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COMPONENTS OF $Hom(\pi_1, PGL(2, \mathbb{R}))$

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In Topological Components of Spaces of Representations [1] Goldman calculated the number of topological components of the varieties $Hom(\pi_1(M), PSL(2,\mathbb{R}))$ and $Hom(\pi_1(M), PSL(2,\mathbb{C}))$ where M is a compact Riemann surface of genus $g > 1$. In this paper, we analyze the variety $Hom(\pi_1(M), PGL(2,\mathbb{R}))$ and show that it has $2^{2g+1} + 4g - 5$ connected components. Copyright © 1996 Elsevier Science Ltd

1. INTRODUCTION AND RESULTS

Let M be a compact Riemann surface of genus $g > 1$. An element σ in the space $Hom(\pi_1(M), PGL(2,\mathbb{R}))$ is a homomorphism from the fundamental group $\pi_1(M)$ to the Lie group $PGL(2,\mathbb{R})$. Such a σ determines a flat principal $PGL(2,\mathbb{R})$ bundle over the surface M. This bundle gives rise to the obstruction classes in $H^n(M, \pi_{n-1}(PGL(2,\mathbb{R})))$. Therefore, for each n, this construction gives rise to an obstruction class map

$$o_n: Hom(\pi_1(M), PGL(2,\mathbb{R})) \to H^n(M, \pi_{n-1}(PGL(2,\mathbb{R}))).$$

o_1 is continuous and hence constant on each connected component. Thus, if two elements of $Hom(\pi_1(M), PGL(2,\mathbb{R}))$ are in the same connected component, their first obstruction classes are the same. Therefore, the first obstruction class allows us to sort the space into sets that are both open and closed. The main result of this paper is the following theorem.

THEOREM 1.1. *Let* $c, c' \in H^1(M, \pi_0(PGL(2,\mathbb{R})))/\{0\}$, *then*
(1) $o_1^{-1}(c)$ *and* $o_1^{-1}(c')$ *are homeomorphic,*
(2) $o_1^{-1}(c)$ *consists of two connected components.*

Remark. $PGL(2,\mathbb{R})$ is isomorphic to $SO(2,1)$. $\pi_0(PGL(2,\mathbb{R}))$ is isomorphic to $\mathbb{Z}_2$, so $H^1(M, \pi_0(PGL(2,\mathbb{R}))) \cong H^1(M, Z_2) \cong \mathbb{Z}_2^{2g}$ which is a $\mathbb{Z}_2$ module.

Goldman showed that $o_1^{-1}(0)$ has $4g - 3$ connected components (see [1]). Combining this with Theorem 1.1, we obtain

COROLLARY 1.1. $Hom(\pi_1(M), PGL(2,\mathbb{R}))$ *has* $2^{2g+1} + 4g - 5$ *connected components.*

For the rest of this paper we will adopt the following notations consistently: Let

$$SL_{\pm}(2,\mathbb{R}) = \{g: g \in GL(2,\mathbb{R}), \det(g) = \pm 1\}$$

$$SL_{\pm}(2,\mathbb{C}) = \{g: g \in GL(2,\mathbb{C}), \det(g) = \pm 1\}$$

G will stand for either $SL_{\pm}(2,\mathbb{R})$ or $SL_{\pm}(2,\mathbb{C})$ and PG the quotient group $G/\{\pm 1\}$. Subscripts will be used to describe different components. For example, G_0 denotes the identity component of G and G_1 is $G\backslash G_0$. Brackets are used to denote the commutator map, i.e.

$$[X, Y] = XYX^{-1}Y^{-1} \quad \text{for } X, Y \in G.$$

Finally, the term "component" always means connected component.

2. COMPONENTS AND THE FIRST OBSTRUCTION CLASS

In this section, G stands for $SL_{\pm}(2, \mathbb{R})$. We first note that $H^1(M, \pi_0(PG))$ and $Hom(\pi_1(M), \pi_0(PG))$ are isomorphic as groups: Let $\sigma \in Hom(\pi_1(M), PG)$ and $f: PG \to \pi_0(PG)$ be the quotient map. Then f composed with σ provides an element in $Hom(\pi_1(M), \pi_0(PG))$. In the other direction, $o_1(\sigma)$ is in $H^1(M, \pi_0(PG))$. Together $f \circ \sigma \leftrightarrow o_1(\sigma)$ is the needed isomorphism.

The group $\pi_1(M)$ is generated by $S = \{A_i, B_i\}_{i=1}^g$ subjected to the relation

$$\prod_{i=1}^{g} [A_i, B_i] = e.$$

The obstruction class $o_1(\sigma) \cong f \circ \sigma$ is determined by its values on S. A set of generators of $\pi_1(M)$ induces a basis for the Z_2 module $Hom(\pi_1(M), \pi_0(PG))$. Let $\alpha_i \in Hom(\pi_1(M), PG)$ such that α_i sends only A_i to the PG_1 component, and the rest to PG_0. We define the β_i's similarly. Let

$$a_i = f \circ \alpha_i \quad \text{and} \quad b_i = f \circ \beta_i.$$

Then $\{a_i, b_i\}_{i=1}^g$ is a basis for $Hom(\pi_1(M), \pi_0(PG)) \cong H^1(M, \pi_0(PG))$, and

$$o_1(\sigma) \cong f \circ \sigma = \sum_{i=1}^{g} (f(\sigma(A_i))a_i + f(\sigma(B_i))b_i).$$

From this point of view, the first obstruction class indicates the component of PG to which σ sends each generator.

PROPOSITION 2.1. *If c, $c' \in H^1(M, \pi_0(PG))\backslash\{0\}$, then $o_1^{-1}(c) \cong o_1^{-1}(c')$.*

Proof. Let $Aut(M)$ denote the group of diffeomorphisms on M, and $Aut(M)_0$ be the subgroup that fixes a base point in M. If $T \in Aut(M)_0$, then $T: M \to M$ induces $T^*: H^1(M, \mathbb{Z}_2) \to H^1(M, \mathbb{Z}_2)$. Thus, $Aut(M)$ acts on $H^1(M, \mathbb{Z}_2)$ by isomorphism, and this action is transitive on $H^1(M, \mathbb{Z}_2)\backslash\{0\}$. Since c, c' are non-zero, there exists a $T \in Aut(M)_0$ taking c to c'. The corresponding isomorphism $T_*: \pi_1(M) \to \pi_1(M)$ induces a homeomorphism:

$$\Psi: Hom(\pi_1(M), PG) \to Hom(\pi_1(M), PG)$$

$$\Psi(\tau) = \tau \circ T_*.$$

Ψ is the desired homeomorphism that takes the component $o_1^{-1}(c)$ to the component $o_1^{-1}(c')$. ■

This proves the first part of Theorem 1.1. and reduces the second part to counting the number of topological components in $o_1^{-1}(a_1)$.

Let $R: PG_1 \times PG_0^{2g-1} \to PG_0$ be the commutator map

$$R(X_1, Y_1, \ldots, X_g, Y_g) = \prod_{i=1}^{g} [X_i, Y_i].$$

Then $o_1^{-1}(a_1)$ is identified with the set $R^{-1}([I])$ where $[I]$ is the identity element in PG_0. We break R down into two maps: $R_1: PG_1 \times PG_0 \to PG_0$ with

$$R_1(X_1, Y_1) = [X_1, Y_1],$$

and $R_2: PG_0^{2g-2} \to PG_0$ with

$$R_2(X_2, Y_2, \ldots, X_g, Y_g) = \prod_{i=2}^{g} [X_i, Y_i].$$

Then the space $R^{-1}([I])$ is

$$\{(X_1, Y_1, \ldots, X_g, Y_g): R_1(X_1, Y_1) = R_2(X_2, Y_2, \ldots, X_g, Y_g)^{-1}\}$$

which we denote $\{R_1 = (R_2)^{-1}\}$. Let

$$\hat{R}(X_1, Y_1, \ldots, X_g) = \textstyle\prod_{i=1}^{g} [X_i, Y_i]$$
$$\hat{R}_1(X_1, Y_1) = [X_1, Y_1]$$
$$\hat{R}_2(X_2, Y_2, \ldots, X_g) = \textstyle\prod_{i=2}^{g} [X_i, Y_i]$$

be the corresponding commutator maps on $G_1 \times G_0^{2g-1}$, $G_1 \times G_0$ and G_0^{2g-3}, respectively. Consider the commutative diagrams

$$\begin{array}{ccc} G_1 \times G_0^{2g-1} & \xrightarrow{\hat{R}} & G_0 \\ \downarrow \pi & & \downarrow \pi' \\ PG_1 \times PG_0^{2g-1} & \xrightarrow{R} & PG_0 \end{array} \quad \begin{array}{ccc} G_1 \times G_0 & \xrightarrow{\hat{R}_1} & G_0 \\ \downarrow \pi & & \downarrow \pi' \\ PG_1 \times PG_0 & \xrightarrow{R_1} & PG_0 \end{array} \quad \begin{array}{ccc} G_0^{2g-3} & \xrightarrow{\hat{R}_2} & G_0 \\ \downarrow \pi & & \downarrow \pi' \\ PG_1 \times PG_0^{2g-3} & \xrightarrow{R_2} & PG_0 \end{array}$$

where π, π' are covering maps. The maps $\hat{R}$, $\hat{R}_1$ and $\hat{R}_2$ are invariant under the actions of their respective covering groups. Hence, there exist

$$\tilde{R}: PG_1 \times PG_0^{2g-1} \to G_0$$
$$\tilde{R}_1: PG_1 \times PG_0 \to G_0$$
$$\tilde{R}_2: PG_0^{2g-3} \to G_0$$

such that

$$\tilde{R} \circ \pi = \hat{R}$$
$$\tilde{R}_1 \circ \pi = \hat{R}_1$$
$$\tilde{R}_2 \circ \pi = \hat{R}_2.$$

Let $C_0 \in G_0$ be a hyperbolic element.

PROPOSITION 2.2. *The map $\tilde{R}_1$ is onto with connected fibres.*

PROPOSITION 2.3. *Every representation in $R^{-1}([I])$ can be deformed to a representation in $\mathscr{HYP} = \{R_1 = R_2^{-1} = [C_0]\}$.*

The proofs of Propositions 2.2 and 2.3 are lengthy and technical, so we defer them to Sections 4 and 5. We now assume both are true and prove Theorem 1.1. Henceforth, all paths constructed are over the interval $[0, 1]$ unless otherwise specified.

Since $\tilde{R}_1$ is onto, so is R_1, and thus R and its lift $\tilde{R}$ are both onto. Hence $R^{-1}([I])$ contains at least two sets that are both open and closed, namely, $\tilde{R}(I)$ and $\tilde{R}(-I)$. Now we prove that $R^{-1}([I])$ has at most two components.

LEMMA 2.1. *Paths in* $G_0 \backslash \{I\}$ *lift to paths in* $G_1 \times G_0$ *via* $\tilde{R}_1$.

Proof. Using the results in [2] and the fact that $\tilde{R}_1$ is onto (Proposition 2.2), we can show that $\tilde{R}_1$ is a submersion onto $G_0 \backslash \{I\}$. The result then follows from Proposition 2.2 and the path-lifting lemma ([1] 1.4). ■

LEMMA 2.2. *The set* $\mathscr{GEN} = PG_0^{2g-3} \backslash \tilde{R}_2^{-1}(\{\pm I\})$ *is connected.*

Proof. Since $\tilde{R}_2$ is a submersion onto $\mathscr{GEN}$ and the set $\{\pm I\}$ has codimension 3 in G_0, $R_2^{-1}(\{\pm I\})$ is a subvariety of codimension 3 in PG_0^{2g-3}. PG_0^{2g-3} is a connected manifold, so $\mathscr{GEN}$ is connected. ■

LEMMA 2.3. *Let* $\psi^+ \in \tilde{R}_1^{-1}(C_0)$, $\psi^- \in \tilde{R}_1^{-1}(-C_0)$ *and* $\phi_0 \in R_2^{-1}([C_0]^{-1})$. *Then for any* $(\psi, \phi) \in \mathscr{HYP}$, *there is a path in* $R^{-1}([I])$ *connecting* (ψ, ϕ) *to either* (ψ^+, ϕ_0) *or* (ψ^-, ϕ_0).

Proof. Since $\mathscr{GEN}$ is connected, there exists path $Q(s) \in \mathscr{GEN}$ such that $Q(0) = \phi$ and $Q(1) = \phi_0$. By the definition of $\mathscr{GEN}$, we have $\tilde{R}_2(Q(s)) \neq \pm I$. Let

$$p(s) = (\tilde{R}_2(Q(s)))^{-1}.$$

Then $p(s) \neq I$ and $-p(s) \neq I$. This implies the paths $p(s)$ and $-p(s)$ can be lifted to paths in $PG_1 \times PG_0$ via $\tilde{R}_1$. If $\tilde{R}_1(\psi) = p(0)$, then we let $P(s)$ be a lift of $p(s)$; however, if $\tilde{R}_1(\psi) = -p(0)$, then we let $P(s)$ be a lift of $-p(s)$ with $P(0) = \psi$ in both cases.

Case 1: $\tilde{R}_1(P(1)) = C_0$. Since $\tilde{R}_1^{-1}(C_0)$ is connected, there is a path $\psi(s) \in \tilde{R}_1^{-1}(C_0)$ that connects $P(1)$ to ψ^+. Hence, the path $(P(s), Q(s))$ followed by $(\psi(s), Q(1))$ connects (ψ, ϕ) to (ψ^+, ϕ_0).

Case 2: $\tilde{R}_1(P(1)) = -C_0$. Since $\tilde{R}_1^{-1}(-C_0)$ is connected, there is a path $\psi(s) \in \tilde{R}_1^{-1}(-C_0)$ that connects $P(1)$ to ψ^-. Hence, the path $(P(s), Q(s))$ followed by $(\psi(s), Q(1))$ connects (ψ, ϕ) to (ψ^-, ϕ_0). ■

Together Lemma 2.3 and Proposition 2.3 give us Theorem 1.1.

Here are some intuitive reasons that $o_1^{-1}(a_1)$ has far fewer components than $o_1^{-1}(0)$. The analogue of Proposition 2.2 for $o_1^{-1}(0)$ is not true. Consider the similar commutator map $R_0 : PG_0^2 \to PG_0$ and its lift $\tilde{R}_0 : PG_0^2 \to G_0$. The latter behaves very differently from $\tilde{R}_1$. One difference is that $\tilde{R}_0$ is not onto (missing $-I$). More importantly, suppose we let

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \theta(s) = \begin{pmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{pmatrix}, \quad \theta_1(s) = \rho\theta(s) \quad \text{where } s \in [0, \pi].$$

Then $\pi_1(PG_0, [I])$ is isomorphic to $\mathbb{Z}$ and is generated by $[\theta(s)]$. Hence,

$$\pi_1(PG_0^2, ([I],[I])) \cong \pi_1(PG_0, [I]) \times \pi_1(PG_0, [I]) \cong \mathbb{Z} \times \mathbb{Z}$$

which is generated by $([I], [\theta,(s)])$ and $([\theta(s)], [I])$. The map R_0 has a constant value of $[I]$ on these two loops. Hence, the induced map

$$R_{0*}: \pi_1(PG_0, [I]) \times \pi_1(PG_0, [I]) \to \pi_1(PG_0, [I])$$

is the trivial map, and so is $\tilde{R}_{0*}$ since $\pi' \circ \tilde{R}_0 = R_0$ (recall that $\pi': G_0 \to PG_0$ is the covering map). Thus, we may further lift $\tilde{R}_0$ to a map that maps into the universal cover of G_0 (see [1]).

In contrast, the fundamental group of $\pi_1(PG_1, [\rho])$ is generated by $[\theta_1(s)]$. It follows that the fundamental group

$$\pi_1(PG_1 \times PG_0, ([\rho],[I])) \cong \mathbb{Z} \times \mathbb{Z}$$

is generated by $([\theta_1(s)],[I])$, $([\rho],[\theta(s)])$. Therefore,

$$R_1([\theta_1(t)],[I]) = [I]$$

$$R_1([\rho],[\theta(s)]) = [\theta(-2s)].$$

This implies that the induced map

$$R_{1*}: \pi_1(PG_1 \times PG_0, ([\rho],[I])) \to \pi_1(PG_0,[I])$$

is

$$R_{1*}(\sigma, \tau) = -2\tau.$$

A direct calculation shows

$$R_{1*}(\pi_1(PG_1 \times PG_0,([\rho],[I]))) = \pi'_*(\pi_1(G_0, I)).$$

This forces us to conclude that, at best, we may lift R_1 to a map that maps into G_0. Indeed, $\tilde{R}_1$ is such a map.

As a result of this difference, the loops in G_0 are not images of loops under the map $\tilde{R}_0$, but are the images of loops under $\tilde{R}_1$. The latter fact is crucial for the construction of paths connecting the various points in $o_1^{-1}(a_1)$.

Remark. We believe $o_1^{-1}(c) \subset Hom(\pi_1, SL_\pm(2,\mathbb{R}))$ has two components where $c \in H^1(M, \pi_0(SL_\pm(2,\mathbb{R}))) \setminus \{0\}$. To prove this, one would need an analogue of Propositions 2.2 and 2.3.

3. THE COMPLEX CASE

Let $G = SL_\pm(2,\mathbb{C})$. There is an analogue of Theorem 1.1 for the complex group PG. The proof for the first part of the theorem is identical to the real case. For the second part, let $\hat{R}_{j,*}: G_j \times G_0^{2g-1} \to G_0$ be

$$\hat{R}_{j,*}(X_1, Y_1, \ldots, X_g, Y_g) = \prod_{i=1}^{g} [X_i, Y_i], \quad j \in \{0,1\}.$$

Similarly, let $R_{j,*}: PG_j \times PG_0^{2g-1} \to PG_0$ be defined as

$$R_{j,*}(X_1, Y_1, \ldots, X_g, Y_g) = \prod_{i=1}^{g} [X_i, Y_i], \quad j \in \{0, 1\}.$$

Then $O_1^{-1}(a_1) = R_{1,*}^{-1}([I])$. Consider the following commutative diagram:

$$\begin{array}{ccc} G_j \times G_0^{2g-1} & \xrightarrow{\hat{R}_{j,*}} & G_0 \\ \downarrow \pi & & \downarrow \pi' \\ PG_j \times PG_0^{2g-1} & \xrightarrow{R_{j,*}} & PG_0 \end{array}$$

where both π, π' are covering maps. Since $\hat{R}_{j,*}$ is invariant under the action of the covering group, there is $\tilde{R}_{j,*}: PG_j \times PG_0^{2g-1} \to G_0$ such that $\tilde{R}_{j,*} \circ \pi = \hat{R}_{j,*}$.

Goldman [1] showed that the fibres $\tilde{R}_{0,*}^{-1}(\pm I)$ are connected and non-empty. We want to show that this is also true for the fibres $\tilde{R}_{1,*}^{-1}(\pm I)$. Let $(X_1, Y_1, \ldots, X_g, Y_g) \in \tilde{R}_{0,*}^{-1}(I)$. Then $(iX_1, Y_1, \ldots, X_g, Y_g) \in \tilde{R}_{1,*}^{-1}(I)$. This shows $\tilde{R}_{1,*}^{-1}(I) \neq 0$. Let

$$(X_1^0, p^0) = (X_1^0, Y_1^0, \ldots, X_g^0, Y_g^0)$$

$$(X_1^1, p^1) = (X_1^1, Y_1^1, \ldots, X_g^1, Y_g^1)$$

be in $\tilde{R}_{1,*}^{-1}(I)$, then $(iX_1^0, p^0), (iX_1^1, p^1) \in \tilde{R}_{0,*}^{-1}(I)$. The fact that $\tilde{R}_{0,*}^{-1}(I)$ is connected implies the existence of a path $(X_1(t), p(t)) \in \tilde{R}_{0,*}^{-1}(I)$ between $(X_1(0), p(0)) = (iX_1^0, p^0)$ and $(X_1(1), p(1)) = (iX_1^1, p^1)$. This means that $(iX_1(t), p(t))$ is in $\tilde{R}_{1,*}^{-1}(I)$ and is a path between (X_1^0, p^0) and (X_1^1, p^1). Hence, $\tilde{R}_{1,*}^{-1}(I)$ is connected. The proof for $\tilde{R}_{1,*}^{-1}(I)$ is the same. ■

Remark. For the simpler case of $Hom(\pi_1, SL_{\pm}(2, \mathbb{C}))$, $\hat{R}_{1,*}^{-1}(I)$ is connected.

4. THE COMMUTATOR MAP $\tilde{R}_1$

Now we prove Proposition 2.2. The restriction of $\tilde{R}_1$ to an appropriate subdomain is a fibration. To make these appropriate choices, we sort the elements in the representation space according to their characters.

4.1. The character map for $G = SL_{\pm}(2, \mathbb{C})$.

Let $\chi: G \times G \to \mathbb{C}^3$ be $\chi(X, Y) = (tr(X), tr(Y), tr(XY))$, and let χ_{ij} denote χ restricting to $G_i \times G_j$. Let $\kappa_{ij}: \mathbb{C}^3 \to \mathbb{C}$ be defined by

$$\kappa_{ij}(x, y, z) = \begin{cases} x^2 + y^2 + z^2 - xyz - 2 & \text{if } i = j = 0 \\ -x^2 + y^2 - z^2 + xyz - 2 & \text{if } i = 1,\ j = 0 \\ x^2 - y^2 - z^2 + xyz - 2 & \text{if } i = 0,\ j = 1 \\ -x^2 - y^2 + z^2 - xyz - 2 & \text{if } i = j = 1. \end{cases}$$

PROPOSITION 4.1. *Suppose $X \in G_i$, $Y \in G_j$, $u = \chi_{ij}(X, Y)$, and $t = \kappa_{ij}(u)$. Then:*

(1) *The map χ determines the character of the representation generated by X, Y.*

(2) *χ is G-invariant and χ_{ij} is surjective.*

(3) $t = tr([X, Y])$.

(4) *$t = 2$ iff the representation generated by X, Y is reducible.*

(5) *If $t \neq 2$, then G_0 acts transitively on $\chi_{ij}^{-1}(u)$.*

Proof. Goldman [1, 3, 4] proved the above statements for $i = j = 0$. We shall prove only the case when $i = 1$ and $j = 0$. The proofs for the other cases are analogous.

(1) Let $w(X, Y)$ be a word of X, X^{-1}, Y and Y^{-1}. Then both iX and Y lie in G_0. So $tr(w(iX, Y))$ is determined by χ_{00}. Since i commutes with X and Y, $tr(w(X, Y)) = (-i)^n tr(w(iX, Y))$ where n is the number of occurrences of X minus the number of occurrences of X^{-1} in the word $w(X, Y)$.

(2) χ is G-invariant follows immediately from the fact that $tr(AB) = tr(BA)$ for all matrices A and B. Let $(x, y, z) \in \mathbb{C}$. The map χ_{00} is onto, so there exists $(X', Y') \in G_0 \times G_0$ such that $\chi_{00}(X', Y') = (-ix, y, -iz)$ but this gives $\chi_{10}(iX', Y') = (x, y, z)$.

(3) Since $(iX, Y) \in G_0^2$, $\kappa_{00}(\chi_{00}(iX, Y)) = tr([iX, Y]) = tr([X, Y])$. On the other hand, $\kappa_{00}(\chi_{00}(iX, Y)) = -tr(X)^2 + tr(Y)^2 - tr(XY)^2 + tr(X)tr(Y)tr(XY) - 2 = \kappa_{10}(\chi_{10}(X, Y))$.

(4) $\langle X, Y\rangle$ is irreducible iff $\langle iX, Y\rangle$ is irreducible iff $tr([iX, Y]) \neq 2$ iff $tr([X, Y]) \neq 2$.

(5) Since the action of G_0 commutes with multiplication by i, G_0 acts transitively on $\chi_{10}^{-1}(u)$ iff it acts transitively on $\chi_{00}^{-1}(\chi_{00}iX, Y))$. ∎

4.2. *The character map for $G = SL_{\pm}(2, \mathbb{R})$*

PROPOSITION 4.2. *Suppose $X \in G_i$, $Y \in G_j$, $u = \chi_{ij}(X, Y)$, and $t = \kappa_{ij}(u)$. Then:*

(1) *The map χ determines the character of the representation generated by X, Y.*

(2) *χ is G-invariant and χ_{ij} is surjective.*

(3) *$t = tr([X, Y])$; in particular, $\kappa_{10} \circ \chi_{10} = tr \circ \hat{R}_1$.*

(4) *$t = 2$ iff the representation generated by X, Y is reducible.*

(5) *If $t \neq 2$, then G acts transitively on $\chi_{ij}^{-1}(u)$.*

Proof. Again we prove the statements only for the case $i = 1, j = 0$.

(1) This follows immediately from the analogous result of the complex case.

(2) Invariance under the action of G comes from the same result of the complex case. Suppose $(x, y, z) \in \mathbb{R}^3$. Let $\delta = -(\kappa_{10}(x, y, z) - 2)$, $\xi^2 = x^2 + 4$, i.e., $\xi = \sqrt{x^2 + 4}$, and let

$$X = \begin{pmatrix} \frac{x-\xi}{2} & 0 \\ 0 & \frac{x+\xi}{2} \end{pmatrix}, \quad Y = \begin{pmatrix} \frac{y}{2} + \frac{xy-2z}{2\xi} & \frac{\delta}{\xi} \\ -\frac{1}{\xi} & \frac{y}{2} - \frac{xy-2z}{2\xi} \end{pmatrix}.$$

Then $\chi_{10}(X, Y) = (x, y, z)$. This is an instance where χ_{10} behaves very differently from χ_{00}. As we recall, χ_{00} is not onto.

(3) This follows immediately from the analogous result in the complex case.

(4) It is immediate that if $\langle X, Y\rangle$ is reducible, then $t = 2$. Suppose $t = 2$. Then $\langle X, Y\rangle$ is reducible as an $SL_{\pm}(2, \mathbb{C})$ representation. By Lemma 1.2.1 of [3], $\langle X, Y\rangle$ being non-abelian implies $\langle X, Y\rangle$ is reducible. (Culler and Shalen proved this for the $SL(2, \mathbb{C})$ representations, but the proof works equally well for the $SL_{\pm}(2, \mathbb{C})$ representations.) Since $X \in G_1$ and $Y \in G_0$, $\langle X, Y\rangle$ being abelian implies $\langle X, Y\rangle$ is reducible. Hence, $t = 2$ iff the representation $\langle X, Y\rangle$ is reducible.

(5) Suppose (X_1, Y_1), $(X_2, Y_2) \in \chi_{10}^{-1}(u)$. Then the parallel result for the complex case ensures the existence of a $g \in GL(2, \mathbb{C})$ that conjugates (X_1, Y_1) to (X_2, Y_2). Since $t \neq 2$, $\langle X, Y\rangle$ is irreducible. Burnside's lemma then states that the matrix algebra $M(2, \mathbb{R})$ is generated by $\{X, Y\}$, so g must conjugate $M(2, \mathbb{R})$ to itself. A direct calculation shows that,

up to a scalar, g must be a matrix of real entries. Hence, (X_1, Y_1) and (X_2, Y_2) are $PGL(2, \mathbb{R})$ conjugates. Intuitively, this says that if g conjugates all real matrices to real matrices, then g must be a multiple of a complex scalar and a real matrix. ■

These results show

$$PGL(2, \mathbb{R}) \to (tr \circ \hat{R}_1)^{-1}(t) \to \kappa_{10}^{-1}(t)$$

is a fibration for $t \neq 2$ (recall $\hat{R}_1 : G_1 \times G_0 \to G_0$ is the commutator map).

4.3. The range and fibres of R_1

We let $G = SL_{\pm}(2, \mathbb{R})$ and shorten our notation by letting κ denote κ_{10} and χ denote χ_{10}.

Recall the commutative diagram:

$$\begin{array}{ccc} G_1 \times G_0 & \xrightarrow{\hat{R}_1} & G_0 \\ \downarrow \pi & & \downarrow \pi' \\ PG_1 \times PG_0 & \xrightarrow{R_1} & PG_0 \end{array}$$

where $\hat{R}_1$ is the commutator map and $\tilde{R}_1 \circ \pi = \hat{R}_1$. G acts on direct products of G by simultaneous conjugation. Let $*$ denote this conjugation action. In particular, $\hat{R}_1$ is equivariant with respect to the action of G. That is, $\hat{R}_1(g * \psi) = g * \hat{R}_1(\psi)$ for all $g \in G$ and $\psi \in G_1 \times G_0$.

The conjugacy classes of $G_0 \backslash \{\pm I\}$ are classified by their traces. We need to show that the image of $\hat{R}_1$ (hence the image of $\tilde{R}_1$) meets each of these conjugacy classes. Let C be the set of matrices with trace t. This is clear for $t \neq \pm 2$ since κ and χ are both onto, and $\kappa \circ \chi = tr \circ \hat{R}_1$. For $t = -2$, C consists of two conjugacy classes: $\{-I\}$ and P_- which is the class of parabolic elements with trace equal to -2. A straightforward calculation shows $\hat{R}_1(X, Y) = -I$ iff $\chi(X, Y) = (0,0,0)$ (see [1] Proposition 5.4). Since χ is onto, $\chi^{-1}(0,0,0)$ is non-empty. For the parabolic class, choose any element in $\chi^{-1}(\kappa^{-1}(-2) \backslash (0,0,0))$ (see Fig. 1). Similarly, for $t = 2$, C consists of two conjugacy classes: $\{I\}$ and P_+ which is the class of parabolic elements with the trace equal to 2. $\hat{R}_1(X, I) = I$ for any $X \in PG_1$ and the class P_+ can be reached by a generic element in $(\kappa \circ \chi)^{-1}(2)$ such as (X, Y) where

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This proves that the map $\hat{R}_1$ is surjective.

Now we prove that the fibres of $\tilde{R}_1$ are connected. The map $\pi : G_1 \times G_0 \to PG_1 \times PG_0$ is a covering map with the covering group $\mathbb{Z}_2 \times \mathbb{Z}_2$. In addition $\hat{R}_1 = \pi \circ \tilde{R}_1$ and $\pi' \circ \hat{R}_1 = R_1$. The strategy is to find an element in each component of $\hat{R}_1^{-1}(C)$ for any $C \in G_0$ and show that all these points are mapped to a single point by π.

The following lemma is a straightforward result from point set topology.

LEMMA 4.1. *Let S and T be topological spaces and $f: S \to T$ a continuous map. Suppose $S = \bigcup_{i=1}^{n} S_i$ where S_i is connected for each i. If, for each i, there exists $x_i \in S_i$ with $f(x_i) = y$ for a fixed $y \in T$, then $f(S)$ is connected.*

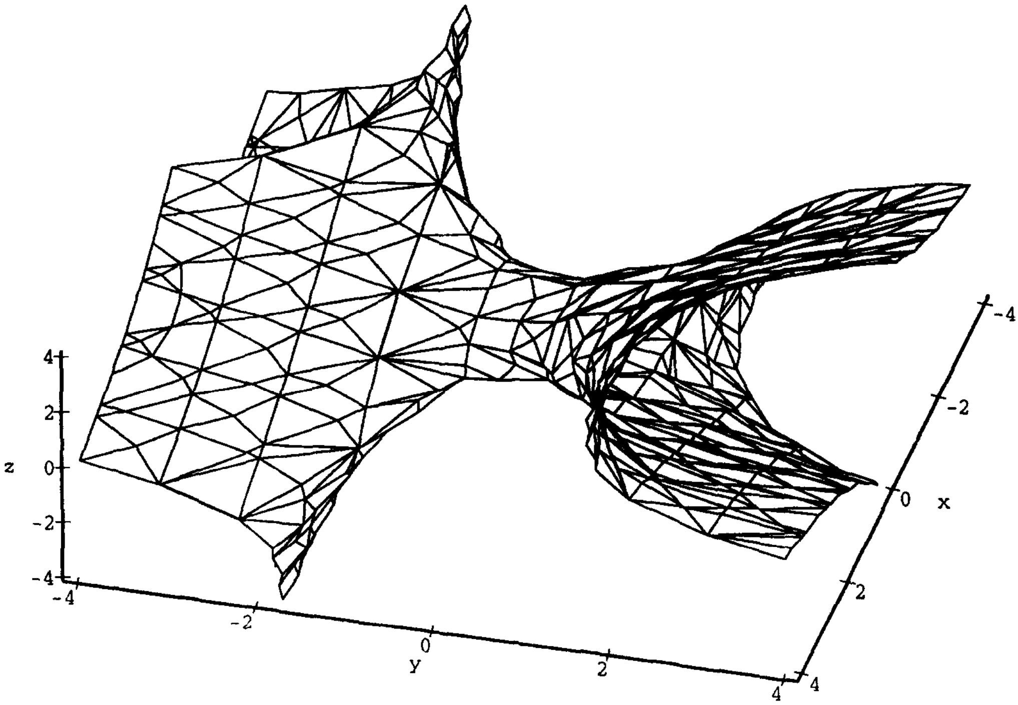

Fig. 1. $t < -2$.

Consider the following diagram where $tr(C) = t$:

$$\begin{array}{ccccc} & & PG = PGL(2,\mathbb{R}) & & \\ & & \downarrow & & \\ \hat{R}_1^{-1}(C) & \longrightarrow & (tr \circ \hat{R}_1)^{-1}(t) & \longrightarrow & tr^{-1}(t) \\ & & \downarrow & & \\ & & \kappa^{-1}(t) & & \end{array}$$

Let ρ and $\theta(s)$ be as in Section 2 but with $s \in [0, 2\pi]$. Let

$$U = \hat{R}_1^{-1}(C)$$

$$V = (tr \circ \hat{R}_1)^{-1}(t)$$

$$W = tr^{-1}(t).$$

Define V_ψ to be the component of V containing ψ, W_g the component of W containing g and let $U_\psi = U \cap V_\psi$.

Remark. Proving $\tilde{R}_1^{-1}(C)$ is connected is the same as proving $\pi(U_\psi)$ is connected for each $\psi \in V$ where $C = \hat{R}_1(\psi)$.

The vertical sequence is a fibration for $t \neq 2$. Since $\hat{R}_1$ is equivariant with respect to the conjugation action of G and this action is transitive on $tr^{-1}(t)$ for $t \neq \pm 2$, the horizontal sequence is a fibration for $t \neq \pm 2$.

Case 1. $t < -2$. $\kappa^{-1}(t)$ is homeomorphic to S^2 minus two disks, and is connected (see Fig. 2). The vertical fibration induces the exact homotopy sequence

$$\pi_1(\kappa^{-1}(t)) \to \pi_0(PG) \to \pi_0(V) \to 0.$$

Since PG has two components, V has at most two components.

Suppose V has two components. Let

$$\psi = (X, Y), \quad \text{where } X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix} \text{ and } t = -u^2 - u^{-2}.$$

A direct calculation shows

$$\hat{R}_1(\psi) = \hat{R}_1(\rho * \psi) = \begin{pmatrix} -1/u^2 & 0 \\ 0 & -u^2 \end{pmatrix}.$$

This implies

$$tr(\hat{R}_1(\psi)) = tr(\hat{R}_1(\rho * \psi)) = t.$$

Hence, we may choose C to be $\hat{R}_1(\psi)$. Since $\rho \in PG \backslash PG_0$, the two components of V must be V_ψ and $V_{\rho * \psi}$.

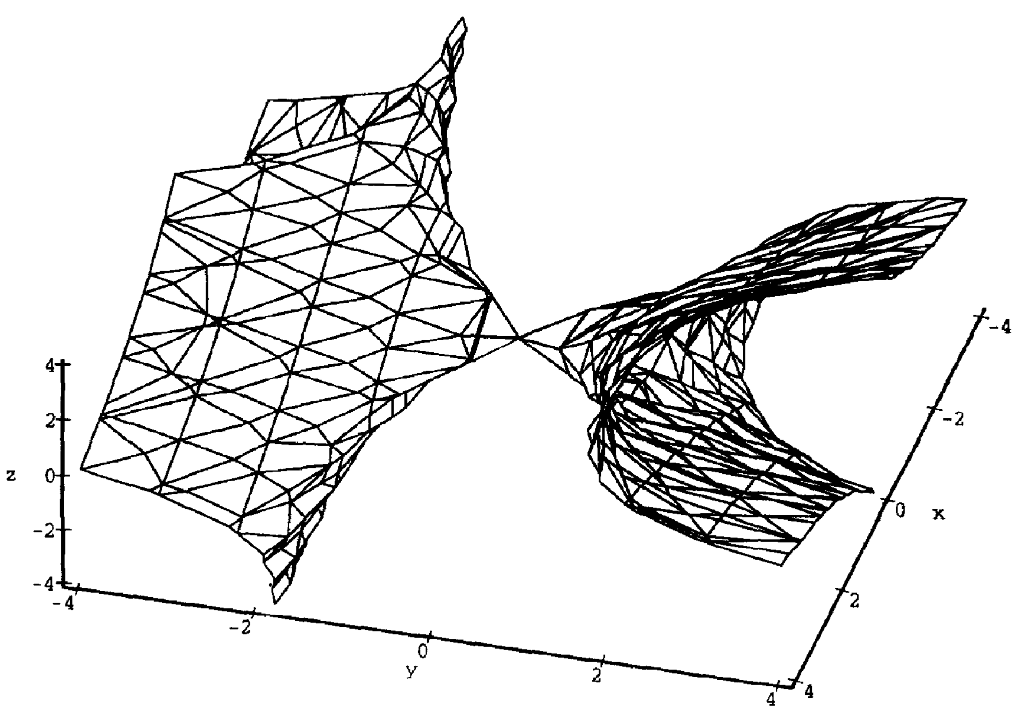

Fig. 2. $t = -2$.

The horizontal fibration then splits into two fibrations

$$U_\psi \to V_\psi \to W$$

$$U_{\rho*\psi} \to V_{\rho*\psi} \to W.$$

From these, we obtain the exact sequences

$$\pi_1(V_\psi, \psi) \to \pi_1(W, C) \to \pi_0(U_\psi, \psi) \to 0$$

$$\pi_1(V_{\rho*\psi} \Lambda\, \rho * \psi) \to \pi_1(W, C) \to \pi_0(U_{\rho*\psi}, \rho * \psi) \to 0.$$

The space W is also homeomorphic to S^2 minus two disks and $\pi_1(W, C)$ is generated by $\theta(s) * C$. Consider

$$\theta(s) * \psi \in \pi_1(V_\psi, \psi) \quad \text{and} \quad \theta(s) * (\rho * \psi) \in \pi_1(V_{\rho*\psi}, \rho * \psi).$$

The induced map $\hat{R}_{1*}$ takes both of them to $\theta(s) * C$. This shows U_ψ and $U_{\rho*\psi}$ are both connected. Since $\pi(\psi) = \pi(\rho * \psi)$, we conclude, from Lemma 4.1, that $\tilde{R}_1^{-1}(C) = \pi(U)$ is connected.

In case V happens to be connected, the induced horizontal sequence is simply

$$\pi_1(V, \psi) \to \pi_1(W, C) \to \pi_0(U, \psi) \to 0.$$

$\theta(s) * \psi$ is then mapped to $\theta(s) * C$ and this means $\hat{R}_1^{-1}(C) = U$ is connected and so is $\tilde{R}_1^{-1}(C) = \pi(U)$.

Remark. We believe V has two components.

Case 2. $t \in (-2, 2)$. In this case, $\kappa^{-1}(t)$ has two components (see Fig. 3) and each is a disk; hence, each component is contractible. From the vertical fibration, we conclude that V has four components. W is also the disjoint union of two disks. Hence, from the exact sequences that come from the horizontal fibration, we conclude that U together with $\rho * U$ has a total of four components. Let

$$\psi = (X, Y) \quad \text{where } X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} \cos(u) & -\sin(u) \\ \sin(u) & \cos(u) \end{pmatrix} \quad \text{and } t = 2\cos(2u).$$

Choose C to be $\hat{R}_1(\psi)$. Then C and $\rho * C$ belong to the two disks of W, respectively.

Each one of the four components of $U \cup \rho * U$ contains one of the four points:

$$\psi, \; -\psi, \; \rho * \psi, \; -\rho * \psi.$$

This means U has two components containing ψ, $-\psi$, respectively (note χ maps ψ, $-\psi$ to different components of $\kappa^{-1}(t)$ which is separated by the plane $y = 0$; see Fig. 3), and $\rho * U$ has two components containing $\rho * \psi$, $-\rho * \psi$, respectively. Finally, π maps ψ, $-\psi$ to a single point and $\rho * \psi$, $-\rho * \psi$ to a single point.

Case 3. $t > 2$. Let

$$\psi = (X, Y) \quad \text{where } X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \text{ and } t = u^2 + u^{-2},$$

and choose $C = \hat{R}_1(\psi)$.

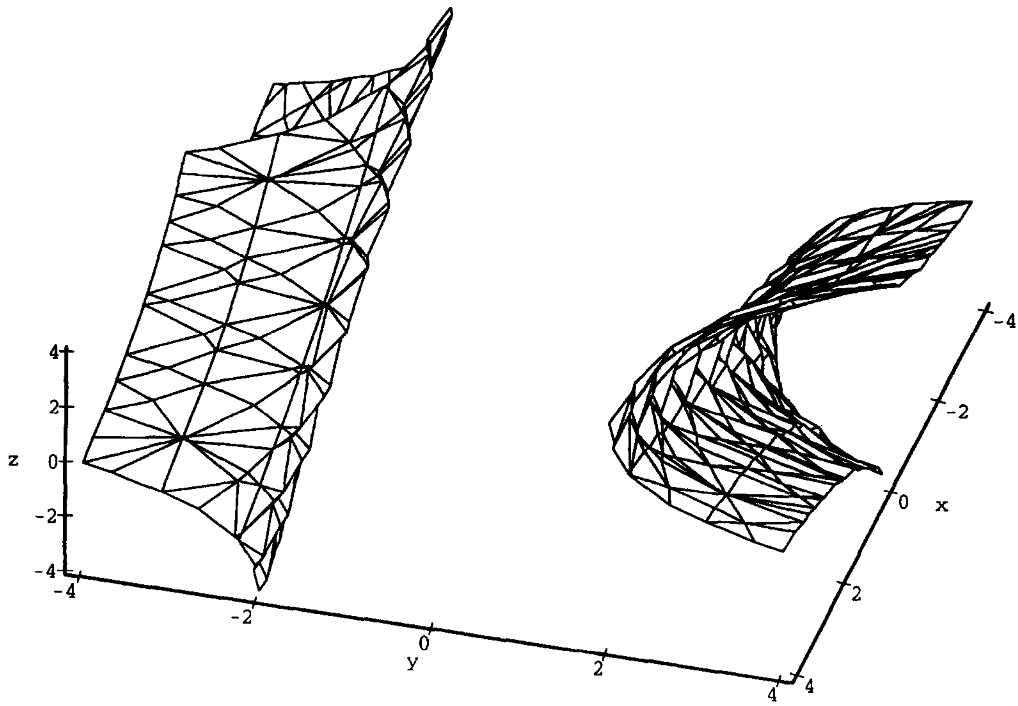

Fig. 3. $t > -2$.

The vertical fibration is exactly the same as in Case 2, and we conclude that V has four components. The horizontal fibration splits into four fibrations

$$\begin{array}{lll} U_{\psi} & \to V_{\psi} & \to W \\ U_{-\psi} & \to V_{-\psi} & \to W \\ U_{\rho*\psi} & \to V_{\rho*\psi} & \to W \\ U_{-\rho*\psi} & \to V_{-\rho*\psi} & \to W. \end{array}$$

These induce homotopy exact sequences

$$\begin{array}{lllll} \pi_1(V_{\psi}, \psi) & \to \pi_1(W, C) \to & \pi_0(U_{\psi}, \psi) & \to & 0 \\ \pi_1(V_{-\psi}, -\psi) & \to \pi_1(W, C) \to & \pi_0(U_{-\psi}, -\psi) & \to & 0 \\ \pi_1(V_{\rho*\psi}, \rho*\psi) & \to \pi_1(W, C) \to & \pi_0(U_{\rho*\psi}, \rho*\psi) & \to & 0 \\ \pi_1(V_{-\rho*\psi}, -\rho*\psi) & \to \pi_1(W, C) \to & \pi_0(U_{-\rho*\psi}, -\rho*\psi) & \to & 0. \end{array}$$

The space W is S^2 minus two disks and $\pi_1(W, C)$ is generated by $\theta(s) * C$. The four homotopy classes

$$\theta(s) * \psi, \ -\theta(s) * \psi, \ \theta(s) * (\rho * \psi), \ -\theta(s) * (\rho * \phi)$$

belong to

$$\pi_1(V_{\psi}, \psi), \ \pi_1(V_{-\psi}, -\psi), \ \pi_1(V_{\rho*\psi}, \rho*\psi), \ \pi_1(V_{-\rho*\psi}, -\rho*\psi)$$

respectively, and all are mapped to $\theta(s) * C$ by $\hat{R}_{1*}$. We conclude that U has four connected pieces U_ψ, $U_{-\psi}$, $U_{\rho*\psi}$, $U_{-\rho*\psi}$. Each component contains one point from the set

$$\{\psi, -\psi, \rho*\psi, -\rho*\psi\},$$

and π maps all these four points to a single point.

Case 4. $t = -2$. Here, we have two PG conjugacy classes: $\{-I\}$ and P_-. Note that $\hat{R}_1(\psi) = -I$ iff $\chi(\psi) = (0,0,0)$. PG acts transitively on $\chi^{-1}(0,0,0)$, so $\chi^{-1}(0,0,0)$ has two components. The two components contain ψ and $\rho*\psi$, respectively, where

$$\psi = (X, Y) \text{ and } X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The covering map, π, maps these two points to a single point.

The parabolic class is $tr^{-1}(-2)\backslash\{-I\}$. The commutative diagram becomes

$$\begin{array}{ccccc} & & PG = PGL(2,\mathbb{R}) & & \\ & & \downarrow & & \\ \hat{R}_1^{-1}(C) & \longrightarrow & (tr\circ\hat{R}_1)^{-1}(-2)\backslash\chi^{-1}(0,0,0) & \longrightarrow & tr^{-1}(-2)\backslash\{-I\} \\ & & \downarrow & & \\ & & \kappa^{-1}(-2)\backslash\{(0,0,0)\} & & \end{array}$$

where $tr(C) = t = -2$ but $C \neq -I$. With this modification, both sequences are then fibrations.

Now we modify slightly the definition of V and W. Let

$$V = (tr\circ\hat{R}_1)^{-1}(-2)\backslash\chi^{-1}(0,0\,0)$$
$$W = tr^{-1}(-2)\backslash\{-I\}$$
$$K = \kappa^{-1}(-2)\backslash\{(0,0,0)\}.$$

The vertical fibration resembles that of Case 1, but here K consists of two punctured disks instead of a tube (note that the singularity $(0,0,0)$ is removed from Fig. 1). Let

$$\psi = (X, Y) \quad \text{where } X = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

and $C = \hat{R}_1(\psi)$. Let K_ψ and $K_{-\psi}$ be the two components of K such that $\chi(\psi) \in K_\psi$ and $\chi(-\psi) \in K_{-\psi}$.

The vertical fibration then splits into the two fibrations

$$PG \to V_\psi \to K_\psi$$
$$PG \to V_{-\psi} \to K_{-\psi}.$$

These induce the homotopy sequences

$$\pi_1(K_\psi) \to \pi_0(PG) \to \pi_0(V_\psi) \to 0$$
$$\pi_1(K_{-\psi}) \to \pi_0(PG) \to \pi_0(V_{-\psi}) \to 0.$$

Since PG has two components, both V_ψ and $V_{-\psi}$ have at most two components. Together, V has at most four components.

Suppose V has four components. As in Case 3, the horizontal fibration splits four ways

$$\begin{array}{ccccc} U_\psi & \to & V_\psi & \to & W_C \\ U_{-\psi} & \to & V_{-\psi} & \to & W_C \\ U_{\rho*\psi} & \to & V_{\rho*\psi} & \to & W_{\rho*C} \\ U_{-\rho*\psi} & \to & V_{-\rho*\psi} & \to & W_{\rho*C}. \end{array}$$

These induce homotopy exact sequences

$$\begin{array}{ccccccc} \pi_1(V_\psi, \psi) & \to & \pi_1(W_C, C) & \to & \pi_0(U_\psi, \psi) & \to & 0 \\ \pi_1(V_{-\psi}, -\psi) & \to & \pi_1(W_C, C) & \to & \pi_0(U_{-\psi}, -\psi) & \to & 0 \\ \pi_1(V_{\rho*\psi}, \rho*\psi) & \to & \pi_1(W_{\rho*C}, \rho*C) & \to & \pi_0(U_{\rho*\psi}, \rho*\psi) & \to & 0 \\ \pi_1(V_{-\rho*\psi}, -\rho*\psi) & \to & \pi_1(W_{\rho*C}, \rho*C) & \to & \pi_0(U_{-\rho*\psi}, -\rho*\psi) & \to & 0. \end{array}$$

The space W consists of two punctured disks with $\pi_1(W_C, C)$ generated by $\theta(s) * C$ and $\pi_1(W_{\rho*C}, \rho * C)$ generated by $\theta(s) * (\rho * C)$. Since

$$\hat{R}_{1*}(\theta(s) * \psi) = \hat{R}_{1*}(-\theta(s) * \psi) = \theta(s) * C$$

$$\hat{R}_{1*}(\theta(s) *(\rho * \psi)) = \hat{R}_{1*}(-\theta(s) *(\rho * \psi)) = \theta(s) *(\rho * C)$$

we conclude that U has two components containing ψ, $-\psi$, respectively, and $\rho * U$ has two components containing $\rho * \psi$, $-\rho * \psi$, respectively. π then maps ψ, $-\psi$ to a single point and $\rho * \psi$, $-\rho * \psi$ to a single point.

Had V only two components, both U and $\rho * U$ would be connected and the conclusion would follow as in Case 1. Again, we believe that V has four components.

Case 5. $t = 2$. By Proposition 4.2, $\langle X, Y\rangle$ is reducible. Hence, conjugating by an element in PG, we may assume X and Y are both upper triangular matrices. Since $\det(X) = -1$, X is diagonalizable. Thus, we may further assume that

$$X = \begin{pmatrix} a & 0 \\ 0 & -1/a \end{pmatrix}, \quad Y = \begin{pmatrix} b & \eta \\ 0 & 1/b \end{pmatrix}.$$

This implies

$$\hat{R}_1(X, Y) = \begin{pmatrix} 1 & -b(a^2+1)\eta \\ 0 & 1 \end{pmatrix}.$$

Thus, we may assume

$$C = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}.$$

This implies $\hat{R}_1^{-1}(C)$ is the variety

$$\{(a, b, \eta) \colon a, b \neq 0, \ -b(a^2+1)\eta = c\} \subset \mathbb{R}^3.$$

This set has four components, and (X, Y), $(-X, Y)$, $(X, -Y)$, $(-X, -Y)$ belong to these four components, respectively. Clearly, π maps these four points to a single point. It follows that $R_1^{-1}(C_0)$ is connected. ■

This completes the proof of Proposition 2.2.

5. AWAY FROM THE SINGULARITY

In this section, we prove Proposition 2.3 and fill a gap in [1]. We begin by adopting the entire set of notations used in Section 2. In addition, let $\tilde{G}_0$ be the universal cover of PG_0 and $\tilde{\tilde{R}}_0 : PG_0^2 \mapsto \tilde{G}_0$ the lift of $\tilde{R}_0$. By [1], Theorem 7.1, the fibre of $\tilde{\tilde{R}}_0$ is connected with its image being the set

$$\mathscr{I} = \{\tilde{I}\} \cup Ell_{\pm 1} \cup Par_0^{\pm} \cup Hyp_0 \cup Par_{\mp 1}^{\pm} \cup Hyp_{\pm 1}.$$

An immediate consequence is that the fibre $\tilde{R}_0^{-1}(I)$ is connected. Furthermore with the results obtained in [2], we conclude that $\tilde{\tilde{R}}_0$ is a submersion onto $\mathscr{I} \backslash \{\tilde{I}\}$. Hence, paths in $\mathscr{I} \backslash \{\tilde{I}\}$ can be lifted to PG_0^2 via $\tilde{\tilde{R}}_0$.

LEMMA 5.1. *Let*

$$T = \{(X_1, Y_1, X_2, Y_2) \colon \tilde{R}_0(X_1, Y_1)\tilde{R}_0(X_2, Y_2) = D\} \subset PG_0^4$$

$$S = \{(X_1, Y_1, X_2, Y_2) \in T \colon \tilde{R}_0(X_1, Y_1) = C\}.$$

Then every point in S can be deformed to a point in $T \backslash S$.

Proof. Let $\Gamma = (X_1, Y_1, X_2, Y_2) \in S$ and $C' = C^{-1}D$.

Case 1. $C \neq 1$ and $C' \neq I$. Let $p(s) \neq I$ be a local path with $p(0) = C$ and $p(s) \neq C$ for $s \in (0, 1]$ and lift the local path $p(s)$ to a path $\tilde{p}(s) \in \tilde{G}_0$ with

$$\tilde{\tilde{R}}_0(X_1, Y_1) = \tilde{p}(0).$$

Then $\tilde{p}(s) \neq \tilde{I}$, so we may lift the local path $\tilde{p}(s)$ to a path $P(s) \in PG_0^2$ via $\tilde{\tilde{R}}_0$. Similarly, we let $q(s) = (p(s))^{-1}$, and lift it to a path $Q(s) \in PG_0^2$ via $\tilde{\tilde{R}}_0$. The path $(P(s), Q(s))$ then provides the required deformation.

Case 2. $C = I$ and $C' \neq I$. Since $\tilde{R}_0^{-1}(I)$ is connected, we may assume

$$X_1 = [I], \qquad Y_1 = \left[\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}\right].$$

Let

$$X_1(s) = \left[\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\right], \qquad Y_1(s) = \left[\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}\right].$$

Then $p(s) = \tilde{R}_0(X_1(s), Y_1(s))$ is a path in G_0 with $p(0) = I$. Let

$$\tilde{C}' = \tilde{\tilde{R}}_0(X_2, Y_2).$$

Since

$$\tilde{R}_0(X_2, Y_2) = C' \neq I$$

$\tilde{\tilde{R}}_0$ is a submersion at $\tilde{C}'$. For small $\varepsilon > 0$, consider the local path

$$q(s) = (p(s))^{-1} C' \neq I \quad \text{for } s \in [0, \varepsilon].$$

We may lift $q(s)$ to a path $\tilde{q}(s)$ in $\tilde{G}$ with

$$\tilde{q}(s) \neq \tilde{I} \quad \text{and} \quad \tilde{q}(0) = \tilde{C}'.$$

We further lift the path $\tilde{q}(s)$ via $\tilde{\tilde{R}}_0$ to a path $(X_2(s), Y_2(s)) \in PG_0^2$ with

$$(X_2(0), Y_2(0)) = (X_2, Y_2).$$

Since

$$\tilde{R}_0(p(s)) \neq I \quad \text{for} \quad s \in (0, \varepsilon]$$

the path

$$(X_1(s), Y_1(s), X_2(s), Y_2(s))$$

provides the required deformation.

Case 3. $C \neq I$ and $C' = I$. This case is an exact mirror image of Case 2.

Case 4. $C = I$ and $C' = I$. Since $\tilde{R}_0^{-1}(I)$ is connected we may assume (X_1, Y_1) to be as defined in Case 2 and set $(X_2, Y_2) = (Y_1, X_1)$. Similarly, let $(X_1(s), Y_1(s))$ be as defined in Case 2 and set $(X_2(s), Y_2(s)) = (Y_1(s), X_1(s))$. Then

$$[X_1(s), Y_1(s)]\,[X_2(s), Y_2(s)] = [X_1, Y_1]\,[X_2, Y_2] = I$$

and $[X_1(s), Y_1(s)] \neq I$ for $s \in (0, 1]$. Hence, the path

$$(X_1(s), Y_1(s), X_2(s), Y_2(s))$$

provides the required deformation. ∎

LEMMA 5.2. *Let*

$$T = \{(X_1, Y_1, X_2, Y_2)\colon \tilde{R}_1(X_1, Y_1)\tilde{R}_0(X_2, Y_2) = D\} \subset PG_1 \times PG_0^3$$

$$S = \{(X_1, Y_1, X_2, Y_2) \in T\colon \tilde{R}_0(X_1, Y_1) = C\}.$$

Then every point in S can be deformed to a point in $T \setminus S$.

Proof. This is very close to the proof for Lemma 5.1, but we provide it here for the purpose of completeness and clarity. Let $\Gamma = (X_1, Y_1, X_2, Y_2) \in S$ and $C' = C^{-1}D$.

Case 1. $C \neq I$ and $C' \neq I$. Let $p(s) \neq I$ be a local path with $p(0) = C$ and $p(s) \neq C, s \in (0, 1]$. Then we may lift the local path $p(s)$ to a path $P(s) \in PG_1 \times PG_0$ via $\tilde{R}_1$. Let $q(s) = (p(s))^{-1}$. Then we may lift it to a path $\tilde{q}(s) \in \tilde{G}_0$ (with $\tilde{\tilde{R}}_0(X_2, Y_2) = \tilde{q}(0)$) and further lift it to a path in $Q(s) \in PG_0^2$ via $\tilde{\tilde{R}}_0$ The path $(P(s), Q(s))$ then provides the required deformation.

Case 2. $C = I$ and $C' \neq I$. $\tilde{R}_1^{-1}(I)$ is connected. Hence, we may assume

$$X_1 = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right], \qquad Y_1 = \left[\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}\right].$$

Let

$$X_1(s) = \left[\begin{pmatrix} 1 & -4s \\ 0 & -1 \end{pmatrix}\right], \qquad Y_1(s) = \left[\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}\right].$$

Then $p(s) = \tilde{R}_1(X_1(s), Y_1(s))$ is a path in G_0 with $p(0) = I$. Let

$$\tilde{C}' = \tilde{\tilde{R}}_0(X_2, Y_2).$$

Since

$$\tilde{R}_0(X_2, Y_2) = C' \neq I$$

$\tilde{\tilde{R}}_0$ is a submersion at $\tilde{C}'$. For small $\varepsilon > 0$, consider the local path

$$q(s) = (p(s))^{-1} C' \neq I \quad \text{for } s \in [0, \varepsilon].$$

We may lift $q(s)$ to a path $\tilde{q}(s)$ in $\tilde{G}$ with

$$\tilde{q}(s) \neq \tilde{I} \quad \text{and} \quad \tilde{q}(0) = \tilde{C}'.$$

We further lift the path $\tilde{q}(s)$ via $\tilde{\tilde{R}}_0$ to a path $(X_2(s), Y_2(s)) \in PG_0^2$ with

$$(X_2(0), Y_2(0)) = (X_2, Y_2).$$

Since

$$\tilde{R}_0(p(s)) \neq I \quad \text{for } s \in (0, \varepsilon]$$

the path

$$(X_1(s), Y_1(s), X_2(s), Y_2(s))$$

provides the required information.

Case 3. $C \neq I$ and $C' = I$. Since $\tilde{R}_0^{-1}(I)$ is connected, we may assume

$$X_2 = [I], \quad Y_2 = \left[\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\right].$$

Let

$$X_2(s) = \left[\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\right], \quad Y_2(s) = \left[\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\right].$$

Then $q(s) = \tilde{R}_0(X_2(s), Y_2(s))$ is a path in G_0 with $q(0) = I$. $\tilde{R}_1(X_1, Y_1) = C \neq I$; hence R_1 is a submersion at C. For small $\varepsilon > 0$, consider the local path

$$p(s) = C(q(s))^{-1} \neq I \quad \text{for } s \in [0, \varepsilon].$$

We may lift the path $p(s)$ via $\tilde{R}_1$ to a path $(X_1(s), Y_1(s)) \in PG_1 \times PG_0$. Since

$$\tilde{R}_1(p(s)) \neq I \quad \text{for } s \in (0, \varepsilon],$$

the path

$$(X_1(s), Y_1(s), X_2(s), Y_2(s))$$

provides the required deformation.

Case 4. $C = I$ and $C' = I$. $\tilde{R}_0^{-1}(I)$ is connected, so we may assume (X_1, Y_1) to be as defined in Case 2 and (X_2, Y_2) as in Case 3. Similarly, let $(X_1(s), Y_1(s))$ be as defined in Case 2 and $(X_2(s), Y_2(s))$ as in Case 3. Then

$$[X_1(s), Y_1(s)]\,[X_2(s), Y_2(s)] = [X_1, Y_1]\,[X_2, Y_2] = I$$

and $[X_1(s), Y_1(s)] \neq I$ for $s \in (0, 1]$. Hence, the path

$$(X_1(s), Y_1(s), X_2(s), Y_2(s))$$

provides the required deformation.

Now we prove Proposition 2.3. Let

$$\Gamma = (X_1, Y_1, \dots, X_g, Y_g) \in o_1^{-1}(a_1)$$
$$D = \tilde{R}_1(X_1, Y_1)\tilde{R}_0(X_2, Y_2).$$

With Lemma 5.2, we may assume

$$\tilde{R}_1(X_1, Y_1) \neq I$$
$$\tilde{R}_0(X_2, Y_2) \neq I$$
$$D \neq \pm C_0.$$

Note, since C_0 is a hyperbolic element, $D = \pm C_0$ implies $g > 2$. Hence, by Lemma 5.1, we may deform (X_2, Y_2, X_3, Y_3) so that after the deformation, the new D is no longer $\pm C_0$.

By [1] Theorem 7.1, we may construct a contractible path $q(s) \in G_0, s \in [0, 1]$ between $(\tilde{R}_1(X_1, Y_1))^{-1}D$ and $C_0^{-1}D$ with the following properties:

(1) $q(s) \neq \pm I$.
(2) $q(s)$ can be lifted to path $\tilde{q}(s) \in \mathscr{I}\backslash\{\tilde{I}\}$.

Let $p(s) = (q(s))^{-1}$. Then, we may lift $p(s)$ and $\tilde{q}(s)$ to paths $P(s) \in PG_1 \times PG_0$ and $Q(s) \in PG_0^2$ (with $(X_1, Y_1) = P(0)$ and $(X_2, Y_2) = Q(0)$) via $\tilde{R}_1$ and $\tilde{\tilde{R}}_0$, respectively. Then the path

$$(P(s), Q(s), X_3, Y_3, \dots, X_g, Y_g))$$

deforms Γ to a representation in

$$\mathscr{HYP} = \{R_1 = (R_2)^{-1} = [C_0]\}. \quad \blacksquare$$

COROLLARY 5.1. *Let*

$$S = \{(X_1, Y_1, \dots, X_g, Y_g) \in PG^{2g}: \exists i \ni R_0(X_i, Y_i) = [I]\}.$$

Then we have

(1) *Every element in PG^{2g} can be deformed to an element in $PG^{2g}\backslash S$.*
(2) *Every element in $o_1^{-1}(0)$ can be deformed to an element in $o_1^{-1}(0)\backslash S$.*

Proof. Let $\Gamma = (X_1, Y_1, \dots, X_g, Y_g) \in S$. Apply Lemma 5.1. iteratively to each pair (X_i, Y_i) with $\tilde{R}_0(X_i, Y_i) = \pm I$.

Remark. The proof of Lemma 8.1 in [1] depends on the statement that a proper subvariety of an irreducible variety is topologically nowhere dense. While the statement is probably true for the complex varieties, it is not for the real varieties. Consider the irreducible variety

$$\{x^2 + y^2 + z^2 - xyz = 0\}.$$

The point $(0, 0, 0)$ is a proper subvariety, but is open in the subspace topology of the variety. Corollary 5.1 is weaker than Lemma 8.1 in [1], but is sufficient for the purpose of [1].

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