varphi(x, t, \xi) + J(x, t, \xi).$$

Therefore, $J(x, t, \xi)$ satisfies the above equation (1.4). We then choose the initial state $J(x, 0, \xi)$ satisfying

$$\int_{-\infty}^{\infty}\int_{R^3}\begin{pmatrix}1\\ \xi_i\\ |\xi|^2\end{pmatrix} J(x, 0, \xi)\,d\xi\,dx = 0, \quad \text{for } i = 1, 2, 3. \tag{1.5}$$

Due to the conservation laws for the macroscopic variables, it follows that

$$\int_{-\infty}^{\infty}\int_{R^3}\begin{pmatrix}1\\ \xi_i\\ |\xi|^2\end{pmatrix} J(x, t, \xi)\,d\xi\,dx = 0, \quad \text{for } i = 1, 2, 3.$$

Such property on macroscopic variables allows one to introduce an anti-derivative variable and the energy method can be applied to study the time-asymptotic stability problem.

Step 3. Derive the equations for applying the energy method. (See Section 3.1 for details.)

Introduce the anti-derivative:

$$W(x,t,\xi) \equiv \int_{-\infty}^{x} J(y,t,\xi)\,\mathrm{d}y.$$

We obtain from (1.4)

$$W_t + \xi_1 W_x = \int_{-\infty}^{x} (Q(\varphi + J, \varphi + J) - Q(\varphi,\varphi) - E(\varphi))\,\mathrm{d}y, \tag{1.6}$$

where $E(\varphi) = \varphi_t + \xi_1\varphi_x - Q(\varphi,\varphi)$.

We need to make use of the shock profile of the Navier-Stokes equation. Let u_{NS} and T_{NS} denote the velocity and temperature for the corresponding shock profile of the Navier-Stokes equation constructed by the same conditions imposed on φ at the far field. We denote the corresponding local Maxwellians by

$$\omega_{\mathrm{tr}}(x,t,\xi) = \frac{e^{-((\xi_1 - u_{\mathrm{NS}})^2 + \xi_2^2 + \xi_3^2)/(2T_{\mathrm{NS}})}}{\sqrt{(2\pi T_{\mathrm{NS}})^3}}$$

and the collision invariants $\psi_i(x,t,\xi)$, $i = 0,\dots,4$, are as follows:

$$\begin{cases} \psi_0(x,t,\xi) = 1, \\ \psi_1(x,t,\xi) = \dfrac{\xi_1 - u_{\mathrm{NS}}}{\sqrt{T_{\mathrm{NS}}}}, \\ \psi_i(x,t,\xi) = \dfrac{\xi_i}{\sqrt{T_{\mathrm{NS}}}}, \quad i = 2,3, \\ \psi_4(x,t,\xi) = \dfrac{1}{\sqrt{6}}\left(\dfrac{(\xi_1 - u_{\mathrm{NS}})^2 + \xi_2^2 + \xi_3^2}{T_{\mathrm{NS}}} - 3\right). \end{cases}$$

We introduce the macroscopic and microscopic variables W_0 and W_1 for W:

$$\begin{aligned} W_0 &\equiv \mathbf{P}_0 W \equiv \sum_{i=0}^{4}\left(\int W\psi_i\,\mathrm{d}\xi\right)\psi_i\omega_{\mathrm{tr}}, \\ W_1 &\equiv \mathbf{P}_1 W \equiv W - W_0, \end{aligned}$$

where $\mathbf{P}_0$ is the projection operator on the space spanned by $\psi_i\omega_{\mathrm{tr}}$, $i = 0,\dots,4$ and $\mathbf{P}_1$ is the orthogonal projection $\mathbf{P}_1 = \mathbf{I} - \mathbf{P}_0$. We also decompose J as

$$J = J_0 + J_1, \quad J_0 \equiv \mathbf{P}_0 J, \quad J_1 \equiv \mathbf{P}_1 J.$$

Applying $\mathbf{P}_0$ to equation (1.6) and $\mathbf{P}_1$ to equation (1.4) separately, we obtain the desired equations as follows:

(†) $\mathbf{P}_0\,\partial_t W_0 + \mathbf{P}_0\xi_1\mathbf{P}_0\,\partial_x W_0 + \mathbf{P}_0\xi_1 J_1 = 0.$

(††) $\mathbf{P}_1\,\partial_t J_0 + \mathbf{P}_1\,\partial_t J_1 + \mathbf{P}_1\xi_1\,\partial_x J_0 + \mathbf{P}_1\xi_1\,\partial_x J_1 - L(J_1) = D(J) + \mathcal{N}(J) - E(\varphi).$

It should be noticed that we only need the component W_0 and J in the proof, and the local existence in time of W_0 and J is shown in Section 3.3.

We note that the projection $\mathbf{P}_0$ to equation (1.6) can be regarded as a linearization around the Navier-Stokes shock profile in the sense that the physical quantities u and T in the background Maxwellian are chosen from the shock profile of the Navier-Stokes equation.

Step 4. Apply the elementary energy method to (†) and (††) to prove that the solution $J(x,t,\xi)$ decays to zero in the $\|\cdot\|_{\mathrm{ref},L^\infty_x(L^2_\xi)}$ norm as $t\to\infty$. Here, the reference norm is defined by

$$\|h\|_{\mathrm{ref},L^\infty_x(L^2_\xi)} = \sup_{x\in R}\int_{R^3} h^2(2\pi T_0)^{3/2}e^{|\xi|^2/(2T_0)}\,\mathrm{d}\xi.$$

(See Section 5 for details.)

We share several technical difficulties with [18]. For the sake of completeness, we address them again. First, we need to apply the transversal wave estimate. When we apply the theory of hyperbolic conservation laws to the macroscopic part, we recover the 3 families of elementary waves. In the present problem, the compressibility of the Navier-Stokes shock profile can be used in the estimate involving the first and third families, but not for the second family which produces transversal terms. Originated from the energy estimate for the stability analysis of a viscous shock profile [10], we refine the transversal wave estimate in the current situation. Secondly, the Boltzmann equation is nonlinear due to the collision operator. We split the collision operator Q into the linear part L and the nonlinear part N. The negative definiteness of L yields the decaying of the microscopic component. But we are left with the nonlinear part N, which produces terms like $\|(1+|\xi|)^{1/2}J_1\|_{L^2_\xi}$. We therefore introduce the norms: $\|\cdot\|_{\mathrm{ref},L^\infty_x(L^2_\xi)}$, $\|\cdot\|_{\mathrm{ref},L^\infty_{x,t}(L^2_\xi)}$. (See Section 3.2.) It leads us to make the right a priori assumption and establish the higher order energy estimate to resolve the nonlinear terms. Finally, we have error terms caused by the approximate solution and the drifting Maxwellians. The facts that the projection $\mathbf{P}_0$ is determined by the physical quantities of the Navier-Stokes shock profile and the local Maxwellians ω_{tr} vary along the same shock profile certainly result in several error terms. When the shock strength is sufficiently small, the Boltzmann shock and the Navier-Stokes shock are close enough, which allows us to control all those errors. In addition, Kawashima's method [13] is applied to control the density term, which is absent in considering the perturbation of a global Maxwellian.

We state the main theorem as follows:

Theorem 1.1. *Consider the hard sphere model of equations* (1.1)–(1.3). *Suppose that the shock strength of* φ *is sufficiently small. Under the condition* (1.5), *there exists* *a constant $\delta_0 > 0$ such that if the initial data are sufficient small:*

$$\sum_{|\alpha|\le 4}\left(\|\partial_x^\alpha W_0\|_{L^\infty_{x,t}(L^2_\xi)}+\|\partial_x^\alpha\partial_t W_0\|_{L^\infty_{x,t}(L^2_\xi)}\right)+\sum_{|\alpha|\le 3}\Big(\|\partial_x^\alpha\{(1+|\xi|)^{1/2}J_1\}\|_{\mathrm{ref},L^\infty_{x,t}(L^2_\xi)}+\|\partial_x^\alpha\partial_t\{(1+|\xi|)^{1/2}J_1\}\|_{\mathrm{ref},L^\infty_{x,t}(L^2_\xi)}\Big)\le\delta_0,$$

then the Cauchy problem (1.4) has a global solution in time, and the solution $J(x,t,\xi)$ decays to zero in the $\|\cdot\|_{\mathrm{ref},L^\infty_x(L^2_\xi)}$ norm as $t\to\infty$.

That is, the solution $F(x,t,\xi)$ to (1.1)–(1.3) tends toward the outgoing shock profile $\varphi(x,t,\xi)$ in the $\|\cdot\|_{\mathrm{ref},L^\infty_x(L^2_\xi)}$ norm as $t\to\infty$.

Remark 1.2. Although both the strength of shocks and the magnitude of perturbations of the initial data are small, these two parameters are chosen independently. When the strength of the shock is sufficiently small, the existence of the Boltzmann shock profile can be guaranteed and the shocks of the Boltzmann equation and the Navier-Stokes equation are close enough. Therefore, we can make use of the compressibility of the Navier-Stokes shock profiles. As for δ_0, the smallness assumption is to close the higher order estimate due to the nonlinearity of the Boltzmann equation. This is also a common short point of the energy method for viscous conservation laws.

This paper is organized as follows. Basic facts about the Boltzmann equation and the shock profiles are summarized in Section 2. In Section 3, the approximate solution for the present problem is constructed and the related equations are derived. Section 4 collects several technical estimates developed to analyze the macroscopic and microscopic equations. The stability analysis is finally shown in Section 5. The Chapman-Enskog expansion and the Navier-Stokes equations are briefly described in Appendix A. The properties of the collision operator for hard spheres are put in the Appendix B.

2. Preliminary

2.1. Macro-micro decomposition. We first recall several important properties of the Boltzmann equation

$$F_t+\xi\cdot\nabla_x F=Q(F,F),\quad (x,t,\xi)\in R^3\times R^+\times R^3.$$

In the present paper we consider the hard sphere as our model, and the collision operator can be written as follows ([11], [12]):

$$Q(g,h)\equiv\int_{R^3}\int_{S^2}\Big[g(\xi')h(\xi'_\star)-g(\xi)h(\xi_\star)\Big]C(\Omega,\xi-\xi_\star)\,\mathrm{d}\Omega\,\mathrm{d}\xi_\star,$$

where

$$\begin{aligned}\xi' &= \xi + (\Omega \cdot (\xi_\star - \xi))\Omega,\\ \xi'_\star &= \xi_\star - (\Omega \cdot (\xi_\star - \xi))\Omega,\\ \Omega &\in S^2.\end{aligned}$$

The function $C(\Omega, \xi - \xi_\star)$ for a hard sphere is

$$C(\Omega, \xi - \xi_\star) \equiv |\Omega \cdot (\xi - \xi_\star)|.$$

The local equilibrium distributions are the distributions F with $Q(F,F) = 0$, for which the only solutions are the Maxwellians

$$F(\xi) = \rho_0 \omega(\xi; u_0, T_0),$$

where

$$\omega(\xi; u_0, T_0) = \frac{e^{-|\xi - u_0|^2/(2T_0)}}{\sqrt{(2\pi T_0)^3}}.$$

The macroscopic density ρ_0, the velocity $u_0 = (u_{01}, u_{02}, u_{03})$, and the temperature T_0 of the local thermal equilibrium state may vary. In the collision process, mass, momentum, and energy are conserved, i.e., for any distributions F and G,

$$\begin{aligned}&\int_{R^3} Q(F,G)\,\mathrm{d}\xi = 0,\\ &\int_{R^3} \xi_i Q(F,G)\,\mathrm{d}\xi = 0, \quad i = 1,2,3,\\ &\int_{R^3} |\xi|^2 Q(F,G)\,\mathrm{d}\xi = 0.\end{aligned} \tag{2.1}$$

Set the collision invariants $\chi_i(\xi)$, $i = 0, \dots, 4$, as follows:

$$\begin{cases}\chi_0(\xi) = 1,\\ \chi_i(\xi) = \dfrac{\xi_i - u_{0i}}{\sqrt{T_0}}, \quad i = 1,2,3,\\ \chi_4(\xi) = \dfrac{1}{\sqrt{6}}\left(\dfrac{|\xi - u_0|^2}{T_0} - 3\right),\end{cases}$$

which are normalized with respect to the Maxwellian state:

$$\int \chi_i \chi_j \omega(\xi)\,\mathrm{d}\xi = \delta_{ij}, \quad i, j = 0, \dots, 4.$$

The linearized collision operator

$$L(h) \equiv Q(\omega, h) + Q(h, \omega)$$

is self-adjoint and non-positive, i.e.,

$$\langle Lg, h\rangle = \langle g, Lh\rangle, \quad \langle Lg, g\rangle \le 0.$$

Here $\langle , \rangle$ is the inner product on the space $L^2(R^3)$ with respect to the variable ξ:

$$\langle g, h\rangle \equiv \int gh \, d\xi.$$

In fact, $\chi_i \omega$, $i = 0, \ldots, 4$, span the null space of L. We denote by $\mathbf{P}_0$ the projection operator on the space spanned by $\chi_i \omega$, $i = 0, \ldots, 4$, and by $\mathbf{P}_1$ the orthogonal projection: $\mathbf{P}_1 = \mathbf{I} - \mathbf{P}_0$. F can be decomposed into the macroscopic part F_0 and the microscopic part F_1:

$$F_0 = \mathbf{P}_0 F = \sum_{i=0}^{4} \left(\int \chi_i F \, d\xi \right) \chi_i \omega,$$
$$F_1 = \mathbf{P}_1 F = F - F_0.$$

It thus follows that

$$L(F_0) = 0 \quad \text{and} \quad L(F) = L(F_1).$$

We introduce new norms for $F(x, t, \xi)$, which will be used in the energy estimates:

$$\|F\|_{L^2_\xi}(x, t) \equiv \langle F, F\rangle^{1/2},$$
$$\|F\|_{L^\infty_{x,t}(L^2_\xi)} \equiv \sup_{(x,t)\in R\times R^+} \|F\|_{L^2_\xi}(x, t).$$

We also note that for hard spheres [5] and Grad's cutoff potentials [11], L is the sum of a multiplication operator and a compact operator K:

$$Lh(\xi) = -\nu(\xi)h(\xi) + K(h)(\xi).$$

The collision frequency $\nu(\xi)$ has a positive lower bound. As a result, L is negative definite on the microscopic part:

$$\int_{R^3} F_1 L(F_1) \, d\xi < -\nu_0 \int_{R^3} F_1^2 \, d\xi$$

for some positive constant ν_0. Let $\mathcal{P}_0 \equiv \ker(L)$ and its orthogonal complement be denoted by $\mathcal{P}_1$. We will write the negative operator restricted to the space $\mathcal{P}_1$ as

$$\bar{L} \equiv L|_{\mathcal{P}_1} \le -\nu_0.$$

Lemma 2.1. *For any* $g_i(x,t,\xi)$ *satisfying* $\mathbf{P}_0 g_i \equiv 0$,

$$|\langle g_1, Lg_2\rangle| \le -\frac{1}{2}\{\gamma\langle g_1, Lg_1\rangle + \gamma^{-1}\langle g_2, Lg_2\rangle\}$$

for any constant $\gamma > 0$.

Proof. By the self-adjoint and non-positive property of L. □

Remark 2.2. For more properties of the collision operator Q, we refer to Appendix B.

2.2. Boltzmann shock profile. Let $\varphi(x-st,\xi)$ be a travelling wave solution of the Boltzmann equation

$$F_t + \xi_1 F_x = Q(F,F), \quad (x,t,\xi) \in R\times R^+ \times R^3. \tag{2.2}$$

φ thus satisfies

$$-s\varphi' + \xi_1\varphi' = Q(\varphi,\varphi). \tag{2.3}$$

Let (ρ, u, T) denote the macroscopic variables of the travelling wave solution:

$$\begin{aligned}
\rho(x-st) &\equiv \int_{R^3} \varphi(x-st,\xi)\,\mathrm{d}\xi,\\
m(x-st) &\equiv \int_{R^3} \xi_1\varphi(x-st,\xi)\,\mathrm{d}\xi, \quad u \equiv \frac{m}{\rho},\\
\mathfrak{E}(x-st) &\equiv \int_{R^3} \frac{|\xi|^2}{2}\varphi(x-st,\xi)\,\mathrm{d}\xi,\\
\left(\frac{m^2}{2\rho} + \rho T\right) &\equiv \mathfrak{E} \equiv \rho\left(\frac{u^2}{2} + e\right).
\end{aligned}$$

Then the states $(\rho_\pm, m_\pm, \mathfrak{E}_\pm) \equiv \lim_{x\to\pm\infty}(\rho, m, \mathfrak{E})(x)$ satisfy the Rankine-Hugoniot condition:

$$\begin{aligned}
s(\rho_- - \rho_+) &= m_- - m_+,\\
s(m_- - m_+) &= (u_- m_- + p_-) - (u_+ m_+ + p_+),\\
s(\mathfrak{E}_- - \mathfrak{E}_+) &= u_-(\mathfrak{E}_- + p_-) - u_+(\mathfrak{E}_+ + p_+),
\end{aligned} \tag{2.4}$$

and the entropy condition

$$p_- - p_+ > 0.$$

Here $p \equiv \rho RT$, $R \equiv 1$. For the existence of the shock profile φ, see [4].

Denote by $\varphi_{\mathcal{T}}(x-st,\xi)$ the corresponding local thermal equilibrium distribution:

$$\varphi_{\mathcal{T}}(x-st,\xi) \equiv \rho(x-st)\frac{e^{-((\xi_1-u(x-st))^2+\xi_2^2+\xi_3^2)/(2T(x-st))}}{\sqrt{(2\pi T(x-st))^3}}.$$

By direct calculations, $\varphi_{\mathcal{T}}$ satisfies the following lemma.

Lemma 2.3.

$$\int_{R^3}(\varphi-\varphi_{\mathcal{T}})\,\mathrm{d}\xi = 0,$$

$$\int_{R^3}\frac{\xi_i-u_i}{\sqrt{T}}(\varphi-\varphi_{\mathcal{T}})\,\mathrm{d}\xi = 0, \quad i=1,2,3,$$

$$\int_{R^3}\left(\frac{|\xi-u|^2}{T}-3\right)(\varphi-\varphi_{\mathcal{T}})\,\mathrm{d}\xi = 0.$$

Here for the shock profile φ of (2.2), $u_1 = u$, $u_2 = u_3 = 0$.

Let $(\rho_{\mathrm{NS}}, u_{\mathrm{NS}}, \mathfrak{E}_{\mathrm{NS}})(x-st)$ be the approximate shock profile of the Navier-Stokes equation obtained by the Chapman-Enskog expansion, which connects the same end states $(\rho_\pm, u_\pm, \mathfrak{E}_\pm)$. The corresponding local Maxwellians are denoted by

$$\omega_{\mathrm{tr}}(x-st,\xi) = \frac{e^{-((\xi_1-u_{\mathrm{NS}})^2+\xi_2^2+\xi_3^2)/(2T_{\mathrm{NS}})}}{\sqrt{(2\pi T_{\mathrm{NS}})^3}},$$

$$\varphi_{\mathrm{tr}}(x-st,\xi) = \rho_{\mathrm{NS}}\frac{e^{-((\xi_1-u_{\mathrm{NS}})^2+\xi_2^2+\xi_3^2)/(2T_{\mathrm{NS}})}}{\sqrt{(2\pi T_{\mathrm{NS}})^3}}.$$

We consider a weak shock φ with strength $\varepsilon \equiv |\rho_- - \rho_+| \ll 1$. The rate of the profile φ converging to the Maxwellian equilibrium states is given in the following theorem.

Theorem 2.4. *On the Boltzmann shock profile $\varphi(x-st,\xi)$, there exist C_1, $C_2 > 1$ and $C_3 \in (0,1)$ such that*

$$|\varphi(x,\xi)-\varphi_{\mathrm{tr}}(x,\xi)| \le C_1\varepsilon^2 e^{-C_3\varepsilon|x|}\rho(x)\frac{e^{-((\xi_1-u(x))^2+\xi_2^2+\xi_3^2)/(2C_2T(x))}}{\sqrt{(T(x))^3}}, \tag{2.5}$$

$$|\partial_x^k\varphi(x,\xi)| \le C_1\varepsilon^{1+k} e^{-C_3\varepsilon|x|}\rho(x)\frac{e^{-((\xi_1-u(x))^2+\xi_2^2+\xi_3^2)/(2C_2T(x))}}{\sqrt{(T(x))^3}}, \quad k = 1,\ldots,10. \tag{2.6}$$

Remark 2.5. The above theorem is proved in [4] and the second inequality is the consequence of the scalings. We also refer readers to Appendix C of [18], where the existence of the Boltzmann shock profile with properties (2.5) and (2.6) has been shown by the weighted energy method.

3. CONSTRUCTION OF THE APPROXIMATE SOLUTION

3.1. The macroscopic and microscopic equations. In this section we construct the approximate solution of (1.1)–(1.3) by superposing two travelling wave solutions moving in the opposite directions with the same speed.

The state $(\rho_+, m_+, \mathfrak{E}_+)$ is given by (1.2) at $x = \infty$. Since (1.3) implies that the macroscopic velocity and momentum are zero at $x = 0$, we first construct a Boltzmann shock profile with these two given conditions. By solving the Rankine-Hugoniot condition (2.4) and equation (2.3), there exist ρ_0, $\mathfrak{E}_0$ and a wave speed $s > 0$ such that $(\rho_+, m_+, \mathfrak{E}_+)$ and $(\rho_0, 0, \mathfrak{E}_0)$ can be connected by a travelling wave solution $\varphi_+(x - st, \xi)$ on the whole space $R = (-\infty, \infty)$. Let $(\rho_+(x - st), m_+(x - st), \mathfrak{E}_+(x - st))$ denote the corresponding macroscopic variables of $\varphi_+(x - st, \xi)$. According to the construction,

$$\begin{aligned}\lim_{x\to\infty} (\rho_+(x), m_+(x), \mathfrak{E}_+(x)) &= (\rho_+, m_+, \mathfrak{E}_+),\\ \lim_{x\to-\infty} (\rho_+(x), m_+(x), \mathfrak{E}_+(x)) &= (\rho_0, 0, \mathfrak{E}_0).\end{aligned}$$

We can also construct the other travelling wave solution $\varphi_-(x+st, \xi)$ in the same way, satisfying

$$\begin{aligned}\lim_{x\to\infty} (\rho_-(x), m_-(x), \mathfrak{E}_-(x)) &= (\rho_0, 0, \mathfrak{E}_0),\\ \lim_{x\to-\infty} (\rho_-(x), m_-(x), \mathfrak{E}_-(x)) &= (\rho_+, -m_+, \mathfrak{E}_+).\end{aligned}$$

In fact, we can construct $\varphi_-(x, \xi_1, \xi_2, \xi_3) = \varphi_+(-x, -\xi_1, \xi_2, \xi_3)$. It thus follows that

$$\begin{aligned}\rho_+(x) &= \rho_-(-x),\\ m_+(x) &= -m_-(-x),\\ \mathfrak{E}_+(x) &= \mathfrak{E}_-(-x),\\ T_+(x) &= T_-(-x).\end{aligned}$$

We now choose the approximate solution $\varphi(x, t, \xi)$ to be

$$\varphi(x, t, \xi) \equiv \varphi_+(x - s(t + t_0), \xi) + \varphi_-(x + s(t + t_0), \xi) - \rho_0\omega(\xi; 0, T_0). \tag{3.1}$$

Here $t_0 \equiv \varepsilon^{-3}$, $\varepsilon \equiv |\rho_+ - \rho_0| \ll 1$. It follows from the above construction that

$$\varphi(x, t, \xi_1, \xi_2, \xi_3) = \varphi(-x, t, -\xi_1, \xi_2, \xi_3).$$

Let F denote a solution of (1.1)–(1.3), extended to the whole space R by setting

$$F(x,t,\xi_1,\xi_2,\xi_3) = F(-x,t,-\xi_1,\xi_2,\xi_3), \quad \text{for } x < 0.$$

We now write

$$F(x,t,\xi) \equiv \varphi(x,t,\xi) + J(x,t,\xi).$$

Then by (1.1) and (3.1), $J(x,t,\xi)$ satisfies

$$J_t + \xi_1 J_x = Q(\varphi + J, \varphi + J) - Q(\varphi,\varphi) - E(\varphi),$$

where

$$E(\varphi) \equiv \varphi_t + \xi_1\varphi_x - Q(\varphi,\varphi).$$

Also,

$$J(x,t,\xi_1,\xi_2,\xi_3) = J(-x,t,-\xi_1,\xi_2,\xi_3).$$

We choose the initial state $J(x,0,\xi)$ satisfying

$$\int_{-\infty}^{\infty}\int_{R^3}\begin{pmatrix}1\\ \xi_i\\ |\xi|^2\end{pmatrix} J(x,0,\xi)\,\mathrm{d}\xi\,\mathrm{d}x = 0, \quad \text{for } i = 1,2,3.$$

Due to the conservation laws for the macroscopic variables, it thus follows that

$$\int_{-\infty}^{\infty}\int_{R^3}\begin{pmatrix}1\\ \xi_i\\ |\xi|^2\end{pmatrix} J(x,t,\xi)\,\mathrm{d}\xi\,\mathrm{d}x = 0, \quad \text{for } i = 1,2,3.$$

We consider the anti-derivative:

$$W(x,t,\xi) \equiv \int_{-\infty}^{x} J(y,t,\xi)\,\mathrm{d}y.$$

We thus have

$$W_t + \xi_1 W_x = \int_{-\infty}^{x}\Big(Q(\varphi + J,\varphi + J) - Q(\varphi,\varphi) - E(\varphi)\Big)\,\mathrm{d}y.$$

Let

$$\Big(\rho_{\mathrm{NS}}^{\pm},\ u_{\mathrm{NS}}^{\pm},\ \mathfrak{E}_{\mathrm{NS}}^{\pm}\Big)(x \mp st)$$

be approximate shock profiles of the Navier-Stokes equation connecting the same end states as those of $\varphi_{\pm}(x \mp st, \xi)$. Define

$$\begin{pmatrix}\rho_{\rm NS}\\ u_{\rm NS}\\ \mathfrak{E}_{\rm NS}\end{pmatrix}(x,t)=\begin{pmatrix}\rho^+_{\rm NS}\\ u^+_{\rm NS}\\ \mathfrak{E}^+_{\rm NS}\end{pmatrix}(x-s(t+t_0))+\begin{pmatrix}\rho^-_{\rm NS}\\ u^-_{\rm NS}\\ \mathfrak{E}^-_{\rm NS}\end{pmatrix}(x+s(t+t_0))-\begin{pmatrix}\rho_0\\ 0\\ \mathfrak{E}_0\end{pmatrix}. \tag{3.2}$$

We denote the corresponding local Maxwellians by

$$\omega_{\rm tr}(x,t,\xi)=\frac{e^{-((\xi_1-u_{\rm NS})^2+\xi_2^2+\xi_3^2)/(2T_{\rm NS})}}{\sqrt{(2\pi T_{\rm NS})^3}}, \tag{3.3}$$

$$\varphi_{\rm tr}(x,t,\xi)=\rho_{\rm NS}\frac{e^{-((\xi_1-u_{\rm NS})^2+\xi_2^2+\xi_3^2)/(2T_{\rm NS})}}{\sqrt{(2\pi T_{\rm NS})^3}}, \tag{3.4}$$

where $\mathfrak{E}_{\rm NS}=\rho_{\rm NS}(\frac{1}{2}u_{\rm NS}^2+T_{\rm NS})$.

In the following we use $\varphi_{\rm tr}$ and $\omega_{\rm tr}$ for the macro-micro decomposition. Let L and $\mathcal{L}$ be the linearized collision operator around $\varphi_{\rm tr}$ and φ respectively:

$$\begin{aligned}L(J)&\equiv Q(\varphi_{\rm tr},J)+Q(J,\varphi_{\rm tr}),\\ \mathcal{L}(J)&\equiv Q(\varphi,J)+Q(J,\varphi).\end{aligned}$$

We introduce the following deviations:

$$\begin{aligned}D(J)&\equiv\mathcal{L}J-LJ,\\ \mathcal{N}(J)&\equiv Q(J,J).\end{aligned}$$

Set the collision invariants $\psi_i(x,t,\xi)$, $i=0,\dots,4$, as follows:

$$\begin{cases}\psi_0(x,t,\xi)=1,\\ \psi_1(x,t,\xi)=\dfrac{\xi_1-u_{\rm NS}}{\sqrt{T_{\rm NS}}},\\ \psi_i(x,t,\xi)=\dfrac{\xi_i}{\sqrt{T_{\rm NS}}},\quad i=2,3,\\ \psi_4(x,t,\xi)=\dfrac{1}{\sqrt{6}}\left(\dfrac{(\xi_1-u_{\rm NS})^2+\xi_2^2+\xi_3^2}{T_{\rm NS}}-3\right).\end{cases}$$

which are normalized with respect to the Maxwellian state $\omega_{\rm tr}$

$$\int\psi_i\psi_j\omega_{\rm tr}\,d\xi=\delta_{ij},\quad i,j=0,\dots,4.$$

Note that the functions $\psi_i \omega_{\mathrm{tr}}$, $i = 0, \dots, 4$, span the kernel of L:

$$L(\psi_i \omega_{\mathrm{tr}}) = 0, \quad \text{for } i = 0, \dots, 4.$$

We now introduce the macroscopic and microscopic variables W_0 and W_1 for W:

$$W_0 \equiv \mathbf{P}_0 W \equiv \sum_{i=0}^{4} \left(\int W \psi_i \, \mathrm{d}\xi \right) \psi_i \omega_{\mathrm{tr}},$$
$$W_1 \equiv \mathbf{P}_1 W \equiv W - W_0,$$

where $\mathbf{P}_0$ is the projection operator on the space spanned by $\psi_i \omega_{\mathrm{tr}}$, $i = 0, \dots, 4$ and $\mathbf{P}_1$ is the orthogonal projection $\mathbf{P}_1 = \mathbf{I} - \mathbf{P}_0$. We also decompose J as

$$J = J_0 + J_1, \quad J_0 \equiv \mathbf{P}_0 J, \quad J_1 \equiv \mathbf{P}_1 J.$$

Applying $\mathbf{P}_0$ to Equation(1.6) and $\mathbf{P}_1$ to equation (1.4) separately, we obtain

$$\mathbf{P}_0 \, \partial_t W_0 + \mathbf{P}_0 \xi_1 \mathbf{P}_0 \, \partial_x W_0 + \mathbf{P}_0 \xi_1 J_1 = \mathbf{P}_0 \left(- \int_{-\infty}^{x} E(\varphi)(y, t, \xi) \, \mathrm{d}y \right), \tag{3.5}$$

$$\mathbf{P}_1 \, \partial_t J_0 + \mathbf{P}_1 \, \partial_t J_1 + \mathbf{P}_1 \xi_1 \, \partial_x J_0 + \mathbf{P}_1 \xi_1 \, \partial_x J_1 - L(J_1) = D(J) + \mathcal{N}(J) - \mathbf{P}_1(E(\varphi)). \tag{3.6}$$

Since φ is constructed by superposing the travelling waves φ_+ and φ_-,

$$\begin{aligned} E(\varphi) &= \varphi_t + \xi_1 \varphi_x - Q(\varphi, \varphi) \\ &= (\partial_t \varphi_+ + \xi_1 \, \partial_x \varphi_+ - Q(\varphi_+, \varphi_+)) + (\partial_t \varphi_- + \xi_1 \, \partial_x \varphi_- - Q(\varphi_-, \varphi_-)) \\ &\qquad + Q(\varphi_+, \varphi_+) + Q(\varphi_-, \varphi_-) - Q(\varphi, \varphi) \\ &= Q(\varphi_+, \varphi_+) + Q(\varphi_-, \varphi_-) - Q(\varphi, \varphi). \end{aligned}$$

By (2.1),

$$\int_{R^3} E(\varphi) \, \mathrm{d}\xi = 0,$$
$$\int_{R^3} \xi_i E(\varphi) \, \mathrm{d}\xi = 0, \quad i = 1, 2, 3,$$
$$\int_{R^3} |\xi|^2 E(\varphi) \, \mathrm{d}\xi = 0.$$

It thus follows that

$$\mathbf{P}_0 \left(\int_{-\infty}^{x} E(\varphi)(y, t, \xi) \, \mathrm{d}y \right) = 0, \quad \mathbf{P}_1(E(\varphi)) = E(\varphi).$$

From (3.6), we have

$$
(3.7)\quad J_1 = L^{-1}(\mathbf{P}_1\,\partial_t J_0 + \mathbf{P}_1\,\partial_t J_1 + \mathbf{P}_1\xi_1\,\partial_x J_0 + \mathbf{P}_1\xi_1\,\partial_x J_1) \\ - L^{-1}D(J) - L^{-1}\mathcal{N}(J) - L^{-1}\mathbf{P}_1(E(\varphi)).
$$

By substituting (3.7) for J_1, we obtain from (3.5)

$$
(3.8)\quad \mathbf{P}_0\,\partial_t W_0 + \mathbf{P}_0\xi_1\mathbf{P}_0\,\partial_x W_0 \\ + \mathbf{P}_0\xi_1 L^{-1}(\mathbf{P}_1\,\partial_t J_0 + \mathbf{P}_1\,\partial_t J_1 + \mathbf{P}_1\xi_1\,\partial_x J_0 + \mathbf{P}_1\xi_1\,\partial_x J_1) \\ - \mathbf{P}_0\xi_1 L^{-1}[D(J) + \mathcal{N}(J) + \mathbf{P}_1(E(\varphi))] = 0.
$$

3.2. Reference macro-micro decomposition and reference norms. We introduce a reference macro-micro decomposition to estimate the derivatives of the microscopic part, which will be used in the higher order energy estimates and especially the estimates for J_1.

To be sufficiently close to the background Maxwellian state ω_{tr}, we actually choose the local Maxwellian state at $x = 0$ for the approximate solution φ. Let $M_0(\xi)$ denote such a Maxwellian state:

$$
M_0(\xi) \equiv \frac{e^{-|\xi|^2/(2T_0)}}{\sqrt{(2\pi T_0)^3}}.
$$

The reference macro-micro decomposition is made with respect to $M_0(\xi)$ as follows:

$$
\langle h_1, h_2\rangle_{\mathrm{ref}} \equiv \int_{R^3} \frac{h_1 h_2}{M_0}\,\mathrm{d}\xi,
$$

$$
\mathbf{P}_0^{\mathrm{ref}} h \equiv \sum_{i=0}^{4} \langle h, \chi_i M_0\rangle_{\mathrm{ref}} \chi_i M_0,
$$

$$
\mathbf{P}_1^{\mathrm{ref}} h \equiv h - \mathbf{P}_0^{\mathrm{ref}},
$$

where χ_i is defined in Section 2.1 with $u_{0i} = 0$ for $i = 1, 2, 3$. The norms in the reference decomposition are defined as follows:

$$
\|h\|_{\mathrm{ref},L^2_\xi} \equiv \langle h, h\rangle_{\mathrm{ref}}^{1/2},
$$

$$
\|h\|_{\mathrm{ref},L^2_x(L^2_\xi)} \equiv \int_R ||h||^2_{\mathrm{ref},L^2_\xi}\,\mathrm{d}x,
$$

$$
\|h\|_{\mathrm{ref},L^\infty_x(L^2_\xi)}(t) \equiv \sup_{x\in R} \|h\|_{\mathrm{ref},L^2_\xi}(x,t),
$$

$$
\|h\|_{\mathrm{ref},L^\infty_{x,t}(L^2_\xi)} \equiv \sup_{(x,t)\in R\times R^+} \|h\|_{\mathrm{ref},L^2_\xi}(x,t).
$$

Remark 3.1. For the macroscopic component $\mathbf{P}_0 h$, the norms $\|\mathbf{P}_0 h\|_{L^2_\xi}$ and $\|\mathbf{P}_0 h\|_{\mathrm{ref},L^2_\xi}$ are equivalent; that is, there exists $K > 1$ such that for any $h(x,t,\xi)$,

$$K^{-1}\|\mathbf{P}_0 h\|_{L^2_\xi} \le \|\mathbf{P}_0 h\|_{\mathrm{ref},L^2_\xi} \le K\|\mathbf{P}_0 h\|_{L^2_\xi}.$$

3.3. Local existence in time. We rewrite equation (1.4) as follows:

$$\begin{aligned} J_t + \xi_1 J_x &= L_0 J + \{(L - L_0)J + Q(J,J) - E(\varphi)\} \\ &= L_0 J + R(J) \end{aligned}$$

where

$$L_0 J \equiv Q(M_0, J) + Q(J, M_0).$$

The local existence of J in time is the necessary first step for a priori estimate. In general such a step is rather standard and had been omitted in the content of a priori energy estimates. Indeed, the local existence theory of J is not the primary interest of this paper. One can refer to the recent survey article [26] about the local existence of J. It provides the necessary background prepared for the energy estimates.

However, one still needs to provide the local existence theory for the macroscopic anti-derivative variable W_0 from J for this shock problem, since the procedure to obtain the estimate is not the standard way, and it can not directly be obtained from the standard local existence theory on J.

We have equation (3.5) for W_0:

$$\mathbf{P}_0\,\partial_t W_0 + \mathbf{P}_0\xi_1\mathbf{P}_0\,\partial_x W_0 + \mathbf{P}_0\xi_1 J_1 = 0.$$

Every W_0 can be decomposed as $\langle\chi_0, W_0\rangle\chi_0 + \langle\chi_1, W_0\rangle\chi_1 + \langle\chi_4, W_0\rangle\chi_4$. By calculating the inner product of the above equation with χ_i separately, $i = 0,\ 1,\ 4$, we obtain a hyperbolic system with a given source term as follows:

$$\begin{aligned} &\partial_t\langle\chi_0, W_0\rangle + \partial_x\langle\chi_0, \xi_1 W_0\rangle + \langle\chi_0, \mathbf{P}_0\xi_1 J_1\rangle = 0, \\ &\partial_t\langle\chi_1, W_0\rangle + \partial_x\langle\chi_1, \xi_1 W_0\rangle + \langle\chi_1, \mathbf{P}_0\xi_1 J_1\rangle = 0, \\ &\partial_t\langle\chi_4, W_0\rangle + \partial_x\langle\chi_4, \xi_1 W_0\rangle + \langle\chi_4, \mathbf{P}_0\xi_1 J_1\rangle = 0, \end{aligned}$$

where χ_i are the collision invariants normalized with respect to M_0:

$$\begin{cases} \chi_0(\xi) = 1, \\ \chi_i(\xi) = \dfrac{\xi_i - u_{0i}}{\sqrt{T_0}}, \quad i = 1,2,3, \\ \chi_4(\xi) = \dfrac{1}{\sqrt{6}}\left(\dfrac{|\xi - u_0|^2}{T_0} - 3\right). \end{cases}$$

Here $u_0 = 0$ and the inner product with χ_2 or χ_3 is zero due to the planar wave assumption.

Note that the above system is a linear strictly hyperbolic system (3×3 system) with finite speed of propagation, and the source term is given in terms of J_1. Its solution can be estimated in terms of J_1, which is a consequence of the local existence theory of J. Therefore, the local existence of W_0 follows directly from the local existence of J in $\|\cdot\|_{\text{ref},L^2_x(L^2_\xi)}$. Thus there exists $C > 0$ and $\tau > 0$ such that

$$||W_0(\cdot, t+\tau)||^2_{L^2_x(L^2_\xi)} \le C||W_0(\cdot, t)||^2_{L^2_x(L^2_\xi)} + \tau C\|J_1(\cdot, t)\|_{L^2_x(L^2_\xi)}.$$

4. BASIC ESTIMATES

In this section we state several basic estimates, which will be used in Section 5 to analyze equation (3.8). We first introduce the following notations.

Definition 4.1. Let $h(x,t,\xi)$ and $g(x,t,\xi)$ be functions defined on $R \times R^+ \times R^3$, and A be an operator on $L^2(R \times R^+ \times R^3)$. Then

$$\langle h \mid g\rangle(x,t) \equiv \int_{R^3} \frac{hg}{\omega_{\text{tr}}}\, d\xi,$$

$$\langle h \mid A \mid g\rangle(x,t) \equiv \int_{R^3} \frac{hAg}{\omega_{\text{tr}}}\, d\xi,$$

where ω_{tr} is defined in (3.3) as

$$\omega_{\text{tr}}(x,t,\xi) = \frac{e^{-((\xi_1-u_{\text{NS}})^2+\xi_2^2+\xi_3^2)/(2T_{\text{NS}})}}{\sqrt{(2\pi T_{\text{NS}})^3}}.$$

4.1. Properties of the macroscopic variables. In the study of the Boltzmann and Navier-Stokes equations the macroscopic dissipation occurs only for the momentum and the energy. The following lemma is used in Section 5 to analyze the dissipation of the macroscopic part.

Lemma 4.2. *There exists $C > 0$, which depends on the Navier-Stokes shock profiles, such that for the macroscopic function $f_0 \equiv \rho(x,t)\psi_0\omega_{\text{tr}} + m(x,t)\psi_1\omega_{\text{tr}} + e(x,t)\psi_4\omega_{\text{tr}}$,*

$$\langle \mathbf{P}_1\xi_1 f_0 \mid \mathbf{P}_1\xi_1 f_0\rangle(x,t) = \left(\frac{7}{3}T_{\text{NS}}\right) m^2(x,t) + \left(\frac{5}{3}T_{\text{NS}}\right) e^2(x,t), \tag{4.1}$$

$$C^{-1}(m^2+e^2) \le |\langle \bar{L}^{-1}\mathbf{P}_1\xi_1 f_0 \mid \bar{L}^{-1}\mathbf{P}_1\xi_1 f_0\rangle| \le C(m^2+e^2). \tag{4.2}$$

Proof. Write

$$\langle \mathbf{P}_1\xi_1 f_0 \mid \mathbf{P}_1\xi_1 f_0\rangle = \langle \xi_1 f_0 \mid \xi_1 f_0\rangle - \langle \mathbf{P}_0\xi_1 f_0 \mid \mathbf{P}_0\xi_1 f_0\rangle.$$

The first equality can be obtained by straightforward computations and the following integrals:

$$\begin{cases} \int \xi_1^2 \omega_{\mathrm{tr}}\, \mathrm{d}\xi = u_{\mathrm{NS}}^2 + T_{\mathrm{NS}}, & \int \xi_1^2 \psi_1^2 \omega_{\mathrm{tr}}\, \mathrm{d}\xi = u_{\mathrm{NS}}^2 + 3T_{\mathrm{NS}}, \\ \int \xi_1^2 \psi_1 \omega_{\mathrm{tr}}\, \mathrm{d}\xi = 2u_{\mathrm{NS}}\sqrt{T_{\mathrm{NS}}}, & \int \xi_1^2 \psi_1 \psi_4 \omega_{\mathrm{tr}}\, \mathrm{d}\xi = \frac{4}{\sqrt{6}} u_{\mathrm{NS}}\sqrt{T_{\mathrm{NS}}}, \\ \int \xi_1^2 \psi_4 \omega_{\mathrm{tr}}\, \mathrm{d}\xi = \frac{2}{\sqrt{6}} T_{\mathrm{NS}}, & \int \xi_1^2 \psi_4^2 \omega_{\mathrm{tr}}\, \mathrm{d}\xi = u_{\mathrm{NS}}^2 + \frac{7}{3} T_{\mathrm{NS}}. \end{cases}$$

The second inequality follows from the fact that $\mathbf{P}_1 \xi_1 (\rho \psi_0 \omega_{\mathrm{tr}}) = 0$ and $L|_{\mathcal{P}_1}$ is invertible. ◻

By straightforward computations and Schwarz's inequality, we can obtain the following result.

Lemma 4.3 ([18]). *There exists $C > 0$, which depends on the Navier-Stokes shock profiles, such that the following holds for all $\mathcal{P}_1$-valued L^2 function f_1:*

$$\int_{R^3} \langle \mathbf{P}_0 \xi_1 f_1 \mid \mathbf{P}_0 \xi_1 f_1 \rangle\, \mathrm{d}x \le C \int_{R^3} \langle f_1 \mid f_1 \rangle\, \mathrm{d}x.$$

4.2. Matrix representation. We now consider the 3×3 matrix $\mathbf{P}_0 \xi_1 \mathbf{P}_0$, where the entries are calculated as follows:

$$\begin{bmatrix} \langle \psi_0 \omega_{\mathrm{tr}} \mid \xi_1 \mid \psi_0 \omega_{\mathrm{tr}} \rangle & \langle \psi_0 \omega_{\mathrm{tr}} \mid \xi_1 \mid \psi_1 \omega_{\mathrm{tr}} \rangle & \langle \psi_0 \omega_{\mathrm{tr}} \mid \xi_1 \mid \psi_4 \omega_{\mathrm{tr}} \rangle \\ \langle \psi_1 \omega_{\mathrm{tr}} \mid \xi_1 \mid \psi_0 \omega_{\mathrm{tr}} \rangle & \langle \psi_1 \omega_{\mathrm{tr}} \mid \xi_1 \mid \psi_1 \omega_{\mathrm{tr}} \rangle & \langle \psi_1 \omega_{\mathrm{tr}} \mid \xi_1 \mid \psi_4 \omega_{\mathrm{tr}} \rangle \\ \langle \psi_4 \omega_{\mathrm{tr}} \mid \xi_1 \mid \psi_0 \omega_{\mathrm{tr}} \rangle & \langle \psi_4 \omega_{\mathrm{tr}} \mid \xi_1 \mid \psi_1 \omega_{\mathrm{tr}} \rangle & \langle \psi_4 \omega_{\mathrm{tr}} \mid \xi_1 \mid \psi_4 \omega_{\mathrm{tr}} \rangle \end{bmatrix}$$

$$= \begin{bmatrix} u_1 & \sqrt{T} & 0 \\ \sqrt{T} & u_1 & \frac{2}{\sqrt{6}}\sqrt{T} \\ 0 & \frac{2}{\sqrt{6}}\sqrt{T} & u_1 \end{bmatrix}.$$

By straightforward computations, its eigenvalues λ_i are

$$\lambda_1 = u_1 - \sqrt{\frac{5T}{3}}, \quad \lambda_2 = u_1, \quad \lambda_3 = u_1 + \sqrt{\frac{5T}{3}}.$$

Here $u_1 = u_{\mathrm{NS}}$, $T = T_{\mathrm{NS}}$ defined in (3.2). Note that ω_{tr} is a local Maxwellian defined around the Navier-Stokes shock profile; therefore, these eigenvalues λ_i, $i = 1, 2, 3$, are defined only for the Navier-Stokes shock profile in this paper. Let $r_j = (r_{j1}, r_{j2}, r_{j3})^t$ and $\ell_j = (\ell_{j1}, \ell_{j2}, \ell_{j3})$ denote the corresponding right and left eigenvectors normalized by $\ell_j \cdot r_i = \delta_{ij}$ and $|r_j| = 1$, $i, j = 1, 2, 3$. We define $\mathbf{r}_j$ and $\boldsymbol{\ell}_j$ as follows:

$$\mathbf{r}_j \equiv (r_{j1}\psi_0 + r_{j2}\psi_1 + r_{j3}\psi_4)\omega_{\mathrm{tr}},$$
$$\boldsymbol{\ell}_j \equiv (\ell_{j1}\psi_0 + \ell_{j2}\psi_1 + \ell_{j3}\psi_4)\omega_{\mathrm{tr}}.$$

Note that $r_j^t = \ell_j$ since $\mathbf{P}_0\xi_1\mathbf{P}_0$ is symmetric. Then for any function $V(x,t,\xi)$, we have

$$\mathbf{P}_0\xi_1\mathbf{r}_j = \lambda_j\mathbf{r}_j,$$
$$\langle \boldsymbol{\ell}_j\xi_1 \mid \mathbf{P}_0 V\rangle = \lambda_j\langle \boldsymbol{\ell}_j \mid V\rangle.$$

4.3. Estimates of the eigenvalues for the Navier-Stokes shock profile. Recall that $\varphi_+(x - st, \xi)$ is the Boltzmann shock profile connecting $(\rho_+, m_+, \mathfrak{E}_+)$ at $x = \infty$ and $(\rho_0, 0, \mathfrak{E}_0)$ at $x = -\infty$. Also, $\varphi_-(x + st, \xi)$ is the Boltzmann shock profile connecting $(\rho_+, -m_+, \mathfrak{E}_+)$ at $x = -\infty$ and $(\rho_0, 0, \mathfrak{E}_0)$ at $x = \infty$. λ_i^+ and λ_i^-, $i = 1, 2, 3$, are the eigenvalues of the corresponding Navier-Stokes shock profiles of φ_+ and φ_- respectively, which are given in Section 4.2 when φ is replaced by φ_+ and φ_- respectively. That is,

$$\lambda_1^+ = u_{\mathrm{NS}}^+ - \sqrt{\frac{5T_{\mathrm{NS}}^+}{3}}, \quad \lambda_2^+ = u_{\mathrm{NS}}^+, \quad \lambda_3^+ = u_{\mathrm{NS}}^+ + \sqrt{\frac{5T_{\mathrm{NS}}^+}{3}}, \tag{4.3}$$

$$\lambda_1^- = u_{\mathrm{NS}}^- - \sqrt{\frac{5T_{\mathrm{NS}}^-}{3}}, \quad \lambda_2^- = u_{\mathrm{NS}}^-, \quad \lambda_3^- = u_{\mathrm{NS}}^- + \sqrt{\frac{5T_{\mathrm{NS}}^-}{3}}. \tag{4.4}$$

We have the following theorem:

Theorem 4.4. *There exist $C_4 \in (0,1)$, $C_3 > 0$ and $C_5 > 1$ such that the eigenvalues of the corresponding Navier-Stokes shock profiles of φ_+ and φ_- given in (4.3) and (4.4) satisfy*

$$C_4\varepsilon^2 e^{-2C_3\varepsilon|x|} \le -\partial_x\lambda_3^+ \le \frac{\varepsilon^2}{C_4}e^{-C_5\varepsilon|x|}, \tag{4.5}$$

$$C_4\varepsilon^2 e^{-2C_3\varepsilon|x|} \le -\partial_x\lambda_1^- \le \frac{\varepsilon^2}{C_4}e^{-C_5\varepsilon|x|}. \tag{4.6}$$

Here $\lambda_1^- = u_{\mathrm{NS}}^- - \sqrt{5T_{\mathrm{NS}}^-/3}$, $\lambda_3^+ = u_{\mathrm{NS}}^+ + \sqrt{5T_{\mathrm{NS}}^+/3}$.

The above inequalities (4.5) and (4.6) are consequences of a well-known fact that the acoustic speed across a weak compressible Navier-Stokes shock profile is strictly monotone. One can obtain it by a two dimensional phase diagram. We refer readers to Appendices A, C of [18] for the proof.

We now estimate $\partial_x\lambda_1(x,t)$ and $\partial_x\lambda_3(x,t)$ for later use. Recall that

$$\lambda_1 = u_{\mathrm{NS}} - \sqrt{\frac{5T_{\mathrm{NS}}}{3}}, \quad \lambda_3 = u_{\mathrm{NS}} + \sqrt{\frac{5T_{\mathrm{NS}}}{3}},$$

where

$$\begin{pmatrix} \rho_{\rm NS} \\ u_{\rm NS} \\ \mathfrak{E}_{\rm NS} \end{pmatrix}(x,t) = \begin{pmatrix} \rho^+_{\rm NS} \\ u^+_{\rm NS} \\ \mathfrak{E}^+_{\rm NS} \end{pmatrix}(x - s(t+t_0)) + \begin{pmatrix} \rho^-_{\rm NS} \\ u^-_{\rm NS} \\ \mathfrak{E}^-_{\rm NS} \end{pmatrix}(x + s(t+t_0)) - \begin{pmatrix} \rho_0 \\ 0 \\ \mathfrak{E}_0 \end{pmatrix},$$

and

$$T_{\rm NS}(x,t) = \frac{\mathfrak{E}_{\rm NS}}{\rho_{\rm NS}} - \frac{u^2_{\rm NS}}{2}.$$

Due to the construction (3.2), we actually have

$$\partial_x \lambda_1^-(x) = \partial_x \lambda_3^+(-x).$$

Since $\rho_{\rm NS}$, $u_{\rm NS}$, $T_{\rm NS}$ are the corresponding density, velocity and temperature profiles of a travelling wave solution of the compressible Navier-Stokes equations, we can measure the decay of $\varphi_{\rm tr}$:

$$\partial_x \varphi_{\rm tr}(x,t,\xi) = O(1)\varepsilon^2 \mathcal{E}(x,t)\{1 + |\xi_1 - u_{\rm NS}| + |\xi_1 - u_{\rm NS}|^2 + \xi_2^2 + \xi_3^2\}\varphi_{\rm tr},$$
$$\partial_t \varphi_{\rm tr}(x,t,\xi) = O(1)\varepsilon^2 \mathcal{E}(x,t)\{1 + |\xi_1 - u_{\rm NS}| + |\xi_1 - u_{\rm NS}|^2 + \xi_2^2 + \xi_3^2\}\varphi_{\rm tr},$$

where

$$\mathcal{E}(x,t) \equiv e^{-C_3\varepsilon|x - s(t+t_0)|} + e^{-C_3\varepsilon|x+s(t+t_0)|}. \tag{4.7}$$

Moreover, by straightforward computations, we can obtain

$$\partial_x \lambda_3(x,t) = \partial_x \lambda_3^+(x - s(t+t_0)) + O(1)\varepsilon^2 \mathcal{E}_-(x,t), \tag{4.8}$$
$$\partial_x \lambda_1(x,t) = \partial_x \lambda_1^-(x + s(t+t_0)) + O(1)\varepsilon^2 \mathcal{E}_+(x,t), \tag{4.9}$$

where

$$\mathcal{E}_+(x,t) \equiv e^{-C_3\varepsilon|x-s(t+t_0)|},$$
$$\mathcal{E}_-(x,t) \equiv e^{-C_3\varepsilon|x+s(t+t_0)|}.$$

5. Stability Analysis

We first derive some technical lemmas for later use. Lower order energy estimates are mainly shown in Subsection 5.1. Transversal wave estimates follow in Subsection 5.2, and the final higher order energy estimates are concluded in Subsection 5.3.

We decompose W and J as follows:

$$W \equiv W_0 + W_1, \quad J \equiv J_0 + J_1,$$

$$W_0 \equiv \sum_{j=1}^{3} w_j \mathbf{r}_j,$$
$$\mathbf{h} \equiv J_0 - \langle \omega_{\mathrm{tr}} \mid J_0 \rangle \omega_{\mathrm{tr}} \equiv (h_1 \psi_1(\xi) + h_2 \psi_4(\xi))\omega_{\mathrm{tr}}.$$

Due to the decomposition, we have

$$J_0 = \mathbf{P}_0\, \partial_x W_0.$$

We introduce the following notation to analyze nonlinear terms:

$$\mathrm{N}(J) \equiv Q(\mathbf{h} + J_1, \mathbf{h} + J_1).$$

Thus,

$$\mathcal{N}(J) = \frac{\rho_J}{\rho_{\mathrm{NS}}} L(\mathbf{h} + J_1) + \mathrm{N}(J) = \frac{\rho_J}{\rho_{\mathrm{NS}}} L(J_1) + \mathrm{N}(J)$$

where $\rho_J \equiv \langle \omega_{\mathrm{tr}} \mid J_0 \rangle$.

We rewrite equations (3.5) and (3.6) again.

$$\mathbf{P}_0\, \partial_t W_0 + \mathbf{P}_0 \xi_1 \mathbf{P}_0\, \partial_x W_0 + \mathbf{P}_0 \xi_1 J_1 = 0, \tag{5.1}$$
$$\mathbf{P}_1\, \partial_t J_0 + \mathbf{P}_1\, \partial_t J_1 + \mathbf{P}_1 \xi_1\, \partial_x J_0 + \mathbf{P}_1 \xi_1\, \partial_x J_1 = \left(1 + \frac{\rho_J}{\rho_{\mathrm{NS}}}\right) L(J_1) + D(J) + N(J) - \mathbf{P}_1(E(\varphi)). \tag{5.2}$$

Write

$$W_0 \equiv (\mathbb{R}(x,t)\psi_0 + \mathbb{M}(x,t)\psi_1 + \mathbb{E}(x,t)\psi_4)\omega_{\mathrm{tr}},$$

and

$$\begin{cases} \mathbb{R}_x = \rho, \\ \mathbb{M}_x = m, \\ \mathbb{E}_x = e. \end{cases}$$

We note that $\rho_J(x,t) = \rho(x,t) + O(1)\varepsilon^2 \mathcal{E}(x,t) \|W_0\|_\infty$, where $\mathcal{E}(x,t)$ is defined in (4.7) and

$$\|W_0\|_\infty \equiv \sup_{(x,t) \in R \times R^+} \Big(|\mathbb{R}| + |\mathbb{M}| + |\mathbb{E}| \Big)(x,t).$$

Motivated by the method in [13], we can prove the following estimate for

$$\int_0^\tau \int_{-\infty}^{\infty} \rho^2 \, \mathrm{d}x \, \mathrm{d}t.$$

Lemma 5.1.

$$
\begin{aligned}
&\int_0^\tau \int_{-\infty}^{\infty} \rho^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = O(1)\Big(\int \rho \mathbb{M}\,\mathrm{d}x\Big)\Big|_{t=0}^{t=\tau} + O(1)\int_0^\tau \int_{-\infty}^{\infty} m^2 + e^2 + \langle J_1 \mid J_1\rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + O(1)\int_0^\tau \int_{-\infty}^{\infty} \varepsilon^4 \mathcal{E}^2(x,t)\langle W_0 \mid W_0\rangle \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Proof. By straightforward calculations, we have

$$
\begin{aligned}
\mathbf{P}_0(\partial_t W_0) &= \mathbb{R}_t \psi_0 \omega + \mathbb{M}_t \psi_1 \omega + \mathbb{E}_t \psi_4 \omega + O(1)\varepsilon^2 \mathcal{E}(x,t)\|W_0\|_\infty \Big(\sum \psi_i \omega\Big), \\
\mathbf{P}_0(\partial_x W_0) &= \mathbb{R}_x \psi_0 \omega + \mathbb{M}_x \psi_1 \omega + \mathbb{E}_x \psi_4 \omega + O(1)\varepsilon^2 \mathcal{E}(x,t)\|W_0\|_\infty \Big(\sum \psi_i \omega\Big).
\end{aligned}
$$

Substituted by the above expressions, (5.1) leads to the following equations:

$$
\mathbb{R}_t + u_{\mathrm{NS}}\mathbb{R}_x + \sqrt{T_{\mathrm{NS}}}\mathbb{M}_x + O(1)\varepsilon^2 \mathcal{E}(x,t)\|W_0\|_\infty = 0, \tag{5.3}
$$

$$
\begin{aligned}
&\mathbb{M}_t + \sqrt{T_{\mathrm{NS}}}\mathbb{R}_x + u_{\mathrm{NS}}\mathbb{M}_x + \frac{2}{\sqrt{6}}\sqrt{T_{\mathrm{NS}}}\mathbb{E}_x \\
&\qquad + \langle \xi_1 J_1, \psi_1\rangle + O(1)\varepsilon^2 \mathcal{E}(x,t)\|W_0\|_\infty = 0,
\end{aligned} \tag{5.4}
$$

$$
\mathbb{E}_t + \frac{2}{\sqrt{6}}\sqrt{T_{\mathrm{NS}}}\mathbb{M}_x + u_{\mathrm{NS}}\mathbb{E}_x + \langle \xi_1 J_1, \psi_4\rangle + O(1)\varepsilon^2 \mathcal{E}(x,t)\|W_0\|_\infty = 0. \tag{5.5}
$$

We have the following estimate by (5.3).

$$
\begin{aligned}
&\int_0^\tau \int_{-\infty}^{\infty} \mathbb{R}_x \mathbb{M}_t \,\mathrm{d}x\,\mathrm{d}t = \\
&= \Big(\int_{-\infty}^{\infty} \mathbb{R}_x \mathbb{M}\,\mathrm{d}x\Big)\Big|_{t=0}^{t=\tau} + \int_0^\tau \int_{-\infty}^{\infty} \mathbb{R}_t \mathbb{M}_x \,\mathrm{d}x\,\mathrm{d}t \\
&= \Big(\int_{-\infty}^{\infty} \rho \mathbb{M}\,\mathrm{d}x\Big)\Big|_{t=0}^{t=\tau} - \int_0^\tau \int_{-\infty}^{\infty} (u_{\mathrm{NS}}\rho + \sqrt{T_{\mathrm{NS}}} m + O(1)\varepsilon^2 \mathcal{E}(x,t)\|W_0\|_\infty) m \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{5.6}
$$

Multiply (5.4) by $\mathbb{R}_x$ and apply (5.6) to obtain:

$$
\begin{aligned}
0 = \int_0^\tau \int_{-\infty}^{\infty} \mathbb{R}_x \Big(&\mathbb{M}_t + \sqrt{T_{\mathrm{NS}}}\mathbb{R}_x + u_{\mathrm{NS}}\mathbb{M}_x + \frac{2}{\sqrt{6}}\sqrt{T_{\mathrm{NS}}}\mathbb{E}_x \\
&+ \langle \xi_1 J_1, \psi_1\rangle + O(1)\varepsilon^2 \mathcal{E}(x,t)\|W_0\|_\infty\Big) \,\mathrm{d}x\,\mathrm{d}t \quad =
\end{aligned} \tag{5.7}
$$

$$
\begin{aligned}
= \left(\int_{-\infty}^{\infty} \rho \mathbb{M}\, dx \right) \Big|_{t=0}^{t=\tau} + \int_0^{\tau} \int_{-\infty}^{\infty} \Big\{ & - u_{\mathrm{NS}} \rho m - \sqrt{T_{\mathrm{NS}}} m^2 \\
& - O(1) \varepsilon^2 m \mathcal{E}(x,t) \|W_0\|_\infty \\
& + \sqrt{T_{\mathrm{NS}}} \rho^2 + u_{\mathrm{NS}} \rho m + \frac{2}{\sqrt{6}} \sqrt{T_{\mathrm{NS}}} \rho e \\
& + \rho \langle \xi_1 J_1, \psi_1 \rangle + O(1) \varepsilon^2 \rho \mathcal{E}(x,t) \|W_0\|_\infty \Big\} \, dx\, dt.
\end{aligned}
$$

By the Schwarz inequality, (5.7) can lead to the following

$$
\begin{aligned}
\int_0^{\tau} \int_{-\infty}^{\infty} \rho^2 \, dx\, dt = O(1) \left(\int \rho \mathbb{M}\, dx \right) \Big|_{t=0}^{t=\tau} & \\
+ O(1) \int_0^{\tau} \int_{-\infty}^{\infty} & m^2 + e^2 + \langle J_1 \mid J_1 \rangle \, dx\, dt \\
+ O(1) \int_0^{\tau} \int_{-\infty}^{\infty} & \varepsilon^4 \mathcal{E}^2(x,t) \langle W_0 \mid W_0 \rangle \, dx\, dt.
\end{aligned}
$$

The proof is complete. ◻

Because of the structure of $\mathrm{N}(J)$ and Lemma B.1, we can obtain the following lemma:

Lemma 5.2.

$$
\|(1+|\xi|)^{-1/2} \mathrm{N}(J)\|_{L^2_\xi} \le O(1) (m^2 + e^2 + \|(1+\xi)^{1/2} J_1\|^2_{L^2_\xi} + \varepsilon^4 \mathcal{E}^2(x,t) \langle W_0 \mid W_0 \rangle).
$$

By Theorem 2.4 and Lemma B.2, we have the following estimate:

Lemma 5.3.

$$
\|L^{-1} D\|_{\mathrm{ref}, L^2_\xi} \le O(1) \varepsilon^2 \mathcal{E}(x,t).
$$

We refer to [18] for the proofs of Lemmas 5.2 and 5.3.

5.1. Lower order energy estimates. In this subsection, we will carry out the lower order energy estimates for W_0 and J separately. We make the following a priori assumption:

$$
\begin{aligned}
(5.8) \quad & \sum_{|\alpha| \le 4} \left(\|\partial_x^\alpha W_0\|_{L^\infty_{x,t}(L^2_\xi)} + \|\partial_x^\alpha \partial_t W_0\|_{L^\infty_{x,t}(L^2_\xi)} \right) \\
& + \sum_{|\alpha| \le 3} \left(\|\partial_x^\alpha \{(1+|\xi|)^{1/2} J_1\}\|_{\mathrm{ref}, L^\infty_{x,t}(L^2_\xi)} + \|\partial_x^\alpha \partial_t \{(1+|\xi|)^{1/2} J_1\}\|_{\mathrm{ref}, L^\infty_{x,t}(L^2_\xi)} \right) \le \delta_0 .
\end{aligned}
$$

Here $\delta_0 \ll 1$ is a given number.

By multiplying equation (3.8) by W_0 and integrating it over $R \times [0,\tau]$, we obtain

$$\int_0^\tau \int_{-\infty}^\infty \Big\langle W_0 \mid \mathbf{P}_0\, \partial_t W_0 + \mathbf{P}_0 \xi_1 \mathbf{P}_0\, \partial_x W_0 + \mathbf{P}_0\xi_1 L^{-1}(\mathbf{P}_1\, \partial_t J_0 + \mathbf{P}_1\, \partial_t J_1 + \mathbf{P}_1 \xi_1\, \partial_x J_0 + \mathbf{P}_1\xi_1\, \partial_x J_1) - \mathbf{P}_0\xi_1 L^{-1}[D(J) + \mathcal{N}(J) + \mathbf{P}_1(E(\varphi))]\Big\rangle\, dx\, dt = 0. \tag{5.9}$$

We separate the above integral into the following 9 terms:

$$\begin{aligned}
0 &= \int_0^\tau \int_{-\infty}^\infty \langle W_0 \mid \mathbf{P}_0\, \partial_t W_0\rangle\, dx\, dt \\
&\quad + \int_0^\tau \int_{-\infty}^\infty \langle W_0 \mid \mathbf{P}_0\xi_1\mathbf{P}_0\, \partial_x W_0\rangle\, dx\, dt \\
&\quad + \int_0^\tau \int_{-\infty}^\infty \langle W_0 \mid \mathbf{P}_0\xi_1 L^{-1}\mathbf{P}_1\, \partial_t J_0\rangle\, dx\, dt \\
&\quad + \int_0^\tau \int_{-\infty}^\infty \langle W_0 \mid \mathbf{P}_0\xi_1 L^{-1}\mathbf{P}_1\, \partial_t J_1\rangle\, dx\, dt \\
&\quad + \int_0^\tau \int_{-\infty}^\infty \langle W_0 \mid \mathbf{P}_0\xi_1 L^{-1}\mathbf{P}_1\xi_1\, \partial_x J_0\rangle\, dx\, dt \\
&\quad + \int_0^\tau \int_{-\infty}^\infty \langle W_0 \mid \mathbf{P}_0\xi_1 L^{-1}\mathbf{P}_1\xi_1\, \partial_x J_1\rangle\, dx\, dt \\
&\quad - \int_0^\tau \int_{-\infty}^\infty \langle W_0 \mid \mathbf{P}_0\xi_1 L^{-1} D(J)\rangle\, dx\, dt \\
&\quad - \int_0^\tau \int_{-\infty}^\infty \langle W_0 \mid \mathbf{P}_0\xi_1 L^{-1} \mathcal{N}(J)\rangle\, dx\, dt \\
&\quad - \int_0^\tau \int_{-\infty}^\infty \langle W_0 \mid \mathbf{P}_0\xi_1 L^{-1}\mathbf{P}_1(E(\varphi))\rangle\, dx\, dt \\
&= I_1 + \cdots + I_9.
\end{aligned}$$

We now estimate each I_i, $i = 1, \ldots, 9$.

$$I_1 \equiv \frac{1}{2}\left(\int_{-\infty}^\infty \langle W_0 \mid W_0\rangle\, dx\right)\Bigg|_{t=0}^{t=\tau}, \tag{5.10}$$

$$\begin{aligned}
I_2 &\equiv \int_0^\tau \int_{-\infty}^\infty \langle W_0 \mid \mathbf{P}_0\xi_1\mathbf{P}_0\, \partial_x W_0\rangle\, dx\, dt \\
&= \int_0^\tau \int_{-\infty}^\infty \sum_{j=1}^3 \lambda_j w_j \Big[w_{jx} + \sum_{k=1}^3 w_k \langle \boldsymbol{\ell}_j \mid \partial_x \mathbf{r}_k\rangle\Big]\, dx\, dt \\
&= \int_0^\tau \int_{-\infty}^\infty \sum_{j=1}^3 \left(-\frac{\partial_x \lambda_j}{2}(w_j)^2 + \lambda_j w_j \sum_{k=1}^3 w_k\langle \boldsymbol{\ell}_j \mid \partial_x \mathbf{r}_k\rangle\right) dx\, dt.
\end{aligned} \tag{5.11}$$

We will estimate I_2 again in Section 5.2.

The term I_3 is defined and estimated as follows:

$$\begin{aligned} I_3 &\equiv \int_0^\tau \int_{-\infty}^{\infty} \langle W_0 \mid \mathbf{P}_0 \xi_1 L^{-1} \mathbf{P}_1 \, \partial_t J_0 \rangle \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_0^\tau \int_{-\infty}^{\infty} \langle \mathbf{P}_1 \xi_1 W_0 \mid L^{-1} \mathbf{P}_1 \, \partial_t J_0 \rangle \, \mathrm{d}x \, \mathrm{d}t. \end{aligned}$$

By integration by parts,

$$I_3 = -\int_0^\tau \int_{-\infty}^{\infty} \langle \mathbf{P}_1 \xi_1 W_0 \mid L^{-1} \mid \mathbf{P}_{1t} J_0 \rangle \, \mathrm{d}x \, \mathrm{d}t.$$

The t-derivative is applied to the profile φ_{tr} to generate a factor $\varphi_{\mathrm{tr}}' = O(1)\varepsilon^2 \mathcal{E}(x,t)$. Then by the Schwarz inequality, there exists a constant $C > 1$ such that for any $\gamma \in (0,1)$,

$$I_3 \le C\varepsilon \int_0^\tau \int_{-\infty}^{\infty} \left\{ \gamma \varepsilon^2 \mathcal{E}^2(x,t) \langle W_0 \mid W_0 \rangle + \frac{1}{\gamma} (\langle \mathbf{h} \mid \mathbf{h} \rangle + \rho_J^2) \right\} \mathrm{d}x \, \mathrm{d}t. \tag{5.12}$$

The term I_5 can be estimated by applying integration by parts as follows:

$$\begin{aligned} I_5 &\equiv \int_0^\tau \int_{-\infty}^{\infty} \langle W_0 \mid \mathbf{P}_0 \xi_1 L^{-1} \mathbf{P}_1 \xi_1 \, \partial_x J_0 \rangle \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_0^\tau \int_{-\infty}^{\infty} \langle \mathbf{P}_1 \xi_1 W_0 \mid L^{-1} \mathbf{P}_1 \xi_1 \, \partial_x J_0 \rangle \, \mathrm{d}x \, \mathrm{d}t \\ &= -\int_0^\tau \int_{-\infty}^{\infty} \langle \mathbf{P}_1 \xi_1 J_0 \mid L^{-1} \mid \mathbf{P}_1 \xi_1 J_0 \rangle \, \mathrm{d}x \, \mathrm{d}t \\ &\quad - \int_0^\tau \int_{-\infty}^{\infty} \langle \mathbf{P}_{1x} \xi_1 W_0 \mid L^{-1} \mathbf{P}_1 \xi_1 J_0 \rangle \, \mathrm{d}x \, \mathrm{d}t \\ &\quad - \int_0^\tau \int_{-\infty}^{\infty} \langle \mathbf{P}_1 \xi_1 \mathbf{P}_1 W_{0x} \mid L^{-1} \mathbf{P}_1 \xi_1 J_0 \rangle \, \mathrm{d}x \, \mathrm{d}t \\ &\quad - \int_0^\tau \int_{-\infty}^{\infty} \langle \mathbf{P}_1 \xi_1 W_0 \mid L_x^{-1} \mid \mathbf{P}_1 \xi_1 J_0 \rangle \, \mathrm{d}x \, \mathrm{d}t \\ &\quad - \int_0^\tau \int_{-\infty}^{\infty} \langle \mathbf{P}_1 \xi_1 W_0 \mid L^{-1} \mathbf{P}_{1x} \xi_1 J_0 \rangle \, \mathrm{d}x \, \mathrm{d}t \\ &\quad - \int_0^\tau \int_{-\infty}^{\infty} \left\langle \frac{\partial_x \omega_{\mathrm{tr}}}{\omega_{\mathrm{tr}}} \mathbf{P}_1 \xi_1 W_0 \mid L^{-1} \mathbf{P}_1 \xi_1 J_0 \right\rangle \mathrm{d}x \, \mathrm{d}t. \end{aligned}$$

The x-derivatives are applied to the profile φ_{tr} to generate a factor $O(1)\varepsilon^2 \mathcal{E}(x,t)$. We thus obtain

$$
\begin{aligned}
(5.13)\qquad & \left| I_5 + \int_0^\tau \int_{-\infty}^{\infty} \langle \mathbf{P}_1\xi_1 J_0 \mid L^{-1} \mid \mathbf{P}_1\xi_1 J_0\rangle \, dx\, dt \right| \\
&\le C\varepsilon\left(\int_0^\tau \int_{-\infty}^{\infty} \varepsilon^2 \mathcal{E}(x,t)^2 \langle W_0 \mid W_0\rangle + \langle \mathbf{P}_1\xi_1 J_0 \mid \mathbf{P}_1\xi_1 J_0\rangle \, dx\, dt \right) \\
&= O(1)\varepsilon\left(\int_0^\tau \int_{-\infty}^{\infty} \varepsilon^2 \mathcal{E}(x,t)^2 \langle W_0 \mid W_0\rangle + \langle \mathbf{h} \mid \mathbf{h}\rangle \, dx\, dt \right).
\end{aligned}
$$

Using integration by parts and the t-derivative of the profile φ_{tr}, we estimate I_4 in the following.

$$
\begin{aligned}
I_4 &\equiv \int_0^\tau \int_{-\infty}^{\infty} \langle W_0 \mid \mathbf{P}_0\xi_1 L^{-1}\mathbf{P}_1\, \partial_t J_1\rangle \, dx\, dt \\
&= \int_0^\tau \int_{-\infty}^{\infty} \langle \mathbf{P}_1\xi_1 W_0 \mid L^{-1}\mathbf{P}_1\, \partial_t J_1\rangle \, dx\, dt \\
&= \left(\int_{-\infty}^{\infty} \langle \mathbf{P}_1\xi_1 W_0 \mid L^{-1} J_1\rangle \, dx \right)\Bigg|_{t=0}^{\tau} \\
&\quad + O(1)\int_0^\tau \int_{-\infty}^{\infty} \gamma\varepsilon^2\mathcal{E}^2(x,t)\langle W_0 \mid W_0\rangle \, dx\, dt \\
&\quad + O(1)\frac{1}{\gamma}\int_0^\tau \int_{-\infty}^{\infty} \|J_1\|^2_{\mathrm{ref},L^2_\xi} \, dx\, dt \\
&\quad - \int_0^\tau \int_{-\infty}^{\infty} \langle \mathbf{P}_1\xi_1 W_{0t} \mid L^{-1} J_1\rangle \, dx\, dt.
\end{aligned}
$$

Note that the third term in the RHS of the above equality is due to the inner product $\langle \text{Macro} \mid \text{Micro}\rangle$, where we apply the following inequality:

$$
\begin{aligned}
\langle \text{Macro} \mid \text{Micro}\rangle &\le O(1)\left| \frac{\sqrt{M_0}}{\sqrt{\omega_{\mathrm{tr}}}}\text{Macro} \right|_{L^2_\xi} \cdot \|\text{Micro}\|_{\mathrm{ref},L^2_\xi} \\
&= O(1)\|\text{Macro}\|_{L^2_\xi}\, \|\text{Micro}\|_{\mathrm{ref},L^2_\xi}.
\end{aligned}
$$

From equation (3.5), we have

$$
W_{0t} = \mathbf{P}_0 W_{0t} + \mathbf{P}_1 W_{0t} = \mathbf{P}_1 W_{0t} - (\mathbf{P}_0\xi_1 J_0 + \mathbf{P}_0\xi_1 J_1).
$$

Applying the above equation and inequality, we can estimate

$$
\int_0^\tau \int_{-\infty}^{\infty} \langle \mathbf{P}_1\xi_1 W_{0t} \mid L^{-1} J_1\rangle \, dx\, dt
$$

as follows:

$$
\begin{aligned}
&\int_0^\tau \int_{-\infty}^\infty \langle \mathbf{P}_1 \xi_1 W_{0t} \mid L^{-1} J_1 \rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^\tau \int_{-\infty}^\infty \langle \mathbf{P}_1 \xi_1 \mathbf{P}_1 W_{0t} \mid L^{-1} J_1 \rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_0^\tau \int_{-\infty}^\infty \langle \mathbf{P}_1 \xi_1 \mathbf{P}_0 \xi_1 J_0 \mid L^{-1} J_1 \rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_0^\tau \int_{-\infty}^\infty \langle \mathbf{P}_1 \xi_1 \mathbf{P}_0 \xi_1 J_1 \mid L^{-1} J_1 \rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le O(1) \int_0^\tau \int_{-\infty}^\infty \varepsilon^2 \mathcal{E}(x,t) \langle W_0 \mid W_0 \rangle^{1/2} \langle J_1 \mid J_1 \rangle^{1/2} \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + C\gamma \int_0^\tau \int_{-\infty}^\infty (\langle \mathbf{h} \mid \mathbf{h} \rangle + \rho_J^2) \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + O(1)(1+\gamma^{-1}) \int_0^\tau \int_{-\infty}^\infty \langle J_1 \mid J_1 \rangle \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

for any $\gamma > 0$. By the above estimate, we obtain

$$
\begin{aligned}
(5.14) \qquad & I_4 - \left(\int_{-\infty}^\infty \langle \mathbf{P}_1 \xi_1 W_0 \mid L^{-1} J_1 \Big\rangle \,\mathrm{d}x \right) \Big|_{t=0}^{\tau} \\
&\quad \le O(1)\gamma \int_0^\tau \int_{-\infty}^\infty \varepsilon^2 \mathcal{E}^2(x,t) \langle W_0 \mid W_0 \rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + C\gamma \int_0^\tau \int_{-\infty}^\infty (\langle \mathbf{h} \mid \mathbf{h} \rangle + \rho_J^2) \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + O(1)(1+\gamma^{-1}) \int_0^\tau \int_{-\infty}^\infty \|J_1\|^2_{\mathrm{ref},L^2_\xi} \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Using integration by parts and the x-derivative of the profile φ_{tr}, we estimate I_6 in the following.

$$
\begin{aligned}
I_6 &\equiv \int_0^\tau \int_{-\infty}^\infty \langle W_0 \mid \mathbf{P}_0 \xi_1 L^{-1} \mathbf{P}_1 \xi_1 \,\partial_x J_1 \rangle \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\tau \int_{-\infty}^\infty \langle \mathbf{P}_1 \xi_1 W_0 \mid L^{-1} \mathbf{P}_1 \xi_1 \,\partial_x J_1 \rangle \,\mathrm{d}x\,\mathrm{d}t \\
&= O(1) \int_0^\tau \int_{-\infty}^\infty \varepsilon^2 \mathcal{E}(x,t) \langle W_0 \mid W_0 \rangle^{1/2} \langle J_1 \mid J_1 \rangle^{1/2} \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_0^\tau \int_{-\infty}^\infty \langle \mathbf{P}_1 \xi_1 W_{0x} \mid L^{-1} \mathbf{P}_1 \xi_1 J_1 \rangle \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Since $\partial_x W_0 = J_0 + \mathbf{P}_1 \partial_x W_0$, it follows that

$$
\begin{aligned}
(5.15)\quad I_6 &= O(1)\int_0^\tau\int_{-\infty}^\infty \varepsilon^2\mathcal{E}(x,t)\langle W_0 \mid W_0\rangle^{1/2}\langle J_1 \mid J_1\rangle^{1/2}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^\tau\int_{-\infty}^\infty \langle \mathbf{P}_1\xi_1 J_0 \mid L^{-1}\mathbf{P}_1\xi_1 J_1\rangle\,\mathrm{d}x\,\mathrm{d}t \\
&= O(1)\int_0^\tau\int_{-\infty}^\infty \varepsilon^2\mathcal{E}(x,t)\langle W_0 \mid W_0\rangle^{1/2}\langle J_1 \mid J_1\rangle^{1/2}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^\tau\int_{-\infty}^\infty \langle \mathbf{P}_1\xi_1 L^{-1}\mathbf{P}_1\xi_1 J_0 \mid J_1\rangle\,\mathrm{d}x\,\mathrm{d}t \\
&\le O(1)\int_0^\tau\int_{-\infty}^\infty \varepsilon^2\mathcal{E}(x,t)\langle W_0 \mid W_0\rangle^{1/2}\langle J_1 \mid J_1\rangle^{1/2}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 2\int_0^\tau\int_{-\infty}^\infty \gamma\langle \mathbf{P}_1\xi_1 L^{-1}\mathbf{P}_1\xi_1 J_0 \mid \mathbf{P}_1\xi_1 L^{-1}\mathbf{P}_1\xi_1 J_0\rangle + \frac{1}{\gamma}\langle J_1 \mid J_1\rangle\,\mathrm{d}x\,\mathrm{d}t \\
&\le O(1)\int_0^\tau\int_{-\infty}^\infty \varepsilon^2\mathcal{E}(x,t)\langle W_0 \mid W_0\rangle^{1/2}\langle J_1 \mid J_1\rangle^{1/2}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 2\int_0^\tau\int_{-\infty}^\infty C\gamma\langle \mathbf{h} \mid \mathbf{h}\rangle + \frac{1}{\gamma}\langle J_1 \mid J_1\rangle\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

We now estimate I_8. By writing $\mathcal{N}(J) = (\rho_J/\rho_{\mathrm{NS}})L(J_1) + \mathrm{N}(J)$,

$$
\begin{aligned}
I_8 &\equiv -\int_0^\tau\int_{-\infty}^\infty \langle W_0 \mid \mathbf{P}_0\xi_1 L^{-1}\mathcal{N}(J)\rangle\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^\tau\int_{-\infty}^\infty \frac{\rho_J}{\rho_{\mathrm{NS}}}\langle \mathbf{P}_1\xi_1 W_0 \mid J_1\rangle\,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau\int_{-\infty}^\infty \langle \mathbf{P}_1\xi_1 W_0 \mid L^{-1}\mathrm{N}(J)\rangle\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

By the Schwarz inequality and Lemma 5.2, there exists $C > 0$ such that

$$
\begin{aligned}
&\int_0^\tau\int_{-\infty}^\infty \langle \mathbf{P}_1\xi_1 W_0 \mid L^{-1}\mathrm{N}(J)\rangle\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^\tau\int_{-\infty}^\infty \langle (1+|\xi|)^{1/2}\mathbf{P}_1\xi_1 W_0 \mid (1+|\xi|)^{-1/2}L^{-1}\mathrm{N}(J)\rangle\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le C\|W_0\|_\infty \int_0^\tau\int_{-\infty}^\infty m^2+e^2+\|(1+\xi)^{1/2}J_1\|^2_{L^2_\xi}+\varepsilon^4\mathcal{E}^2(x,t)\langle W_0 \mid W_0\rangle\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Also,

$$
\int_0^\tau\int_{-\infty}^\infty \frac{\rho_J}{\rho_{\mathrm{NS}}}\langle \mathbf{P}_1\xi_1 W_0 \mid J_1\rangle\,\mathrm{d}x\,\mathrm{d}t \le C\|\mathbf{P}_1\xi_1 W_0\|_{L^\infty_{x,t}(L^2_\xi)}\int_0^\tau\int_{-\infty}^\infty \rho_J^2+\langle J_1 \mid J_1\rangle\,\mathrm{d}x\,\mathrm{d}t.
$$

By applying Lemma 5.1 to the term $\int_0^\tau\int_{-\infty}^{\infty}\rho_J^2\,\mathrm{d}x\,\mathrm{d}t$, it follows that

$$I_8 \le C\|W_0\|_\infty\int_0^\tau\int_{-\infty}^{\infty} m^2+e^2+\|(1+\xi)^{1/2}J_1\|_{L^2_\xi}^2 + \varepsilon^4\mathcal{E}^2(x,t)\langle W_0\mid W_0\rangle\,\mathrm{d}x\,\mathrm{d}t + C\|W_0\|_\infty\left|\left(\int_{-\infty}^{\infty}\rho\mathbb{M}\,\mathrm{d}x\right)\Big|_{t=0}^{t=\tau}\right|. \tag{5.16}$$

Note that in the estimates of I_4, I_6, and I_8 above, we can also estimate J_1 with respect to the norm $\|\cdot\|_{\mathrm{ref},L^2_\xi}$.

By Lemma 4.2, there exists a constant $C>0$ such that

$$C\langle \mathbf{h}\mid\mathbf{h}\rangle \le |\langle L^{-1}\mathbf{P}_1\xi_1J_0\mid L^{-1}\mathbf{P}_1\xi_1J_0\rangle|. \tag{5.17}$$

We now combine (5.14), (5.15), (5.16), and (5.17) under the smallness assumption (5.8) to obtain the following inequality:

$$\begin{aligned}
&\left|I_4+I_6+I_8-\left(\int_{-\infty}^{\infty}\langle\mathbf{P}_1\xi_1W_0\mid L^{-1}J_1\rangle\,\mathrm{d}x\right)\Big|_{t=0}^{t=\tau}\right.\\
&\qquad\left.+\int_0^\tau\int_{-\infty}^{\infty}\langle\mathbf{P}_1\xi_1J_0\mid L^{-1}\mid\mathbf{P}_1\xi_1J_0\rangle\,\mathrm{d}x\,\mathrm{d}t\right|\\
&\quad\le C\gamma\int_0^\tau\int_{-\infty}^{\infty}\varepsilon^2\mathcal{E}^2(x,t)\langle W_0\mid W_0\rangle\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\frac{1}{\gamma}\int_0^\tau\int_{-\infty}^{\infty}\|J_1\|^2_{\mathrm{ref},L^2_\xi}\,\mathrm{d}x\,\mathrm{d}t + C(\|W_0\|_\infty+\gamma)\left|\left(\int_{-\infty}^{\infty}(\rho^2+\mathbb{M}^2)\,\mathrm{d}x\right)\Big|_{t=0}^{t=\tau}\right|
\end{aligned}\tag{5.18}$$

for some $\gamma\in(0,1)$.

We now estimate I_7.

$$\begin{aligned}
I_7 &\equiv -\int_0^\tau\int_{-\infty}^{\infty}\langle W_0\mid\mathbf{P}_0\xi_1L^{-1}D(J)\rangle\,\mathrm{d}x\,\mathrm{d}t\\
&= -\int_0^\tau\int_{-\infty}^{\infty}\langle\mathbf{P}_1\xi_1W_0\mid L^{-1}D\mathbf{P}_0W_{0x}\rangle\,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau\int_{-\infty}^{\infty}\langle\mathbf{P}_1\xi_1W_0\mid L^{-1}DJ_1\rangle\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_0^\tau\int_{-\infty}^{\infty}O(1)\langle\mathbf{h}\mid L^{-1}DW_0\rangle + O(1)\varepsilon^2\mathcal{E}(x,t)\|D\|_{\mathrm{ref}}\langle W_0\mid W_0\rangle\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad - \int_0^\tau\int_{-\infty}^{\infty}\langle\mathbf{P}_1\xi_1W_0\mid L^{-1}DJ_1\rangle\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

By the Schwarz inequality and Lemma 5.3, there exists a constant $C > 1$ such that for any $\gamma \in (0,1)$,

$$I_7 \le C \int_0^\tau \int_{-\infty}^\infty \gamma \varepsilon^3 \mathcal{E}(x,t) \langle W_0 \mid W_0 \rangle + \frac{\varepsilon}{\gamma} (\langle J_1 \mid J_1 \rangle + \langle \mathbf{h} \mid \mathbf{h} \rangle) \, dx \, dt. \tag{5.19}$$

The last term I_9 is defined as follows:

$$\begin{aligned} I_9 &\equiv - \int_0^\tau \int_{-\infty}^\infty \langle W_0 \mid \mathbf{P}_0 \xi_1 L^{-1} \mathbf{P}_1 (E(\varphi)) \rangle \, dx \, dt \\ &= - \int_0^\tau \int_{-\infty}^\infty \langle \mathbf{P}_1 \xi_1 W_0 \mid L^{-1} E(\varphi) \rangle \, dx \, dt \\ &\le C \int_0^\tau \int_{-\infty}^\infty \langle W_0 \mid W_0 \rangle^{1/2} \langle E(\varphi) \mid E(\varphi) \rangle^{1/2} \, dx \, dt. \end{aligned}$$

To estimate I_9, we need to estimate the error term $E(\varphi)$ first. For $x > 0$, we write $E(\varphi)$ as

$$\begin{aligned} E(\varphi) = -Q(\varphi_+, \varphi_- - \varphi_0) - Q(\varphi_- - \varphi_0, \varphi_+ + \varphi_- - \varphi_0) \\ + \partial_t (\varphi_- - \varphi_0) + \xi_1 \, \partial_x (\varphi_- - \varphi_0) \end{aligned}$$

where $\varphi_0 = \rho_0 \omega(\xi; 0; T_0)$. Mainly due to the exponential decay of the shock profile φ_-, we obtain by (2.6) and Lemma B.1 that

$$\langle E(\varphi) \mid E(\varphi) \rangle^{1/2} (x,t) \le C \varepsilon^2 e^{-c_0 \varepsilon |x + s(t+t_0)|}, \quad x > 0,$$

for some constants $C, c_0 > 0$. By similar arguments, we also obtain

$$\langle E(\varphi) \mid E(\varphi) \rangle^{1/2} (x,t) \le C \varepsilon^2 e^{-c_0 \varepsilon |x - s(t+t_0)|}, \quad x < 0.$$

Combining the two estimates above, we have

$$\begin{aligned} \langle E(\varphi) \mid E(\varphi) \rangle^{1/2} (x,t) &\le C \varepsilon^2 e^{-c_0 \varepsilon (|x| + |s(t+t_0)|)} \\ &\le \hat{C} \varepsilon^4 e^{-c_0/2 \varepsilon (|x| + |s(t+t_0)|)} \end{aligned} \tag{5.20}$$

for some constant $\hat{C} > 0$.

By the Schwarz inequality and (5.20), for $\gamma > 0$,

$$\begin{aligned} I_9 \le C \int_0^\tau \int_{-\infty}^\infty \gamma \varepsilon^3 e^{-(c_0/2)\varepsilon(|x| + |s(t+t_0)|)} \langle W_0 \mid W_0 \rangle \, dx \, dt \\ + C \int_0^\tau \int_{-\infty}^\infty \frac{1}{\gamma} \{ \varepsilon^3 e^{-(c_0/2)\varepsilon(|x| + |s(t+t_0)|)} \}^{-1} \langle E(\varphi) \mid E(\varphi) \rangle \, dx \, dt. \end{aligned} \tag{5.21}$$

Finally, we combine all the estimates of I_i's: (5.10), (5.11), (5.12), (5.13), (5.18), (5.19), and (5.21) to conclude that for $\gamma > 0$,

$$\begin{aligned}
(5.22)\quad & \left(\int_{-\infty}^{\infty} \frac{1}{2}(\langle W_0 \mid W_0\rangle + \langle \mathbf{P}_1\xi_1 W_0 \mid L^{-1}J_1\rangle)\,\mathrm{d}x\right)\Big|_{t=0}^{t=\tau} \\
& - 3\int_0^\tau \int_{-\infty}^{\infty} \langle \mathbf{P}_1\xi_1 J_0 \mid L^{-1} \mid \mathbf{P}_1\xi_1 J_0\rangle\,\mathrm{d}x\,\mathrm{d}t + I_2 \\
\le & \int_0^\tau \int_{-\infty}^{\infty} C\gamma\varepsilon^2 \mathcal{E}(x,t)\|W_0\|_{L^2_\xi}^2 + \frac{C}{\gamma}\|J_1\|_{\mathrm{ref},L^2_\xi}^2\,\mathrm{d}x\,\mathrm{d}t \\
& + C(\|W_0\|_\infty + \gamma)\left|\left(\int_{-\infty}^{\infty}(\rho^2 + \mathbb{M}^2)\,\mathrm{d}x\right)\Big|_{t=0}^{t=\tau}\right| \\
& + C\int_0^\tau \int_{-\infty}^{\infty} \frac{1}{\gamma}\{\varepsilon^3 e^{-(c_0/2)\varepsilon(|x|+|s(t+t_0)|)}\}^{-1}\langle E(\varphi) \mid E(\varphi)\rangle\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

To estimate terms involving $\|J_1\|^2_{\mathrm{ref},L^2_\xi}$, we now turn to the Boltzmann equation (1.4). We have from Equation (1.4) that

$$\int_0^\tau \int_{-\infty}^{\infty} \langle J, J_t + \xi_1 J_x - L(J) - D(J) - \mathcal{N}(J) + E(\varphi)\rangle_{\mathrm{ref}}\,\mathrm{d}x\,\mathrm{d}t = 0.$$

It thus follows that

$$\begin{aligned}
\frac{1}{2}\left(\int_{-\infty}^{\infty}\langle J, J\rangle_{\mathrm{ref}}\,\mathrm{d}x\right)\Big|_{t=0}^{t=\tau} &- \int_0^\tau \int_{-\infty}^{\infty}\left(1 + \frac{\rho_J}{\rho_{\mathrm{NS}}}\right)\langle J_1, L(J_1)\rangle_{\mathrm{ref}}\,\mathrm{d}x\,\mathrm{d}t \\
&- \int_0^\tau \int_{-\infty}^{\infty}\langle J_1, D(J)\rangle_{\mathrm{ref}}\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_{-\infty}^{\infty}\langle J_1, E(\varphi)\rangle_{\mathrm{ref}}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\tau \int_{-\infty}^{\infty}\langle J_1, N(J)\rangle_{\mathrm{ref}}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

By Lemma 5.2,

$$\begin{aligned}
&\int_0^\tau \int_{-\infty}^{\infty}\langle J_1, N(J)\rangle_{\mathrm{ref}}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \int_0^\tau \int_{-\infty}^{\infty}\langle (1+|\xi|)^{1/2}J_1, (1+|\xi|)^{-1/2}N(J)\rangle_{\mathrm{ref}}\,\mathrm{d}x\,\mathrm{d}t \quad \le
\end{aligned}$$

$$
\begin{aligned}
&\leq O(1)\int_0^\tau\int_{-\infty}^\infty \Big(\|(1+|\xi|)^{1/2}J_1\|_{\mathrm{ref},L^\infty_{x,t}(L^2_\xi)} \\
&\quad\times (m^2+e^2+\varepsilon^4\mathcal{E}^2(x,t)\langle W_0 \mid W_0\rangle)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + O(1)\int_0^\tau\int_{-\infty}^\infty \|J_1\|_{\mathrm{ref},L^\infty_{x,t}(L^2_\xi)}\,\|(1+|\xi|)^{1/2}J_1\|^2_{\mathrm{ref},L^2_\xi}\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}
$$

Finally, we obtain from the estimates above that

$$
\begin{aligned}
(5.23)\qquad &\frac{1}{2}\Big(\int_{-\infty}^\infty \langle J,J\rangle_{\mathrm{ref}}\,\mathrm{d}x\Big)\Big|_{t=0}^{t=\tau} \\
&- \int_0^\tau\int_{-\infty}^\infty \Big(1+\frac{\rho_J}{\rho_{\mathrm{NS}}}\Big)\langle J_1, L(J_1)\rangle_{\mathrm{ref}}\,\mathrm{d}x\,\mathrm{d}t \\
&- \int_0^\tau\int_{-\infty}^\infty \langle J_1, D(J)\rangle_{\mathrm{ref}}\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_{-\infty}^\infty \langle J_1, E(\varphi)\rangle_{\mathrm{ref}}\,\mathrm{d}x\,\mathrm{d}t \\
&\leq O(1)\int_0^\tau\int_{-\infty}^\infty \Big(\|(1+|\xi|)^{1/2}J_1\|_{\mathrm{ref},L^\infty_{x,t}(L^2_\xi)} \\
&\quad\times (m^2+e^2+\varepsilon^4\mathcal{E}^2(x,t)\langle W_0 \mid W_0\rangle)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + O(1)\int_0^\tau\int_{-\infty}^\infty \|J_1\|_{\mathrm{ref},L^\infty_{x,t}(L^2_\xi)}\,\|(1+|\xi|)^{1/2}J_1\|^2_{\mathrm{ref},L^2_\xi}\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}
$$

Remark 5.4. We remind that (5.22) and (5.23) are intermediary results, which will be used in the following sections.

5.2. Transversal wave estimates. We focus on the term I_2 in this subsection. We rewrite I_2 as follows:

$$
\begin{aligned}
I_2 &\equiv \int_0^\tau\int_{-\infty}^\infty \langle W_0 \mid \mathbf{P}_0\xi_1\mathbf{P}_0\,\partial_x \mid W_0\rangle\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\tau\int_{-\infty}^\infty \sum_{j=1}^3\Big(-\frac{\partial_x\lambda_j}{2}(w_j)^2+\lambda_j w_j\sum_{k=1}^3 w_k\langle \boldsymbol{\ell}_j \mid \partial_x\mathbf{r}_k\rangle\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\tau\int_0^\infty \sum_{j=1}^3\Big(-\frac{\partial_x\lambda_j}{2}(w_j)^2+\lambda_j w_j\sum_{k=1}^3 w_k\langle \boldsymbol{\ell}_j \mid \partial_x\mathbf{r}_k\rangle\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau\int_{-\infty}^0 \sum_{j=1}^3\Big(-\frac{\partial_x\lambda_j}{2}(w_j)^2+\lambda_j w_j\sum_{k=1}^3 w_k\langle \boldsymbol{\ell}_j \mid \partial_x\mathbf{r}_k\rangle\Big)\,\mathrm{d}x\,\mathrm{d}t = I_{21}+I_{22}.
\end{aligned}
$$

The estimates (4.8) and (4.9) give that

$$\partial_x\lambda_3(x,t) = \partial_x\lambda_3^+(x - s(t+t_0)) + O(1)\varepsilon^2\mathcal{E}_-(x,t),$$
$$\partial_x\lambda_1(x,t) = \partial_x\lambda_1^-(x + s(t+t_0)) + O(1)\varepsilon^2\mathcal{E}_+(x,t),$$

which imply that $-\partial_x\lambda_3(x,t)$ is strictly positive when $x > 0$, and $-\partial_x\lambda_1(x,t)$ is strictly positive when $x < 0$. Here $\lambda_1 = -|O(1)|$, $\lambda_2 = O(1)\varepsilon$ and $\lambda_3 = |O(1)|$. We can write

$$I_{21} = \int_0^\tau\int_0^\infty \frac{|\partial_x\lambda_3|}{2}(w_3)^2 + O(1)|\partial_x\lambda_3|(w_1^2 + w_2^2)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_0^\infty O(1)|\partial_x\lambda_3|\sum_{j,k=1}^{3}\lambda_j w_j w_k\,\mathrm{d}x\,\mathrm{d}t,$$

where the term $\partial_x\mathbf{r}_k$ is bounded by $O(1)|\partial_x\lambda_3|$. Thus there exists $C > 0$ such that I_{21} satisfies

$$\int_0^\tau\int_0^\infty \frac{|\partial_x\lambda_3|}{4}\{w_3^2 - C(w_1^2 + w_2^2)\}\,\mathrm{d}x\,\mathrm{d}t \le I_{21},$$

and I_{22} can be estimated in the same way:

$$\int_0^\tau\int_{-\infty}^0 \frac{|\partial_x\lambda_1|}{4}\{w_1^2 - C(w_2^2 + w_3^2)\}\,\mathrm{d}x\,\mathrm{d}t \le I_{22}.$$

Combining the two estimates above, we obtain

$$I_2 \ge \int_0^\tau\int_0^\infty \frac{|\partial_x\lambda_3|}{4}\{w_3^2 - C(w_1^2 + w_2^2)\}\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_{-\infty}^0 \frac{|\partial_x\lambda_1|}{4}\{w_1^2 - C(w_2^2 + w_3^2)\}\,\mathrm{d}x\,\mathrm{d}t. \tag{5.24}$$

Thus, (5.22) together with (5.24) leads to the following estimate. For any $\gamma_1 > 0$,

$$\left(\int_{-\infty}^\infty \frac{1}{2}(\langle W_0 \mid W_0\rangle + \langle \mathbf{P}_1\xi_1 W_0 \mid L^{-1}J_1\rangle)\,\mathrm{d}x\right)\Bigg|_{t=0}^{t=\tau} - C\int_0^\tau\int_{-\infty}^\infty \langle \mathbf{P}_1\xi_1 J_0 \mid L^{-1} \mid \mathbf{P}_1\xi_1 J_0\rangle\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_0^\infty \frac{|\partial_x\lambda_3|}{4}\{w_3^2 - C(w_1^2 + w_2^2)\}\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_{-\infty}^0 \frac{|\partial_x\lambda_1|}{4}\{w_1^2 - C(w_2^2 + w_3^2)\}\,\mathrm{d}x\,\mathrm{d}t \quad \le \tag{5.25}$$

$$\leq \int_0^\tau \int_{-\infty}^{\infty} C\gamma_1 \varepsilon^2 \mathcal{E}(x,t) \|W_0\|_{L^2_\xi}^2 + \frac{C}{\gamma_1} \|J_1\|_{\mathrm{ref},L^2_\xi}^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$+ C(\|W_0\|_\infty + \gamma_1) \left| \left(\int_{-\infty}^{\infty} (\rho^2 + \mathbb{M}^2)\,\mathrm{d}x \right) \Big|_{t=0}^{t=\tau} \right|$$
$$+ C \int_0^\tau \int_{-\infty}^{\infty} \frac{1}{\gamma_1} \{\varepsilon^3 e^{-(c_0/2)\varepsilon(|x|+|s(t+t_0)|)}\}^{-1} \langle E(\varphi) \mid E(\varphi)\rangle \,\mathrm{d}x\,\mathrm{d}t.$$

It remains to estimate the non-positive term

$$\int_0^\tau \int_0^\infty \frac{|\partial_x \lambda_3|}{4} \{-C(w_1^2 + w_2^2)\}\,\mathrm{d}x\,\mathrm{d}t.$$

Originated from the transversal wave estimates for the stability analysis of a viscous shock profile [11], [16], we instead consider the following integral:

$$\begin{aligned}
\text{(i)} &\equiv \int_0^\tau \int_0^\infty \partial_x(\lambda_3 - s)\langle W_0 \mid (\xi^1 - s)W_0\rangle \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\tau \int_0^\infty \partial_x(\lambda_3 - s)\Big(\sum_{i=1}^3 (\lambda_i - s) w_i^2\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\geq \int_0^\tau \int_0^\infty |\partial_x \lambda_3| \Big(\sum_{i=1}^3 C(w_1^2 + w_2^2) - C\varepsilon w_3^2\Big)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

We note that $\lambda_3 - s = O(1)\varepsilon$ and $\lambda_1 + s = O(1)\varepsilon$. Similarly, we consider the following integral:

$$\begin{aligned}
\text{(ii)} &\equiv \int_0^\tau \int_{-\infty}^0 \partial_x(\lambda_1 + s)\langle W_0 \mid (\xi^1 + s)W_0\rangle \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\tau \int_{-\infty}^0 \partial_x(\lambda_1 + s)\Big(\sum_{i=1}^3 (\lambda_i + s) w_i^2\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\geq \int_0^\tau \int_{-\infty}^0 |\partial_x \lambda_1| \Big(\sum_{i=1}^3 C(w_2^2 + w_3^2) - C\varepsilon w_1^2\Big)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\text{(5.26)}\qquad \text{(i)} + \text{(ii)} \geq &\int_0^\tau \int_0^\infty |\partial_x \lambda_3| \Big(\sum_{i=1}^3 C(w_1^2 + w_2^2) - C\varepsilon w_3^2\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau \int_{-\infty}^0 |\partial_x \lambda_1| \Big(\sum_{i=1}^3 C(w_2^2 + w_3^2) - C\varepsilon w_1^2\Big)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

However, apply integration by parts to (i) and (ii) separately,

$$\begin{aligned}
\text{(i)} &= -\int_0^\tau \int_0^\infty (\lambda_3 - s)\, \partial_x \langle W_0 \mid (\xi^1 - s) W_0 \rangle \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^\tau ((\lambda_3 - s)\langle W_0 \mid (\xi^1 - s) W_0 \rangle)\big|_{x=0}^{x=\infty} \, \mathrm{d}t, \\
\text{(ii)} &= -\int_0^\tau \int_{-\infty}^0 (\lambda_1 + s)\, \partial_x \langle W_0 \mid (\xi^1 + s) W_0 \rangle \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^\tau ((\lambda_1 + s)\langle W_0 \mid (\xi^1 + s) W_0 \rangle)\big|_{x=-\infty}^{x=0} \, \mathrm{d}t.
\end{aligned}$$

When $x = 0$, $\lambda_1 = -\lambda_3$, $\lambda_2 = 0$, and $w_1 = -w_3$ by the symmetry of the approximate solution. We thus have

$$\begin{aligned}
\text{(i)} + \text{(ii)} &= -\int_0^\tau \int_0^\infty (\lambda_3 - s)\, \partial_x \langle W_0 \mid (\xi^1 - s) W_0 \rangle \, \mathrm{d}x \, \mathrm{d}t \\
&\quad - \int_0^\tau \int_{-\infty}^0 (\lambda_1 + s)\, \partial_x \langle W_0 \mid (\xi^1 + s) W_0 \rangle \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^\tau 2(\lambda_1 + s)\lambda_1 (w_1^2 - w_3^2)\big|_{x=0} \, \mathrm{d}t \\
&= -\int_0^\tau \int_0^\infty (\lambda_3 - s)\, \partial_x \langle W_0 \mid (\xi^1 - s) W_0 \rangle \, \mathrm{d}x \, \mathrm{d}t \\
&\quad - \int_0^\tau \int_{-\infty}^0 (\lambda_1 + s)\, \partial_x \langle W_0 \mid (\xi^1 + s) W_0 \rangle \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}$$

Due to the symmetric property of the approximate solution again and equation (5.1),

$$\begin{aligned}
\text{(i)} + \text{(ii)} &= -\int_0^\tau \int_0^\infty (\lambda_3 - s) 2 \langle W_0 \mid (\xi^1 - s) \mathbf{P}_0 \, \partial_x W_0 \rangle \, \mathrm{d}x \, \mathrm{d}t \\
&\quad - \int_0^\tau \int_{-\infty}^0 (\lambda_1 + s) 2 \langle W_0 \mid (\xi^1 + s) \mathbf{P}_0 \, \partial_x W_0 \rangle \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^\tau \int_{-\infty}^\infty O(1) \varepsilon^3 \mathcal{E}(x,t) \langle W_0 \mid W_0 \rangle \, \mathrm{d}x \, \mathrm{d}t \\
&= -\int_0^\tau \int_0^\infty (\lambda_3 - s) 2 \langle W_0 \mid -\partial_t W_0 - \xi_1 J_1 \rangle \, \mathrm{d}x \, \mathrm{d}t \\
&\quad - \int_0^\tau \int_0^\infty (\lambda_3 - s) 2 \langle W_0 \mid -s \, \partial_x W_0 \rangle \, \mathrm{d}x \, \mathrm{d}t \\
&\quad - \int_0^\tau \int_{-\infty}^0 (\lambda_1 + s) 2 \langle W_0 \mid -\partial_t W_0 - \xi_1 J_1 \rangle \, \mathrm{d}x \, \mathrm{d}t \\
&\quad - \int_0^\tau \int_{-\infty}^0 (\lambda_1 + s) 2 \langle W_0 \mid s \, \partial_x W_0 \rangle \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^\tau \int_{-\infty}^\infty O(1) \varepsilon^3 \mathcal{E}(x,t) \langle W_0 \mid W_0 \rangle \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}$$

Also due to the construction in Section 3, $(\lambda_3 - s)(x) = -(\lambda_1 + s)(-x)$ and

$$\langle W_0(x) \mid \partial_x W_0(x) \rangle = \langle W_0(-x) \mid \partial_x (W_0(-x)) \rangle.$$

It thus follows from the above equality that

$$\begin{aligned}
(5.27) \qquad & \text{(i)} + \text{(ii)} + \int_0^\tau \int_0^\infty (\lambda_3 - s) 2 \langle W_0 \mid -\partial_t W_0 - \xi_1 J_1 \rangle \, dx \, dt \\
&+ \int_0^\tau \int_{-\infty}^0 (\lambda_1 + s) 2 \langle W_0 \mid -\partial_t W_0 - \xi_1 J_1 \rangle \, dx \, dt \\
&+ \int_0^\tau \int_{-\infty}^\infty O(1) \varepsilon^3 \mathcal{E}(x,t) \langle W_0 \mid W_0 \rangle \, dx \, dt = 0.
\end{aligned}$$

Note that (5.27) also implies that

$$\begin{aligned}
\text{(i)} + \text{(ii)} = & \left(\int_0^\infty (\lambda_3 - s) \langle W_0 \mid W_0 \rangle \, dx \right) \Big|_{t=0}^{t=\tau} \\
&+ 2 \int_0^\tau \int_0^\infty (\lambda_3 - s) \langle W_0 \mid \xi_1 J_1 \rangle \, dx \, dt \\
&+ \left(\int_{-\infty}^0 (\lambda_1 + s) \langle W_0 \mid W_0 \rangle \, dx \right) \Big|_{t=0}^{t=\tau} \\
&+ 2 \int_0^\tau \int_{-\infty}^0 (\lambda_1 + s) \langle W_0 \mid \xi_1 J_1 \rangle \, dx \, dt \\
&+ \int_0^\tau \int_0^\infty \partial_t (\lambda_3 - s) \langle W_0 \mid W_0 \rangle \, dx \, dt \\
&+ \int_0^\tau \int_{-\infty}^0 \partial_t (\lambda_1 + s) \langle W_0 \mid W_0 \rangle \, dx \, dt \\
&+ \int_0^\tau \int_{-\infty}^\infty O(1) \varepsilon^3 \mathcal{E}(x,t) \langle W_0 \mid W_0 \rangle \, dx \, dt.
\end{aligned}$$

We finally obtain

$$\begin{aligned}
\text{(i)} + \text{(ii)} = & \left(\int_0^\infty (\lambda_3 - s) \langle W_0 \mid W_0 \rangle \, dx \right) \Big|_{t=0}^{t=\tau} \\
&+ 2 \int_0^\tau \int_0^\infty (\lambda_3 - s) \langle W_0 \mid \xi_1 J_1 \rangle \, dx \, dt \\
&+ \left(\int_{-\infty}^0 (\lambda_1 + s) \langle W_0 \mid W_0 \rangle \, dx \right) \Big|_{t=0}^{t=\tau} \\
&+ 2 \int_0^\tau \int_{-\infty}^0 (\lambda_1 + s) \langle W_0 \mid \xi_1 J_1 \rangle \, dx \, dt \\
&+ \int_0^\tau \int_{-\infty}^\infty O(1) \varepsilon^3 \mathcal{E}(x,t) \langle W_0 \mid W_0 \rangle \, dx \, dt.
\end{aligned}$$

Therefore, (5.26) together with the above equality leads to the following estimate

$$
\begin{aligned}
(5.28)\qquad & \int_0^\tau \int_0^\infty |\partial_x \lambda_3| \Big(\sum_{i=1}^3 C(w_1^2 + w_2^2) - C\varepsilon w_3^2 \Big) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau \int_{-\infty}^0 |\partial_x \lambda_1| \Big(\sum_{i=1}^3 C(w_2^2 + w_3^2) - C\varepsilon w_1^2 \Big) \,\mathrm{d}x\,\mathrm{d}t \\
&\le \Big(\int_0^\infty (\lambda_3 - s) \langle W_0 \mid W_0 \rangle \,\mathrm{d}x \Big) \Big|_{t=0}^{t=\tau} \\
&\quad + 2 \int_0^\tau \int_0^\infty (\lambda_3 - s) \langle W_0 \mid \xi_1 J_1 \rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \Big(\int_{-\infty}^0 (\lambda_1 + s) \langle W_0 \mid W_0 \rangle \,\mathrm{d}x \Big) \Big|_{t=0}^{t=\tau} \\
&\quad + 2 \int_0^\tau \int_{-\infty}^0 (\lambda_1 + s) \langle W_0 \mid \xi_1 J_1 \rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau \int_{-\infty}^\infty O(1) \varepsilon^3 \mathcal{E}(x,t) \langle W_0 \mid W_0 \rangle \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

However, $\lambda_3 - s = O(1)\varepsilon$ and $\lambda_1 + s = O(1)\varepsilon$; we thus obtain by (3.7) that

$$
\begin{aligned}
(5.29)\qquad & \int_0^\tau \int_0^\infty (\lambda_3 - s) \langle W_0 \mid \xi_1 J_1 \rangle \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau \int_{-\infty}^0 (\lambda_1 + s) \langle W_0 \mid \xi_1 J_1 \rangle \,\mathrm{d}x\,\mathrm{d}t = O(1)\varepsilon (I_3 + \cdots + I_9).
\end{aligned}
$$

By applying (5.28), (5.29) and the previous lower order energy estimates of I_i, $i = 3, \dots, 9$ in Section 5.1, it thus follows that for $\gamma_1 > 0$,

$$
\begin{aligned}
(5.30)\qquad & \int_0^\tau \int_0^\infty |\partial_x \lambda_3| \Big(\sum_{i=1}^3 C(w_1^2 + w_2^2) - C\varepsilon w_3^2 \Big) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau \int_{-\infty}^0 |\partial_x \lambda_1| \Big(\sum_{i=1}^3 C(w_2^2 + w_3^2) - C\varepsilon w_1^2 \Big) \,\mathrm{d}x\,\mathrm{d}t \\
&\le O(1)\varepsilon \int_{-\infty}^\infty (\|W_0\|_{L^2_\xi}^2 + \|J_1\|_{\mathrm{ref},L^2_\xi}^2)\,|_{t=0} + (\|W_0\|_{L^2_\xi}^2 + \|J_1\|_{\mathrm{ref},L^2_\xi}^2)\,|_{t=\tau} \,\mathrm{d}x \\
&\quad + \int_0^\tau \int_{-\infty}^\infty \Big\{ C(1+\gamma_1)\varepsilon^3 \mathcal{E}(x,t) \|W_0\|_{L^2_\xi}^2 + \frac{C}{\gamma_1} \varepsilon \|J_1\|_{\mathrm{ref},L^2_\xi}^2 \Big\} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\varepsilon (\|W_0\|_\infty + \gamma_1) \Big| \Big(\int_{-\infty}^\infty \rho^2 \,\mathrm{d}x \Big) \Big|_{t=0}^{t=\tau} \Big| \\
&\quad + C \int_0^\tau \int_{-\infty}^\infty \frac{\varepsilon}{\gamma_1} \{\varepsilon^3 e^{-(c_0/2)\varepsilon(|x| + |s(t+t_0)|)}\}^{-1} \langle E(\varphi) \mid E(\varphi) \rangle \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

We now choose γ_1 and γ_2 to satisfy

$$0 < \varepsilon \ll \gamma_1 \ll 1, \quad 0 < \gamma_2 \ll \gamma_1.$$

Under the smallness assumption (5.8) with $\delta_0 \ll \gamma_2$, we compute

$$(5.25) + \frac{1}{\gamma_1}(5.30) + \frac{1}{\gamma_2}(5.23)$$

to obtain the final lower order energy estimate:

$$\begin{aligned}
(5.31)\quad & \left(\int_{-\infty}^{\infty} \left(\frac{1}{2} + C_2 \frac{\varepsilon}{\gamma_1} \right) ||W_0||^2_{L^2_\xi} \right. \\
&\quad \left. + \left(\frac{1}{4} + C_2 \frac{\varepsilon}{\gamma_1} + \frac{1}{2\gamma_2} \right) ||J||^2_{\mathrm{ref},L^2_\xi} \, \mathrm{d}x \right) \Bigg|_{t=\tau} \\
&\quad - C_0 \int_0^\tau \int_{-\infty}^{\infty} \langle \mathbf{P}_1 \xi_1 J_0 \mid L^{-1} \mid \mathbf{P}_1 \xi_1 J_0 \rangle \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^\tau \int_0^\infty C_1 |\partial_x \lambda_3| \left\{ w_3^2 + \frac{1}{\gamma_1}(w_1^2 + w_2^2) \right\} \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^\tau \int_{-\infty}^0 C_1 |\partial_x \lambda_1| \left\{ w_1^2 + \frac{1}{\gamma_1}(w_2^2 + w_3^2) \right\} \mathrm{d}x \, \mathrm{d}t \\
&\le 2 \left(\int_{-\infty}^{\infty} \left(\frac{1}{2} + C_2 \frac{\varepsilon}{\gamma_1} \right) ||W_0||^2_{L^2_\xi} \right. \\
&\quad \left. + \left(1 + C_2 \frac{\varepsilon}{\gamma_1} + \frac{1}{2\gamma_2} \right) ||J||^2_{\mathrm{ref},L^2_\xi} \, \mathrm{d}x \right) \Bigg|_{t=0} \\
&\quad + C_0 \int_0^\tau \int_{-\infty}^{\infty} \frac{1}{\gamma_1} \{ \varepsilon^3 e^{-(c_0/2)\varepsilon(|x| + |s(t+t_0)|)} \}^{-1} \langle E(\varphi) \mid E(\varphi) \rangle \, \mathrm{d}x \, \mathrm{d}t
\end{aligned}$$

where C_0, C_1, and C_2 are positive constants.

In summary, we arrive at the following result.

Proposition 5.5 (Lower order estimate)**.** *Let W_0 and J be the solutions of the following equations:*

$$\mathbf{P}_0 \, \partial_t W_0 + \mathbf{P}_0 \xi_1 \mathbf{P}_0 \, \partial_x W_0 + \mathbf{P}_0 \xi_1 J_1 = 0,$$

$$\begin{aligned}
&\mathbf{P}_1 \, \partial_t J_0 + \mathbf{P}_1 \, \partial_t J_1 + \mathbf{P}_1 \xi_1 \, \partial_x J_0 + \mathbf{P}_1 \xi_1 \, \partial_x J_1 \\
&\quad = \left(1 + \frac{\rho_J}{\rho_{\mathrm{NS}}} \right) L(J_1) + D(J) + N(J) - \mathbf{P}_1(E(\varphi)).
\end{aligned}$$

Suppose that the initial state $J(x,0,\xi)$ satisfies

$$\int_{-\infty}^{\infty}\int_{R^3}\begin{pmatrix}1\\ \xi_i\\ |\xi|^2\end{pmatrix} J(x,0,\xi)\,\mathrm{d}\xi\,\mathrm{d}x = 0, \quad \text{for } i = 1,2,3.$$

Under the smallness assumption

$$\begin{aligned}&\sum_{|\alpha|\le 4}\Big(\|\partial_x^\alpha W_0\|_{L^\infty_{x,t}(L^2_\xi)} + \|\partial_x^\alpha \partial_t W_0\|_{L^\infty_{x,t}(L^2_\xi)}\Big)\\ &\quad + \sum_{|\alpha|\le 3}\Big(\|\partial_x^\alpha\{(1+|\xi|)^{1/2}J_1\}\|_{\mathrm{ref},L^\infty_{x,t}(L^2_\xi)} + \|\partial_x^\alpha \partial_t\{(1+|\xi|)^{1/2}J_1\}\|_{\mathrm{ref},L^\infty_{x,t}(L^2_\xi)}\Big)\\ &\qquad \le \delta_0,\end{aligned}$$

here $\delta_0 \ll 1$, there exist positive constants C_0, C_1 and C_2 independent of τ such that for $\tau > 0$,

$$\begin{aligned}&\bigg(\int_{-\infty}^{\infty}\Big(\frac{1}{2} + C_2\frac{\varepsilon}{\gamma_1}\Big)||W_0||^2_{L^2_\xi} + \Big(\frac{1}{4} + C_2\frac{\varepsilon}{\gamma_1} + \frac{1}{2\gamma_2}\Big)||J||^2_{\mathrm{ref},L^2_\xi}\,\mathrm{d}x\bigg)\bigg|_{t=\tau}\\ &\quad - C_0\int_0^\tau\int_{-\infty}^{\infty}\langle \mathbf{P}_1\xi_1 J_0 \mid L^{-1} \mid \mathbf{P}_1\xi_1 J_0\rangle\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_0^\tau\int_0^{\infty} C_1|\partial_x\lambda_3|\Big\{w_3^2 + \frac{1}{\gamma_1}(w_1^2 + w_2^2)\Big\}\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_0^\tau\int_{-\infty}^{0} C_1|\partial_x\lambda_1|\Big\{w_1^2 + \frac{1}{\gamma_1}(w_2^2 + w_3^2)\Big\}\,\mathrm{d}x\,\mathrm{d}t\\ &\le 2\bigg(\int_{-\infty}^{\infty}\Big(\frac{1}{2} + C_2\frac{\varepsilon}{\gamma_1}\Big)||W_0||^2_{L^2_\xi} + \Big(1 + C_2\frac{\varepsilon}{\gamma_1} + \frac{1}{2\gamma_2}\Big)||J||^2_{\mathrm{ref},L^2_\xi}\,\mathrm{d}x\bigg)\bigg|_{t=0}\\ &\quad + C_0\int_0^\tau\int_{-\infty}^{\infty}\frac{1}{\gamma_1}\{\varepsilon^3 e^{-(c_0/2)\varepsilon(|x|+|s(t+t_0)|)}\}^{-1}\langle E(\varphi) \mid E(\varphi)\rangle\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where $\langle E(\varphi) \mid E(\varphi)\rangle^{1/2}(x,t) \le \hat{C}\varepsilon^4 e^{-(c_0/2)\varepsilon(|x|+|s(t+t_0)|)}$ for some constant $\hat{C} >$ 0.

Remark 5.6. In the above lower order energy estimate, we obtain the integral $(\int_{-\infty}^{\infty}\|J\|^2_{\mathrm{ref},L^2_\xi}\,\mathrm{d}x)|_{t=\tau}$, instead of $(\int_{-\infty}^{\infty}\|(1+|\xi|)^{1/2}J\|^2_{\mathrm{ref},L^2_\xi}\,\mathrm{d}x)|_{t=\tau}$ which is needed in the a priori assumption (5.8). We shall resolve this problem in the next section.

5.3. Higher order energy estimates. To establish the higher order energy estimates, we now consider the combination of equations (3.8), (1.4), and (5.27):

$$\sum_{0\le i\le 6}\frac{1}{\gamma^i}\int_0^\tau\int_R\langle\partial_x^i W_0 \mid \partial_x^i(3.8)\rangle\,\mathrm{d}x\,\mathrm{d}t$$

$$+\sum_{0\le i\le 6}\frac{1}{\gamma^{i+1}}\int_0^\tau\int_R\langle\partial_x^i J,\partial_x^i(1.4)\rangle_{\mathrm{ref}}\,\mathrm{d}x\,\mathrm{d}t$$

$$+\sum_{0\le i\le 6}\frac{1}{\gamma^i}\int_0^\tau\int_R\langle\partial_x^i\,\partial_t W_0 \mid \partial_x^i\,\partial_t(3.8)\rangle\,\mathrm{d}x\,\mathrm{d}t$$

$$+\sum_{0\le i\le 6}\frac{1}{\gamma^{i+1}}\int_0^\tau\int_R\langle\partial_x^i\,\partial_t J,\partial_x^i\,\partial_t(1.4)\rangle_{\mathrm{ref}}\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{\gamma_0}(5.27).$$

We state the equations (3.8), (1.4), and (5.27) again.

$$\begin{aligned}(3.8)\qquad & \mathbf{P}_0\,\partial_t W_0+\mathbf{P}_0\xi_1\mathbf{P}_0\,\partial_x W_0\\ &\quad+\mathbf{P}_0\xi_1 L^{-1}(\mathbf{P}_1\,\partial_t J_0+\mathbf{P}_1\,\partial_t J_1+\mathbf{P}_1\xi_1\,\partial_x J_0+\mathbf{P}_1\xi_1\,\partial_x J_1)\\ &\quad-\mathbf{P}_0\xi_1 L^{-1}[D(J)+\mathcal{N}(J)+\mathbf{P}_1(E(\varphi))]=0,\end{aligned}$$

$$(1.4)\qquad J_t+\xi_1 J_x-L(J)-D(J)-\mathcal{N}(J)+E(\varphi)=0,$$

$$\begin{aligned}(5.27)\qquad & \int_0^\tau\int_0^\infty\partial_x(\lambda_3-s)\langle W_0 \mid (\xi^1-s)W_0\rangle\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_0^\tau\int_{-\infty}^0\partial_x(\lambda_1+s)\langle W_0 \mid (\xi^1+s)W_0\rangle\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_0^\tau\int_0^\infty(\lambda_3-s)2\langle W_0 \mid -\partial_t W_0-\xi_1 J_1\rangle\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_0^\tau\int_{-\infty}^0(\lambda_1+s)2\langle W_0 \mid -\partial_t W_0-\xi_1 J_1\rangle\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_0^\tau\int_{-\infty}^\infty O(1)\varepsilon^3\mathcal{E}(x,t)\langle W_0 \mid W_0\rangle\,\mathrm{d}x\,\mathrm{d}t=0.\end{aligned}$$

In this subsection, we will carry out the estimates of the following equality:

$$
\begin{aligned}
(\star)\quad 0 = &\sum_{0\le i\le 6}\frac{1}{\gamma^i}\int_0^\tau\int_R \langle \partial_x^i W_0 \mid \partial_x^i(\mathbf{P}_0\,\partial_t W_0 + \mathbf{P}_0\xi_1\mathbf{P}_0\,\partial_x W_0)\rangle\,\mathrm{d}x\,\mathrm{d}t\\
&+\sum_{0\le i\le 6}\frac{1}{\gamma^i}\int_0^\tau\int_R \Big\langle \partial_x^i W_0 \mid \partial_x^i\Big\{\mathbf{P}_0\xi_1 L^{-1}\Big(\mathbf{P}_1\,\partial_t J_0 + \mathbf{P}_1\,\partial_t J_1 + \mathbf{P}_1\xi_1\,\partial_x J_0\\
&\qquad\qquad + \mathbf{P}_1\xi_1\,\partial_x J_1\Big)\Big\}\Big\rangle\,\mathrm{d}x\,\mathrm{d}t\\
&+\sum_{0\le i\le 6}\frac{1}{\gamma^i}\int_0^\tau\int_R \Big\langle \partial_x^i W_0 \mid \partial_x^i\Big\{-\mathbf{P}_0\xi_1 L^{-1}\Big[D(J) + \mathcal{N}(J)\\
&\qquad\qquad + \mathbf{P}_1(E(\varphi))\Big]\Big\}\Big\rangle\,\mathrm{d}x\,\mathrm{d}t\\
&+\sum_{0\le i\le 6}\frac{1}{\gamma^i}\int_0^\tau\int_R \langle \partial_x^i\,\partial_t W_0 \mid \partial_x^i\,\partial_t(\mathbf{P}_0\,\partial_t W_0 + \mathbf{P}_0\xi_1\mathbf{P}_0\,\partial_x W_0)\rangle\,\mathrm{d}x\,\mathrm{d}t\\
&+\sum_{0\le i\le 6}\frac{1}{\gamma^i}\int_0^\tau\int_R \Big\langle \partial_x^i\,\partial_t W_0 \mid \partial_x^i\,\partial_t\Big\{\mathbf{P}_0\xi_1 L^{-1}\Big(\mathbf{P}_1\,\partial_t J_0 + \mathbf{P}_1\,\partial_t J_1\\
&\qquad\qquad + \mathbf{P}_1\xi_1\,\partial_x J_0 + \mathbf{P}_1\xi_1\,\partial_x J_1\Big)\Big\}\Big\rangle\,\mathrm{d}x\,\mathrm{d}t\\
&+\sum_{0\le i\le 6}\frac{1}{\gamma^i}\int_0^\tau\int_R \Big\langle \partial_x^i\,\partial_t W_0 \mid \partial_x^i\,\partial_t\Big\{-\mathbf{P}_0\xi_1 L^{-1}\Big[D(J) + \mathcal{N}(J)\\
&\qquad\qquad + \mathbf{P}_1(E(\varphi))\Big]\Big\}\Big\rangle\,\mathrm{d}x\,\mathrm{d}t\\
&+\sum_{0\le i\le 6}\frac{1}{\gamma^{i+1}}\int_0^\tau\int_R \Big\langle \partial_x^i J, \partial_x^i\Big(J_t + \xi_1 J_x - L(J) - D(J) - \mathcal{N}(J)\\
&\qquad\qquad + E(\varphi)\Big)\Big\rangle_{\mathrm{ref}}\,\mathrm{d}x\,\mathrm{d}t\\
&+\sum_{0\le i\le 6}\frac{1}{\gamma^{i+1}}\int_0^\tau\int_{\mathbb{R}} \Big\langle \partial_x^i\,\partial_t J, \partial_x^i\,\partial_t\Big(J_t + \xi_1 J_x - L(J) - D(J) - \mathcal{N}(J)\\
&\qquad\qquad + E(\varphi)\Big)\Big\rangle_{\mathrm{ref}}\,\mathrm{d}x\,\mathrm{d}t\\
&+\frac{1}{\gamma_0}\int_0^\tau\int_0^\infty \partial_x(\lambda_3 - s)\langle W_0 \mid (\xi^1 - s)W_0\rangle\,\mathrm{d}x\,\mathrm{d}t\\
&+\frac{1}{\gamma_0}\int_0^\tau\int_{-\infty}^0 \partial_x(\lambda_1 + s)\langle W_0 \mid (\xi^1 + s)W_0\rangle\,\mathrm{d}x\,\mathrm{d}t\\
&+\frac{1}{\gamma_0}\int_0^\tau\int_0^\infty (\lambda_3 - s)2\langle W_0 \mid -\partial_t W_0 - \xi_1 J_1\rangle\,\mathrm{d}x\,\mathrm{d}t\\
&+\frac{1}{\gamma_0}\int_0^\tau\int_{-\infty}^0 (\lambda_1 + s)2\langle W_0 \mid -\partial_t W_0 - \xi_1 J_1\rangle\,\mathrm{d}x\,\mathrm{d}t\\
&+\frac{1}{\gamma_0}\int_0^\tau\int_{-\infty}^\infty O(1)\varepsilon^3\mathcal{E}(x,t)\langle W_0 \mid W_0\rangle\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

In what follows, we will calculate the derivatives of those terms shown in the above equality. Since some estimates are analogous to the corresponding ones appearing in the lower order energy estimates, we only write down those to be treated carefully.

Lemma 5.7. *For $i \geq 0$,*

$$\langle \mathbf{P}_1 \xi_1 \, \partial_x J_0 \mid \mathbf{P}_1 \xi_1 \, \partial_x J_0 \rangle = O(1) \langle \partial_x \mathbf{h} \mid \partial_x \mathbf{h} \rangle + O(1) \rho_J^2 \varepsilon^4 \mathcal{E}^2(x,t),$$

$$||\partial_x^i \mathbf{h}||_{L^2_\xi}^2 = O(1) \sum_{j=0}^{i} (|\partial_x^j m|^2 + |\partial_x^j e|^2) + O(1)\varepsilon^4 \mathcal{E}^2(x,t) ||W_0||_\infty^2,$$

$$||\partial_x^i \, \partial_t \mathbf{h}||_{L^2_\xi}^2 = O(1) \sum_{j=0}^{i} (|\partial_x^j \, \partial_t m|^2 + |\partial_x^j \, \partial_t e|^2 + |\partial_x^j m|^2 + |\partial_x^j e|^2) + O(1)\varepsilon^4 \mathcal{E}^2(x,t) ||W_0||_\infty^2.$$

Proof. Since

$$\mathbf{h} = m\psi_1 \omega_{\mathrm{tr}} + e\psi_4 \omega_{\mathrm{tr}} + O(1)\varepsilon^2 \mathcal{E}(x,t) \|W_0\|_\infty \Big(\sum \psi_i \omega \Big),$$

the estimates can be verified by Lemma 4.2 and direct calculations. □

Lemma 5.8. *For $i \geq 0$,*

$$\begin{aligned}
&\langle \partial_x^i J_1, \partial_x^i N(J) \rangle_{\mathrm{ref}} \\
&\leq O(1) \sum_{0 \leq \beta \leq [i/2]} \Big\{ \|\partial_x^\beta J_1\|_{\mathrm{ref}, L^\infty_{x,t}(L^2_\xi)} + \|\partial_x^\beta m\|_\infty + \|\partial_x^\beta e\|_\infty \Big\} \\
&\quad \cdot \sum_{0 \leq \beta \leq [i/2]} \Big\{ ||(1+|\xi|)^{1/2} \, \partial_x^{i-\beta} J_1||_{\mathrm{ref},(L^2_\xi)}^2 + |\partial_x^{i-\beta} m|^2 + |\partial_x^{i-\beta} e|^2 + \mathcal{R}(x,t) \Big\} \\
&\quad + O(1) \sum_{0 \leq \beta \leq [i/2]} \Big\{ \|(1+|\xi|)^{1/2} \, \partial_x^\beta J_1\|_{\mathrm{ref}, L^\infty_{x,t}(L^2_\xi)} + \|\partial_x^\beta m\|_\infty + \|\partial_x^\beta e\|_\infty \Big\} \\
&\quad \cdot \sum_{0 \leq \beta \leq [i/2]} \Big\{ ||\partial_x^{i-\beta} J_1||_{\mathrm{ref},(L^2_\xi)}^2 + |\partial_x^{i-\beta} m|^2 + |\partial_x^{i-\beta} e|^2 + \mathcal{R}(x,t) \Big\},
\end{aligned}$$

and

$$
\begin{aligned}
&\langle \partial_x^i \partial_t J_1, \partial_x^i \partial_t N(J)\rangle_{\mathrm{ref}} \\
&\le O(1)\Big\{ \sum_{0\le\beta\le[i/2]} \Big(\|\partial_x^\beta \partial_t J_1\|_{\mathrm{ref},L^\infty_{x,t}(L^2_\xi)} + \|\partial_x^\beta \partial_t m\|_\infty + \|\partial_x^\beta \partial_t e\|_\infty \\
&\quad + \|\partial_x^\beta J_1\|_{\mathrm{ref},L^\infty_{x,t}(L^2_\xi)} + \|\partial_x^\beta m\|_\infty + \|\partial_x^\beta e\|_\infty \Big)\Big\} \\
&\quad \cdot \Big\{ \sum_{0\le\beta\le[i/2]} \Big(||(1+|\xi|)^{1/2} \partial_x^{i-\beta} \partial_t J_1||^2_{\mathrm{ref},(L^2_\xi)} + |\partial_x^{i-\beta} \partial_t m|^2 + |\partial_x^{i-\beta} \partial_t e|^2 \\
&\quad + ||(1+|\xi|)^{1/2} \partial_x^{i-\beta} J_1||^2_{\mathrm{ref},(L^2_\xi)} + |\partial_x^{i-\beta} m|^2 + |\partial_x^{i-\beta} e|^2 + \mathcal{R}(x,t)\Big)\Big\} \\
&\quad + O(1)\Big\{ \sum_{0\le\beta\le[i/2]} \Big(\|(1+|\xi|)^{1/2} \partial_x^\beta \partial_t J_1\|_{\mathrm{ref},L^\infty_{x,t}(L^2_\xi)} + \|\partial_x^\beta \partial_t m\|_\infty \\
&\quad + \|\partial_x^\beta \partial_t e\|_\infty + \|(1+|\xi|)^{1/2} \partial_x^\beta J_1\|_{\mathrm{ref},L^\infty_{x,t}(L^2_\xi)} + \|\partial_x^\beta m\|_\infty + \|\partial_x^\beta e\|_\infty \Big)\Big\} \\
&\quad \cdot \Big\{ \sum_{0\le\beta\le[i/2]} \Big(||\partial_x^{i-\beta} \partial_t J_1||^2_{\mathrm{ref},(L^2_\xi)} [0.2cm] + |\partial_x^{i-\beta} \partial_t m|^2 + |\partial_x^{i-\beta} \partial_t e|^2 \\
&\quad + ||\partial_x^{i-\beta} J_1||^2_{\mathrm{ref},(L^2_\xi)} + |\partial_x^{i-\beta} m|^2 + |\partial_x^{i-\beta} e|^2 + \mathcal{R}(x,t)\Big)\Big\},
\end{aligned}
$$

where $\mathcal{R}(x,t) = \varepsilon^4 \mathcal{E}^2(x,t)\|W_0\|^2_\infty$ *and* $[\ell]$ *denotes the largest integer less than* ℓ.

Proof. Write

$$
\langle J_1, N(J)\rangle_{\mathrm{ref}} = \langle (1+|\xi|)^{1/2} J_1, (1+|\xi|)^{-1/2} N(J)\rangle_{\mathrm{ref}}.
$$

Since $N(J) = Q(\mathbf{h} + J_1, \mathbf{h} + J_1)$, and

$$
\partial_x^i Q(J,J) = O(1)[Q(\partial_x^i J, J) + Q(\partial_x^{i-1} J, \partial_x J) + \cdots + Q(\partial_x^{[i/2]+1} J, \partial_x^{[i/2]} J)],
$$

we can obtain the estimates by direct calculations and applying Lemma B.1, Lemma 5.2 and Lemma 5.7. ◻

In the following estimates of (a) to (f), we just point out the derivatives of those terms which should be calculated carefully.

(a) When a derivative is applied to the profile φ_{tr}, a factor $\varphi'_{\mathrm{tr}} = O(1)\varepsilon^2 \mathcal{E}(x,t)$ is generated. We obtain the following by calculations:

$$
\begin{aligned}
(5.32)\qquad &\langle \partial_x^i W_0 \mid \partial_x^i (\mathbf{P}_0 \partial_t W_0)\rangle \\
&= \frac{1}{2}\partial_t (|\partial_x^i \mathbb{R}|^2 + |\partial_x^i \mathbb{M}|^2 + |\partial_x^i \mathbb{E}|^2) \\
&\quad + O(1)\varepsilon^2 \mathcal{E}(x,t) \langle \partial_x^i W_0 \mid (\partial_t W_0 + \cdots + \partial_x^{i-1} \partial_t W_0)\rangle.
\end{aligned}
$$

$$
\begin{aligned}
(5.33)\qquad &\langle \partial_x^i \partial_t W_0 \mid \partial_x^i \partial_t (\mathbf{P}_0 \partial_t W_0)\rangle \\
&= \frac{1}{2}\partial_t (|\partial_x^i \partial_t \mathbb{R}|^2 + |\partial_x^i \partial_t \mathbb{M}|^2 + |\partial_x^i \partial_t \mathbb{E}|^2) \\
&\quad + O(1)\varepsilon^2 \mathcal{E}(x,t)\langle \partial_x^i \partial_t W_0 \mid (\partial_t W_0 + \cdots + \partial_x^{i-1}\partial_t W_0)\rangle,
\end{aligned}
$$

where

$$
W_0 = (\mathbb{R}(x,t)\psi_0 + \mathbb{M}(x,t)\psi_1 + \mathbb{E}(x,t)\psi_4)\omega_{\mathrm{tr}}.
$$

(b) Write

$$
\begin{aligned}
\langle \partial_x^i W_0 \mid \partial_x^i (\mathbf{P}_0 \xi_1 \mathbf{P}_0 \partial_x W_0)\rangle &= \langle \partial_x^i W_0 \mid \mathbf{P}_0 \xi_1 \mathbf{P}_0 \partial_x (\partial_x^i W_0)\rangle \\
&\quad + \sum_{j>1} \langle \partial_x^i W_0 \mid \partial_x^{i+1-j} (\mathbf{P}_0 \xi_1 \mathbf{P}_0)\, \partial_x^j W_0\rangle.
\end{aligned}
$$

Since $\mathbf{P}_0 \xi_1 \mathbf{P}_0$ is symmetric, we apply integration by parts to obtain

$$
\begin{aligned}
(5.34)\qquad &\int_{-\infty}^{\infty} \langle \partial_x^i W_0 \mid \partial_x^i (\mathbf{P}_0 \xi_1 \mathbf{P}_0 \partial_x W_0)\rangle \,\mathrm{d}x \\
&\leq O(1) \sum_{[i/2]\leq j\leq i} \int_{-\infty}^{\infty} (|\partial_x^j \mathbb{R}|^2 + |\partial_x^j \mathbb{M}|^2 + |\partial_x^j \mathbb{E}|^2) \cdot \Big(\sum_{1\leq k\leq [i/2]} \varepsilon^{k+1}\mathcal{E}(x,t)\Big)\,\mathrm{d}x.
\end{aligned}
$$

(c) As for $\langle \partial_x^i W_0 \mid \partial_x^i (\mathbf{P}_0 \xi_1 L^{-1} \mathbf{P}_1 \xi_1 \partial_x J_0)\rangle$, we write

$$
\begin{aligned}
(5.35)\qquad &\langle \partial_x^i W_0 \mid \partial_x^i (\mathbf{P}_0 \xi_1 L^{-1} \mathbf{P}_1 \xi_1 \partial_x J_0)\rangle \\
&= \langle \partial_x^i W_0 \mid \partial_x^{i-1} (\mathbf{P}_0 \xi_1 L^{-1} \mathbf{P}_1 \xi_1 \partial_x J_0)\rangle_x \\
&\quad - \{\langle \mathbf{P}_1 \xi_1 \partial_x^i J_0 \mid L^{-1} \mid \mathbf{P}_1 \xi_1 \partial_x^i J_0\rangle + \cdots\}.
\end{aligned}
$$

Then we can apply Lemma 5.7 and the negative definiteness of L to the RHS, which are similar to the corresponding ones in the lower order energy estimate.

(d) We have

$$
\begin{aligned}
\partial_x^i D(J) &= O(1)\Big[Q(\partial_x^i(\varphi - \varphi_{\mathrm{tr}}), J) + \cdots + Q(\varphi - \varphi_{\mathrm{tr}}, \partial_x^i J) \\
&\qquad + Q(\partial_x^i J, \varphi - \varphi_{\mathrm{tr}}) + \cdots + Q(J, \partial_x^i(\varphi - \varphi_{\mathrm{tr}}))\Big], \\
\partial_x^i \mathcal{N}(J) &= \partial_x^i \left[\frac{\rho_J}{\rho_{\mathrm{NS}}} L(J_1)\right] + \partial_x^i N(J).
\end{aligned}
$$

By Lemma B.1, we can obtain

$$
(5.36)\qquad \|\partial_x^i D(J)\|_{\mathrm{ref},L^2_\xi} \leq C \sum_{k=0}^{i} \Big(\varepsilon^{i+1-k}\mathcal{E}(x,t)\|(1+|\xi|)^{1/2}\partial_x^k J\|_{\mathrm{ref},L^2_\xi}\Big),
$$

$$\|\partial_x^i L(J_1)\|_{\mathrm{ref},L^2_\xi} \le C \sum_{k=0}^{i} \Big(\varepsilon^{i+1-k}\mathcal{E}(x,t)\|(1+|\xi|)^{1/2}\,\partial_x^k J_1\|_{\mathrm{ref},L^2_\xi}\Big). \tag{5.37}$$

(e) We write

$$\langle \partial_x^i W_0 \mid \partial_x^i \mathbf{P}_0 \xi_1 L^{-1} D(J)\rangle = \langle \partial_x^i W_0 \mid \mathbf{P}_0 \xi_1 L^{-1}(\partial_x^i D(J))\rangle + \cdots, \tag{5.38}$$

$$\langle \partial_x^i W_0 \mid \partial_x^i \mathbf{P}_0 \xi_1 L^{-1} \mathcal{N}(J)\rangle = \langle \partial_x^i W_0 \mid \mathbf{P}_0 \xi_1 L^{-1}(\partial_x^i \mathcal{N}(J))\rangle + \cdots. \tag{5.39}$$

The RHS can then be estimated by (5.36) and (5.37) and straightforward calculations.

(f) Finally, we have the following estimate

$$\langle \partial^i E(\varphi) \mid \partial^i E(\varphi)\rangle^{1/2}(x,t) = O(1)\varepsilon^{i+2} e^{-\widehat{c_0}\varepsilon(|x|+|s(t+t_0)|)}, \tag{5.40}$$

which can be derived by arguments similar to the estimate (5.20) in Section 5.1.

For the convenience of writing, we use the following notation:

$$\partial_x^i\,\partial_t^j \widehat{W_0} \equiv (\partial_x^i\,\partial_t^j \mathbb{R})\psi_0\omega_{\mathrm{tr}} + (\partial_x^i\,\partial_t^j \mathbb{M})\psi_1\omega_{\mathrm{tr}} + (\partial_x^i\,\partial_t^j \mathbb{E})\psi_4\omega_{\mathrm{tr}},$$

where $i = 0, 1, \dots, 6$, and $j = 0, 1$.

By applying the estimates of (5.32)–(5.40) to $(\star)$ and using arguments similar to those shown in Section 5.1, it follows that under the smallness assumption (5.8)

$$\begin{aligned}
&\frac{1}{C}\sum_{0\le i\le 6}\Big\{\int_{-\infty}^{\infty} \gamma^{-i}(\|\partial_x^i \widehat{W_0}\|^2_{L^2_\xi} + \|\partial_x^i\,\partial_t \widehat{W_0}\|^2_{L^2_\xi}) \\
&\qquad + \gamma^{-i-1}(\|\partial_x^i J\|^2_{\mathrm{ref},L^2_\xi} + \|\partial_x^i\,\partial_t J\|^2_{\mathrm{ref},L^2_\xi})\,dx\Big\}\Big|_{t=\tau} \\
&\qquad + \frac{1}{C}\sum_{0\le i\le 6}\int_0^{\tau}\int_{-\infty}^{\infty} \gamma^{-i}(\|\partial_x^i \mathbf{h}\|^2_{L^2_\xi} + \|\partial_x^i\,\partial_t \mathbf{h}\|^2_{L^2_\xi})\,dx\,dt \\
&\qquad + \frac{1}{C}\sum_{0\le i\le 6}\int_0^{\tau}\int_{-\infty}^{\infty} \gamma^{-i-1}\Big(\|(1+|\xi|)^{1/2}\,\partial_x^i J_1\|^2_{\mathrm{ref},L^2_\xi} \\
&\qquad + \|(1+|\xi|)^{1/2}\,\partial_x^i\,\partial_t J_1\|^2_{\mathrm{ref},L^2_\xi}\Big)\,dx\,dt \\
&\qquad + \frac{1}{C}\int_0^{\tau}\int_{-\infty}^{\infty} \varepsilon^2\mathcal{E}(x,t)\|W_0\|^2_{L^2_\xi}\,dx\,dt \quad \le
\end{aligned} \tag{5.41}$$

$$\leq C \sum_{0\leq i\leq 6} \Big\{ \int_{-\infty}^{\infty} \gamma^{-i} (\|\partial_x^i \widehat{W_0}\|_{L_\xi^2}^2 + \|\partial_x^i\, \partial_t \widehat{W_0}\|_{L_\xi^2}^2)$$

$$+ \gamma^{-i-1} (\|\partial_x^i J\|_{\mathrm{ref},L_\xi^2}^2 + \|\partial_x^i\, \partial_t J\|_{\mathrm{ref},L_\xi^2}^2)\, \mathrm{d}x \Big\} \Big|_{t=0}$$

$$+ C_0 \int_0^{\tau} \int_{-\infty}^{\infty} \{\varepsilon^3 e^{-(c_0/2)\varepsilon(|x|+|s(t+t_0)|)}\}^{-1}$$

$$\times \sum_{0\leq i\leq 6} \{\|\partial_x^i E(\varphi)\|_{L_\xi^2}^2 + \|\partial_x^i\, \partial_t E(\varphi)\|_{L_\xi^2}^2\}\, \mathrm{d}x\, \mathrm{d}t.$$

It remains to estimate the term $(\int_R \|(1+|\xi|)^{1/2}\, \partial_x^i J_1\|_{\mathrm{ref},L_\xi^2}^2\, \mathrm{d}x)|_{t=\tau}$. We need the following inequality:

$$(5.42)\quad \left(\int_R \|(1+|\xi|)^{1/2}\, \partial_x^i J_1\|_{\mathrm{ref},L_\xi^2}^2\, \mathrm{d}x \right)\Big|_{t=\tau}$$
$$\leq \left(\int_R \|(1+|\xi|)^{1/2}\, \partial_x^i J_1\|_{\mathrm{ref},L_\xi^2}^2\, \mathrm{d}x \right)\Big|_{t=0}$$
$$+ \int_0^{\tau} \int_R \|(1+|\xi|)^{1/2}\, \partial_x^i J_1\|_{\mathrm{ref},L_\xi^2}\, \|(1+|\xi|)^{1/2}\, \partial_x^i\, \partial_t J_1\|_{\mathrm{ref},L_\xi^2}\, \mathrm{d}x\, \mathrm{d}t$$
$$\leq \left(\int_R \|(1+|\xi|)^{1/2}\, \partial_x^i J_1\|_{\mathrm{ref},L_\xi^2}^2\, \mathrm{d}x \right)\Big|_{t=0}$$
$$+ \int_0^{\tau} \int_R (\|(1+|\xi|)^{1/2}\, \partial_x^i J_1\|_{\mathrm{ref},L_\xi^2}^2 + \|(1+|\xi|)^{1/2}\, \partial_x^i\, \partial_t J_1\|_{\mathrm{ref},L_\xi^2}^2)\, \mathrm{d}x\, \mathrm{d}t.$$

Combine (5.41) and (5.42) to obtain the final estimate:

Proposition 5.9 (Higher order estimate). *Let W_0 and J be the solutions of the following equations:*

$$\mathbf{P}_0\, \partial_t W_0 + \mathbf{P}_0 \xi_1 \mathbf{P}_0\, \partial_x W_0 + \mathbf{P}_0 \xi_1 J_1 = 0,$$

$$\mathbf{P}_1\, \partial_t J_0 + \mathbf{P}_1\, \partial_t J_1 + \mathbf{P}_1 \xi_1\, \partial_x J_0 + \mathbf{P}_1 \xi_1\, \partial_x J_1$$
$$= \left(1 + \frac{\rho_J}{\rho_{\mathrm{NS}}}\right) L(J_1) + D(J) + N(J) - \mathbf{P}_1(E(\varphi)).$$

Suppose that the initial state $J(x, 0, \xi)$ satisfies

$$\int_{-\infty}^{\infty} \int_{R^3} \begin{pmatrix} 1 \\ \xi_i \\ |\xi|^2 \end{pmatrix} J(x, 0, \xi)\, \mathrm{d}\xi\, \mathrm{d}x = 0, \quad \text{for } i = 1, 2, 3.$$

Under the smallness assumption

$$\sum_{|\alpha|\le 4}\Big(\|\partial_x^\alpha W_0\|_{L^\infty_{x,t}(L^2_\xi)}+\|\partial_x^\alpha\partial_t W_0\|_{L^\infty_{x,t}(L^2_\xi)}\Big)+\sum_{|\alpha|\le 3}\Big(\|\partial_x^\alpha\{(1+|\xi|)^{1/2}J_1\}\|_{\mathrm{ref},L^\infty_{x,t}(L^2_\xi)}+\|\partial_x^\alpha\partial_t\{(1+|\xi|)^{1/2}J_1\}\|_{\mathrm{ref},L^\infty_{x,t}(L^2_\xi)}\Big)\le\delta_0,$$

here $\delta_0\ll 1$*, then there exist positive constants* C_0*,* C*, and a sufficiently small number* γ*, which are independent of* τ*, such that the following holds for* $\tau>0$*,*

$$\begin{aligned}
&\frac{1}{C}\sum_{0\le i\le 6}\Big\{\int_{-\infty}^{\infty}\gamma^{-i}(\|\partial_x^i\widehat{W_0}\|^2_{L^2_\xi}+\|\partial_x^i\partial_t\widehat{W_0}\|^2_{L^2_\xi})\\
&\quad+\gamma^{-i-1}(\|(1+|\xi|)^{1/2}\partial_x^iJ\|^2_{\mathrm{ref},L^2_\xi}+\|(1+|\xi|)^{1/2}\partial_x^i\partial_tJ\|^2_{\mathrm{ref},L^2_\xi})\,\mathrm{d}x\Big\}\Big|_{t=\tau}\\
&\quad+\frac{1}{C}\sum_{0\le i\le 6}\int_0^\tau\int_{-\infty}^{\infty}\gamma^{-i}(\|\partial_x^i\mathbf{h}\|^2_{L^2_\xi}+\|\partial_x^i\partial_t\mathbf{h}\|^2_{L^2_\xi})\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\frac{1}{C}\sum_{0\le i\le 6}\int_0^\tau\int_{-\infty}^{\infty}\gamma^{-i-1}\Big(\|(1+|\xi|)^{1/2}\partial_x^iJ_1\|^2_{\mathrm{ref},L^2_\xi}\\
&\quad+\|(1+|\xi|)^{1/2}\partial_x^i\partial_tJ_1\|^2_{\mathrm{ref},L^2_\xi}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\frac{1}{C}\int_0^\tau\int_{-\infty}^{\infty}\varepsilon^2\mathcal{E}(x,t)\|W_0\|^2_{L^2_\xi}\,\mathrm{d}x\,\mathrm{d}t\\
&\le C\sum_{0\le i\le 6}\Big\{\int_{-\infty}^{\infty}\gamma^{-i}(|\partial_x^i\widehat{W_0}|^2_{L^2_\xi}+\|\partial_x^i\partial_t\widehat{W_0}\|^2_{L^2_\xi})\\
&\quad+\gamma^{-i-1}(\|(1+|\xi|)^{1/2}\partial_x^iJ\|^2_{\mathrm{ref},L^2_\xi}+\|(1+|\xi|)^{1/2}\partial_x^i\partial_tJ\|^2_{\mathrm{ref},L^2_\xi})\,\mathrm{d}x\Big\}\Big|_{t=0}\\
&\quad+C_0\int_0^\tau\int_{-\infty}^{\infty}\{\varepsilon^3e^{-(c_0/2)\varepsilon(|x|+|s(t+t_0)|)}\}^{-1}\\
&\quad\times\sum_{0\le i\le 6}\{\|\partial_x^iE(\varphi)\|^2_{L^2_\xi}+\|\partial_x^i\partial_tE(\varphi)\|^2_{L^2_\xi}\}\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where $\sum_{0\le i\le 6}\{\|\partial_x^iE(\varphi)\|^2_{L^2_\xi}+\|\partial_x^i\partial_tE(\varphi)\|^2_{L^2_\xi}\}\le\hat{C}\varepsilon^4e^{-(c_0/2)\varepsilon(|x|+|s(t+t_0)|)}$ *for some constant* $\hat{C}>0$.

Now we are ready to prove the main theorem.

Proof of Theorem 1.1. By integrating (1.4), we have

$$\int_{-\infty}^{\infty} \langle J, J_t + \xi_1 J_x - L(J) - D(J) - \mathcal{N}(J) + E(\varphi)\rangle_{\mathrm{ref}}\, dx = 0.$$

Following the estimate of (5.23), we can obtain

$$\frac{d}{dt}\int_{-\infty}^{\infty} \|J\|^2_{\mathrm{ref},L^2_\xi}\, dx \le C\Big(\int_{-\infty}^{\infty} \|(1+|\xi|)^{1/2}J\|^2_{\mathrm{ref},L^2_\xi}\, dx + \int_{-\infty}^{\infty} \|E(\varphi)\|^2_{\mathrm{ref},L^2_\xi}\, dx\Big).$$

Thus,

$$\Big|\int_{-\infty}^{\infty} \|J(t_2)\|^2_{\mathrm{ref},L^2_\xi}\, dx - \int_{-\infty}^{\infty} \|J(t_1)\|^2_{\mathrm{ref},L^2_\xi}\, dx\Big|$$
$$\le C\Big(\int_{t_1}^{t_2}\int_{-\infty}^{\infty} \|(1+|\xi|)^{1/2}J\|^2_{\mathrm{ref},L^2_\xi}\, dx\, dt + \int_{t_1}^{t_2}\int_{-\infty}^{\infty} \|E(\varphi)\|^2_{\mathrm{ref},L^2_\xi}\, dx\, dt\Big).$$

By Proposition 5.9, we can obtain that

$$\int_{t_1}^{t_2}\int_{-\infty}^{\infty} \|(1+|\xi|)^{1/2}J\|^2_{\mathrm{ref},L^2_\xi}\, dx\, dt + \int_{t_1}^{t_2}\int_{-\infty}^{\infty} \|E(\varphi)\|^2_{\mathrm{ref},L^2_\xi}\, dx\, dt \to 0, \quad \text{as } t_1 \to \infty.$$

Therefore,

$$\int_{-\infty}^{\infty} \|J(t_1)\|^2_{\mathrm{ref},L^2_\xi}\, dx \to 0, \quad \text{as } t_1 \to \infty.$$

By the Sobolev inequality,

$$\|J(\cdot,t)\|_{\mathrm{ref},L^\infty_x(L^2_\xi)} \le C\|J(\cdot,t)\|^{1/2}_{\mathrm{ref},L^2_x(L^2_\xi)}\, \|J_x(\cdot,t)\|^{1/2}_{\mathrm{ref},L^2_x(L^2_\xi)}.$$

We thus conclude that

$$\|J(\cdot,t)\|_{\mathrm{ref},L^\infty_x(L^2_\xi)} \to 0, \quad \text{as } t \to \infty.$$

The proof is complete. □

Appendix A. Chapman-Enskog Expansion and Navier-Stokes Shock Profiles

Consider the Boltzmann equation

$$\frac{\partial F}{\partial t} + \xi\cdot\nabla_x F = \frac{1}{\kappa}Q(F,F),$$

where $\kappa > 0$ is the mean free path. We introduce the fluid variables as follows:

$$\begin{aligned}
\text{mass density:}\quad & \rho \equiv \int F(x,t,\xi)\,\mathrm{d}\xi,\\
\text{fluid flow velocity:}\quad & u_i \equiv \frac{1}{\rho}\int \xi_i F(x,t,\xi)\,\mathrm{d}\xi,\\
\text{momentum:}\quad & m_i \equiv \rho u_i\\
\text{pressure tensor:}\quad & p_{ij} \equiv \int (\xi_i - u_i)(\xi_j - u_j)F(x,t,\xi)\,\mathrm{d}\xi,\\
\text{pressure:}\quad & p \equiv \sum_k \frac{1}{3}p_{kk},\\
\text{internal energy per unit mass:}\quad & e \equiv \frac{1}{\rho}\int \frac{1}{2}|\xi - u|^2 F(x,t,\xi)\,\mathrm{d}\xi,\\
\text{absolute temperature:}\quad & T \equiv \frac{2}{3R}e,\\
\text{total energy:}\quad & \mathfrak{E} \equiv \rho e + \frac{1}{2}\rho|u|^2 = \int \frac{1}{2}|\xi|^2 F(x,t,\xi)\,\mathrm{d}\xi.
\end{aligned}$$

Expand the solution F and the operator $\partial/\partial t$ in a power series of κ:

$$\begin{aligned}
F &= F_0 + \kappa F_1 + \kappa^2 F_2 + \cdots,\\
\frac{\partial}{\partial t} &= \frac{\partial_0}{\partial t} + \kappa\frac{\partial_1}{\partial t} + \kappa^2\frac{\partial_2}{\partial t} + \cdots,
\end{aligned}$$

where $\partial_i/\partial t$ are the differential operators on the fluid variables ρ, u, T, ... It is well known that the first approximation F_0 in the above Chapman-Enskog expansion is the local Maxwellian, i.e.,

$$F_0(x,t,\xi) \equiv \rho\frac{e^{-|\xi-u|^2/(2RT)}}{\sqrt{(2\pi RT)^3}},$$

and

$$\begin{aligned}
\frac{\partial_0}{\partial t}\rho &\equiv -\operatorname{div} m,\\
\frac{\partial_0}{\partial t}m_i &\equiv -\sum_{j=1}^{3}\frac{\partial(m_i u_j)}{\partial x_j} - \frac{\partial p}{\partial x_i}, \qquad i = 1,2,3,\\
\frac{\partial_0}{\partial t}\mathfrak{E} &\equiv -\sum_{j=1}^{3}\frac{\partial(\mathfrak{E}u_j + pu_j)}{\partial x_j},
\end{aligned}$$

where $m \equiv (m_1, m_2, m_3)$, $m_i = \rho u_i$, $\mathfrak{E} \equiv \rho(|u|^2/2 + e)$, $e = \frac{3}{2}RT$, $p = \rho RT$. (We set the gas constant $R \equiv 1$.)

F_1 is uniquely solved by the following equations.

$$\text{(A.1)} \qquad Q(F_0, F_1) + Q(F_1, F_0) = \frac{\partial_0}{\partial t} F_0 + \xi \cdot \nabla_x F_0,$$
$$\int_{R^3} F_1 \, d\xi = 0,$$
$$\int_{R^3} \xi_i F_1 \, d\xi = 0, \quad i = 1, 2, 3,$$
$$\int_{R^3} |\xi|^2 F_1 \, d\xi = 0.$$

Notice that F_1 is purely microscopic and $|\partial_{x_i}^k F_1| < O(1) \sup_{1 \le j \le k} |\partial_{x_i}^k \partial_{x_j} F_0|$.

The operator $\partial_1/\partial t$ is defined by the following:

$$\frac{\partial_1}{\partial t} \rho \equiv -\int_{R^3} \xi \cdot \nabla_x F_1 \, d\xi = 0,$$

$$\frac{\partial_1}{\partial t} \rho u_i \equiv -\int_{R^3} \xi_i \xi \cdot \nabla_x F_1 \, d\xi$$
$$= \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left(\mu(T) \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_k} \delta_j^i \right] \right), \quad i = 1, 2, 3,$$

$$\frac{\partial_1}{\partial t} \rho \left(\frac{|u|^2}{2} + e \right) \equiv -\int_{R^3} |\xi|^2 \xi \cdot \nabla_x F_1 \, d\xi,$$
$$= \sum_{1 \le j, k \le 3} \frac{\partial}{\partial x_j} \left\{ \mu(T) \left[u_k \left(\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} \right) - \frac{2}{3} u_j \frac{\partial u_k}{\partial x_j} \right] \right\}$$
$$+ \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left(\lambda(T) \frac{\partial T}{\partial x_j} \right),$$

where $\mu(T) > 0$ is the coefficient of viscosity and $\lambda(T) > 0$ is the coefficient of the heat conductivity. By the definition of $\partial_1/\partial t$, we know that $(\partial_1/\partial t)F_0 + \xi \cdot \nabla_x F_1$ is purely microscopic.

Consider the first two terms in the expansion of $\partial/\partial t$:

$$\left(\frac{\partial}{\partial t} - \frac{\partial_0}{\partial t} - \kappa \frac{\partial_1}{\partial t} \right) \rho = 0,$$
$$\left(\frac{\partial}{\partial t} - \frac{\partial_0}{\partial t} - \kappa \frac{\partial_1}{\partial t} \right) \rho u_i = 0, \quad i = 1, 2, 3,$$
$$\left(\frac{\partial}{\partial t} - \frac{\partial_0}{\partial t} - \kappa \frac{\partial_1}{\partial t} \right) \rho \left(\frac{|u|^2}{2} + e \right) = 0,$$

which lead to the compressible Navier-Stokes equation corresponding to the second approximation $F_0 + \kappa F_1$:

$$\rho_t + \operatorname{div} m = 0, \tag{A.2}$$

$$(m_i)_t + \sum_{j=1}^{3} (m_i u_j)_{x_j} + p_{x_i} \tag{A.3}$$

$$= \kappa \sum_{j=1}^{3} \left(\mu(T) \left[(u_i)_{x_j} + (u_j)_{x_i} - \frac{2}{3} \sum_{k=1}^{3} (u_k)_{x_k} \delta_j^i \right] \right)_{x_j}, \quad i = 1, 2, 3,$$

$$\mathfrak{E}_t + \sum_{j=1}^{3} (u_j[\mathfrak{E} + p])_{x_j} \tag{A.4}$$

$$= \kappa \sum_{1 \le j,k \le 3} \left(\mu(T) \left[u_k((u_k)_{x_j} + (u_j)_{x_k}) - \frac{2}{3} u_j (u_k)_{x_j} \right] \right)_{x_j} + \kappa \sum_{j=1}^{3} (\lambda(T) T_{x_j})_{x_j}.$$

Assume that $\kappa = 1$. Let $(\rho, m, \mathfrak{E})(x, t) = (\bar{\rho}, \bar{m}, \bar{\mathfrak{E}})(x_1 - st)$ be a travelling wave solution of (A.2)–(A.4) connecting the two end states $(\rho_-, m_-, \mathfrak{E}_-)$ and $(\rho_+, m_+, \mathfrak{E}_+)$. Since we consider plane entropy shock waves moving in the x_1 direction, the Navier-Stokes shock profile satisfies the following equations:

$$-s\bar{\rho}_{x_1} + (\bar{m}_1)_{x_1} = 0, \tag{A.5}$$

$$-s(\bar{m}_1)_{x_1} + (\bar{m}_1 \bar{u}_1)_{x_1} + \bar{p}_{x_1} = \frac{4}{3} (\mu(\bar{T})(\bar{u}_1)_{x_1})_{x_1} \tag{A.6}$$

$$-s\bar{\mathfrak{E}}_{x_1} + (\bar{u}_1[\bar{\mathfrak{E}} + \bar{p}])_{x_1} = \frac{4}{3} (\mu(\bar{T}) \bar{u}_1 (\bar{u}_1)_{x_1})_{x_1} + (\lambda(\bar{T}) \bar{T}_{x_1})_{x_1}, \tag{A.7}$$

$$\begin{aligned} &\lim_{x \to \infty} (\bar{\rho}, \bar{m}, \bar{\mathfrak{E}})(x) = (\rho_+, m_+, \mathfrak{E}_+), \\ &\lim_{x \to -\infty} (\bar{\rho}, \bar{m}, \bar{\mathfrak{E}})(x) = (\rho_-, m_-, \mathfrak{E}_-), \end{aligned} \tag{A.8}$$

where s is the speed of the shock wave. We note that when the strength of the shock wave, $|\rho_- - \rho_+| + |m_- - m_+| + |\mathfrak{E}_- - \mathfrak{E}_+|$, is sufficiently small, the acoustic speed of the travelling solution is monotone; that is,

$$\partial_x \left(\bar{u}_1 + \sqrt{\frac{5\bar{T}}{3}} \right) < 0.$$

We refer to Appendices A and C in [18] for the estimates.

Appendix B. Estimates on Collision Operators

In the present paper we consider the hard sphere as our model, and the collision operator can be written as follows (see [11], [12]):

$$L(h) \equiv Q(\omega, h) + Q(h, \omega),$$
$$Q(g, h) \equiv \int_{R^3} \int_{S^2} [g(\xi')h(\xi'_\star) - g(\xi)h(\xi_\star)] C(\Omega, \xi - \xi_\star) \, \mathrm{d}\Omega \, \mathrm{d}\xi_\star,$$

where

$$\xi' = \xi + (\Omega \cdot (\xi_\star - \xi))\Omega,$$
$$\xi'_\star = \xi_\star - (\Omega \cdot (\xi_\star - \xi))\Omega,$$
$$\Omega \in S^2.$$

The function $C(\Omega, \xi - \xi_\star)$ for a hard sphere is

$$C(\Omega, \xi - \xi_\star) \equiv |\Omega \cdot (\xi - \xi_\star)|.$$

We actually have

$$L(h)(\xi) = -\nu(\xi)h(\xi) + \int_{R^3} [-k_1(\xi, \eta) + k_2(\xi, \eta)] h(\eta) \, \mathrm{d}\eta,$$

and

$$\nu(\xi) \equiv \sqrt{2\pi} b_0 \left(e^{-|\xi|^2/2} + |\xi|^2 \int_0^1 e^{-u^2|\xi|^2/2} \, \mathrm{d}u \right),$$

$$k_1(\xi, \eta) = \frac{b_0}{2\sqrt{2\pi}} |\xi - \eta| e^{-(|\xi|^2 + |\eta|^2)/4},$$

$$k_2(\xi, \eta) = \frac{2b_0}{\sqrt{2\pi}} \frac{1}{|\xi - \eta|} e^{-(|\xi|^2 + |\eta|^2)/8 - ((|\xi|^2 - |\eta|^2)^2)/(8|\xi - \eta|^2)},$$

where b_0 is a positive constant.

We need the following lemmas for the energy estimates. See Appendix B in [18] for the proofs.

Lemma B.1. *Let $\omega_0(\xi)$ be a given Maxwellian distribution. There exists a constant $K > 0$ such that the following inequality holds for any g and h satisfying $\|(1 + |\xi|)^{1/2} g\|_{L^2(R^3)} < \infty$, $\|(1 + |\xi|)^{1/2} h\|_{L^2(R^3)} < \infty$:*

$$\int_{R^3} \frac{Q(\omega_0 g, \omega_0 h)^2 + Q(\omega_0 h, \omega_0 g)^2}{(1 + |\xi|)\omega_0(\xi)^2} \, \mathrm{d}\xi$$

$$\leq K \int_{R^3} (1 + |\xi|) g(\xi)^2 \, \mathrm{d}\xi \int_{R^3} h(\xi)^2 \, \mathrm{d}\xi + K \int_{R^3} (1 + |\xi|) h(\xi)^2 \, \mathrm{d}\xi \int_{R^3} g(\xi)^2 \, \mathrm{d}\xi.$$

Lemma B.2. *Let* $\omega_0(\xi)$ *be a given Maxwellian distribution and* $d(\xi)$ *be a given function satisfying* $d(\xi) \le C_0\omega_0(\xi)^\alpha$, $\alpha \in [\frac{1}{6}, \frac{1}{3}]$. $\mathbf{D}h$ *is a linear operator defined by*

$$\mathbf{D}h \equiv \frac{1}{\omega_0}[Q(\omega_0 d, \omega_0 h) + Q(\omega_0 h, \omega_0 d)].$$

Then there exists a constant $K_0 > 0$ *such that*

$$\int_{R^3} h\mathbf{D}h \, d\xi \le K_0 C_0 \int_{R^3} h(\xi)^2(1 + |\xi|) \, d\xi.$$

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